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ORIGINAL RESEARCH

A seed-based structural model for constructing rhombic quasilattice with 7-fold symmetry

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Abstract

A seed-based theoretical model with built-in local degree of freedom for constructing rhombic quasilattice with 7-fold symmetry is presented. This new approach mitigates a key limitation with existing structural models for describing quasicrystals, which do not incorporate atomic fluctuations or phasonic flips in their approaches. Here, we propose a structural model that works in concert with the seed-initiated nucleation growth models of quasicrystals and incorporates a degree of flexibility that allows the lattice to rearrange locally without affecting the global long-range order. This approach suggests that the position of high-symmetry motifs locally and globally is defined by one long-range framework and not based on local rules (i.e., inflation, deflation, substitution, matching, overlapping, etc.). The proposed model is based on building a hierarchical network that allows the self-similar quasilattice to expand infinitely without any gaps, overlaps, or mismatches. The use of a global relational logic provides scientists, artists, and teachers with a simple method for creating a wide variety of complicated hierarchical quasilattice formations without the need for any specialized software or complicated mathematics and could possibly provide a deeper understanding of how the atoms interact to form their complicated long-range quasicrystalline formations.

Keywords Seven-fold symmetry · Quasilattice · Quasicrystals · Phasonic flips · Structural model · Self-similarity

Introduction

Quasiperiodic structures with long-range orientational order but lacking translational symmetry have been reported in a wide range of materials [1–8]. This new state of matter, also known as quasicrystals, exhibits rotational symmetries, which were thought to be impossible for the periodic structures within the traditional framework of crystallography. A key characteristic of these formations is their self-similar hierarchical structures which allow the same patterns to recur at multiple scales. The most commonly occurring rotational symmetries in quasicrystals are 5-, 8-, 10-, and 12-fold, which have been widely reported and studied in the literature. In contrast, rotational symmetries with 7-, 9-, 11-, 13-, 14-, or higher have rarely been observed [9, 10]. While it is still unclear to why these symmetries are harder to form, they hold special significance because of their complicated but homogeneous lattice arrangements. One intriguing potential of their unique quasilattice formations is the ability to mold, orchestrate, or prevent the flow of light or sound energy, which could find applications across a wide range of light and sound-based technologies. Recently, increased effort has been dedicated to exploring the possibility of designing and fabricating artificial two-dimensional and three-dimensional quasilattice structures for their potential applications as photonic devices, heterostructures, and optical metamaterials [11–18]. Roichman and Grier were able to assemble 7-fold quasicrystalline heterostructures using holographic optical trapping technique [12]. Jules Michael and his colleagues employed colloidal particles in laser fields to investigate the geometric constrains that affect the formation of 7- and 9-fold quasiperiodic symmetries and concluded that such probability is connected to the frequency of the local high-symmetry motifs [13]. Additionally, when comparing light-induced colloidal quasicrystals with different symmetries, it was found that the laser intensities needed to induce high rotational symmetry is much higher than their lower counterparts [13, 14]. Dong and colleagues showed that quasiperiodic lattice with rotational symmetries higher than 5-fold can behave as zero-refractive-

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index materials [15]. In other words, light propagates though such lattice without any phase accumulation "leaping through the medium as if it were completely invisible" [16]. While there still significant challenges to overcome before we fabricate the first invisible material, such findings open up new opportunities for exploring a wide range of unexamined high rotational geometries beyond the conventional quasiperiodic lattice systems [15, 16]. Martinsons and colleagues found that phasonic drifts can help facilitate the formation of two-dimensional quasicrystals with 14-fold symmetry [17]. Phasonic flips are inherent unique feature of quasicrystal structure that provide an additional degree of freedom by allowing correlated particle rearrangements [17], which could facilitate the growth of quasicrystals [18].

Today, the search for methods to design and fabricate structures with uncommon rotational symmetries is on full force. Because of the strong link between the structure of matter and related physical behavior, theoretical models for the formulation of unusual quasilattice can be very instrumental in advancing such agenda. It is the goal of this paper to introduce a new theoretical model for constructing two-dimensional 7-fold quasiperiodic lattice (analogue of Penrose), without appealing to any of the existing approaches (i.e., projection from higher dimensions, substitution, inflation/deflation, tiling, etc.). The new model, which is based on the global hierarchical nature of quasicrystals is able to explain the long-range order, which allows the quasilattice to expand ad infinitum without any overlaps, gaps or mismatches. Before we introduce the new model, it is beneficial to survey the literature for existing structural models for constructing 7- and 14-fold quasilattices.

The interest in 7-fold rotational symmetry and its relation to crystallography has been ongoing even before the discovery of quasicrystals [19–21]. However, the search for self-similar quasicrystalline lattice that exhibits 7-fold symmetry has proven to be a challenging pursuit. As compared to 5-fold Penrose quasicrystalline tilings, matching rules for 7-fold tilings may nevertheless exist, though they do not force the formation of unique defectless structures [22, 23]. To date, many different variations of self-similar tilings with 7-fold symmetries in two-dimensions have been proposed, most of which are derived from two principal approaches—projection from higher dimensions and various substitution and inflation-deflation techniques [24–35]. In 1993, Lancon and Billard obtained variations of binary and random binary tilings with 7-fold rotational symmetry using iteration of a substitution [24]. Franco proposed a heptagonal tiling of the plane using five basic triangular prototiles [25]. In 1996, Danzer derived the first 2D quasiperiodic tiling with 7-fold symmetry using self-consistent inflation rules of three triangles and their mirror images [26]. Using similar approach, Garcia-Escudero derived variations of the 2D 7-fold self-similar patterns using four triangular prototiles [27]. Harriss used inflation rules to devise a rhombic 7-fold tiling; however, the initial patch that he used does not appear in the inflated patches, which makes his tiling "singular" and not recurrent [28]. Derived from historical precedents, some researchers generated 7-fold self-similar designs by incorporating traditional motives into the rhombic inflation rules [29, 30]. Gähler and his colleagues employed computer algorithm to search for substitution rules on a set of triangles and found that there are many different inflation factors for 7-fold rotational symmetries [31]. Kari and Rissanen were successful in devising non-singular substitution rules to create 7-fold quasiperiodic tiling; however, they had to use a very large inflation factor to accomplish their goal [32]. Schoen was able to use inflation rules to create variations of 7-fold self-similar rhombic tilings; however, his rules created overlapping regions that needed to be replaced with "ad hoc arrangements" [22, 33]. Cyclotomic substitution rules were also used to produce variations of triangular and rhombic 7-fold tilings [23, 34]. Such approach allows the possibility of some patterns to cyclically morph into different patches and return back to the initial pattern after several iterations. In the case of rhombic tiling, Madison was unable to reduce the number of the nine basic tiles, and questions if some are redundant [23]. Both Madison and Savard used similar approaches to design 14-fold tilings [23, 35].

While these methods have contributed greatly to our understanding of quasilattices of high symmetry, there are still many challenges to devising rhombic 7-fold quasilattice system that is able to satisfy the clear and rigorous requirements, analogous of Penrose 5-fold quasilattice. Therefore, the goal of this paper is to present a new structural model for generating 7-fold rhombic quasilattice without appealing to any of the existing approaches (substitution, deflation/inflation, higher dimensions, edge-edge tiling, etc.). The new model utilizes a global hierarchical framework to orchestrate both the local and global 7-fold symmetry and is able to satisfy the following strict requirements:

1. The quasilattice can be expanded ad infinitum without any gaps, overlaps, or mismatches.
2. The quasilattice is self-similar, self-consistent, and not singular. The different hierarchical levels contain repeatedly appearing patches with the same original motifs, including 7-fold symmetry.
3. The quasilattice does not require any ad hoc *rearrangements*.
4. Generalized inflation/deflation rules can be derived using a small number of prototiles.

In addition, one of the key challenges in devising structural models for quasicrystals is the problem of incorporating atomic fluctuations or phasonic flips in the structural models of quasicrystals [36]. The proposed model offers a new insight into mitigating such challenge by incorporating a degree of local freedom without scarifying the global quasiperiodic order of 7-fold high-symmetry motifs. Derived from the traditional methods of Islamic geometry, the presented method is a continuation of our pervious effort which successfully used the global framework approach to construct a number of quasicrystalline formations including: Penrose 5-fold quasiperiodic tilings [37], 10-fold quasiperiodic tilings [38], Ammann–Beenker 8-fold quasiperiodic tilings [39] and 12-fold quasiperiodic tilings [40]. The proposed model is based on a hierarchical clustering order of quasicrystals [41–43]. In this order, the global quasilattice formation is constructed by building an infinite progression of hierarchical clusters, in which every new hierarchy is built on the previous level. The process of constructing the infinite 7-fold quasilattice is demonstrated in the following three sections: "The hierarchical framework," "Cluster configurations," and "Growing the quasilattice empire."

The hierarchical framework

The construction process of 7-fold rhombic quasilattice global empire is governed by a proportional system that is derived from the traditional method of Islamic geometry and is based on a tightly constructed proportional logic that is derived from the nature of the compass and a straightedge construction [37–40]. The construction process does not rely on any specific quantities, angles, distances, or follow localized procedures; instead, its logic utilizes the relationships inherent within the relational system, allowing a higher level of organization to happen intuitively and without using any complicated or abstract mathematics.

In general, constructing the 7-fold hierarchical quasilattice is based on combining two basic design elements: an underlying framework and a limited number of repeated *seed* clusters (units). In this system, the arrangements of clusters within the global empire as well as the arrangements of local building units within each cluster are determined entirely by one global framework. A framework of nested tetradecagrams serves as the underlying hidden grid for guiding the construction process of the high-symmetry motifs throughout the quasilattice structure. This framework is constructed by drawing a polar array of lines through connecting points of equal distances on the initial tetradecagon (Fig. 1). The hierarchical framework is attained by building a progression of nested tetradecagrams, in which every hierarchy is proportionally built on the previous one, resulting in a self-similar network (Fig. 1a–c). This progression of nested tetradecagrams servers a critical role in maintaining a relational aspect ratio between the different levels, which is the key to resolving the 7-fold quasiperiodic structure. If we denote the radius of the nth tetradecagon by $rad(n)$ and the next larger radius by $rad(n+1)$, then the ratio $rad(n+1)/rad\ (n)$ is equal to $1 + 2\cos(2\pi/7)$.

$$rad(n+1)/rad(n) = 1 + 2\cos(2\pi/7) = 2.24697960\ldots$$

Cluster configurations

According to the proposed model, for every hierarchy, three main clusters with unique 7-fold rotational symmetry exist, each can serve as the central *seed* cluster for initiating the growth of the infinite 7-fold quasilattice without any gaps, overlaps, or mismatches. A unique quality of these clusters is that they cyclically morph into each other with different-level hierarchies. Each cluster exhibits unique local symmetry formation which plays a vital role in generating the global quasilattice. The process for constructing the three main clusters of the first hierarchy is demonstrated in Fig. 2. Each cluster in this hierarchy is composed of three smaller *seed* units (clusters) with 7-fold rotational symmetry (Fig. 2a), and a limited number of connecting formations (fragments) (Fig. 2b). The size of the central *seed* unit is strictly derived from the progression sequence of the nested tetradecagrams (Fig. 1d). If we denote the radius of the *seed* unit by $rad(n)$ and the radius of the first hierarchy framework by $rad(n+2)$, then the ratio $rad(n+2)/rad(n)$ is equal to $(1 + 2\cos(2\pi/7))^2$.

$$rad(n+2)/rad(n) = (1 + 2\cos(2\pi/7))^2 = 5.048917339\ldots$$

The locations of the high-symmetry *seed* units are determined entirely by the intersection points generated by the framework of nested tetradecagrams as shown in Fig. 2c, d. Please note that for each cluster, the center of rotational symmetry is occupied by one of the three main *seed* units, while the remaining positions of high symmetry are occupied by a unique formation of the remaining two *seed* units. The gaps between the three distributed *seed* units are filled with unique but flexible connecting formations (Fig. 2d). The three final generated clusters of the first hierarchy are shown in Fig. 2e, f.

If we examine the final generated three main clusters in Fig. 2f, it is important to highlight that the main connecting arrangements can be rotated or flipped in place without dislocating the points of the high symmetry or changing the relative distances or orientations between the *seeds* of high symmetry (Fig. 3b–d). This flexibility provides an extra degree of freedom of rearrangement at the local level without scarifying the global order. Such flexibility could provide a theoretical grounding for explaining phasonic flips or the local fluctuations that are unique to quasicrystals.

Growing the quasilattice empire

Growing the global empire of the 7-fold quasilattice formations requires building a progression of multilevel hierarchical clusters, in which every higher-order cluster is built on the previous one. Accordingly, generating the next higher-level clusters follows the same process used to generate the three clusters of the first

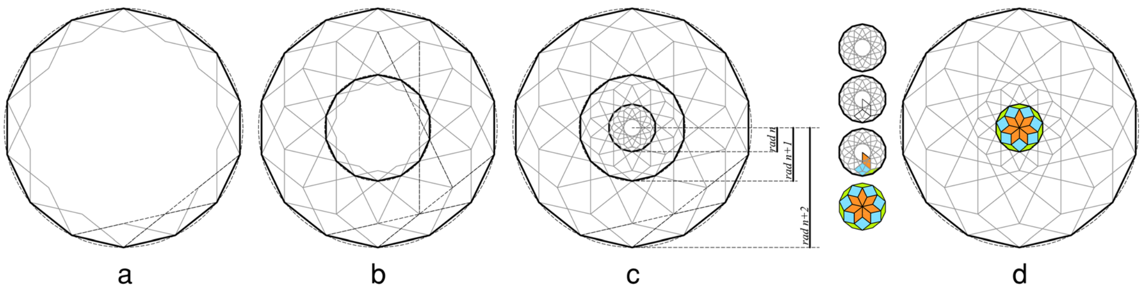

Fig. 1 A framework of nested tetradecagrams serves as the underlying hidden grid for guiding the construction process of the quasilattice system. **a**–**c** The hierarchical framework is attained by building a progression of nested tetradecagrams, in which every hierarchy is proportionally built on the previous one; resulting in a self-similar network. **d** The size of the central seed unit is strictly derived from the progression sequence of the nested tetradecagrams

hierarchy. In the 7-fold hierarchical system, the construction of the second higher level is governed by a new generation of the nested tetradecagrams which grows proportionally according to the ratio $(1 + 2\cos(2\pi/7))^2$. Similar to the first hierarchy, three main clusters with unique 7-fold rotational symmetry exist for the second hierarchy, each can serve as the central *seed* for growing the infinite 7-fold quasilattice without any gaps, overlaps, or mismatches. Each cluster exhibits unique local symmetry formation which plays a vital role in generating the global quasilattice.

The process for constructing the three main high-symmetry clusters of the second-level hierarchy is demonstrated in Figs. 4, 5, and 6. In this second level, the final generated clusters of the first hierarchy shown in Fig. 2f serve as the *seed* clusters (units) for the second hierarchy. Each cluster in the second-level hierarchy is composed of three high-symmetry smaller *seed* clusters

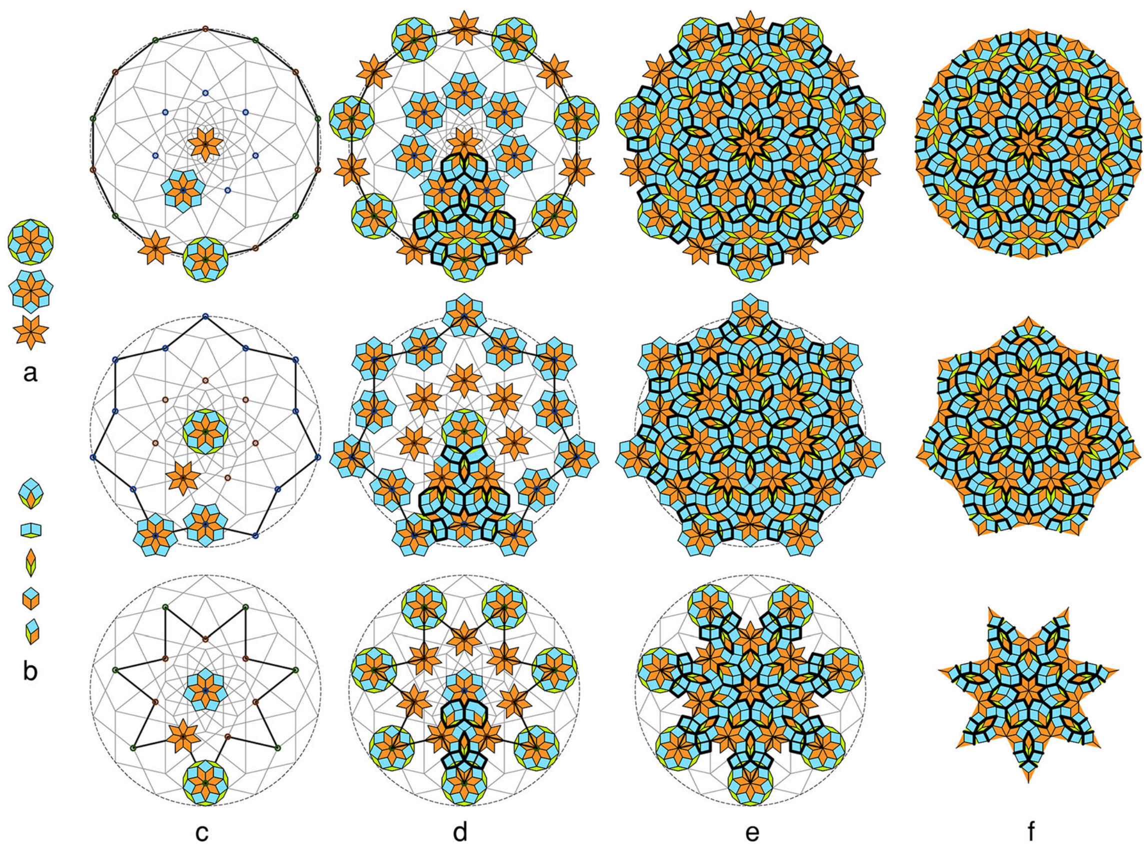

Fig. 2 The process for constructing the three main high-symmetry clusters of the first hierarchy of the 7-fold quasilattice. **a** The three *seed* units (clusters) with 7-fold rotational symmetry. **b** The connecting formations. **c**, **d** The locations of the high-symmetry *seed* units are determined entirely by the intersection points generated by the framework of nested tetradecagrams. **e**, **f** The three final generated clusters of the first hierarchy

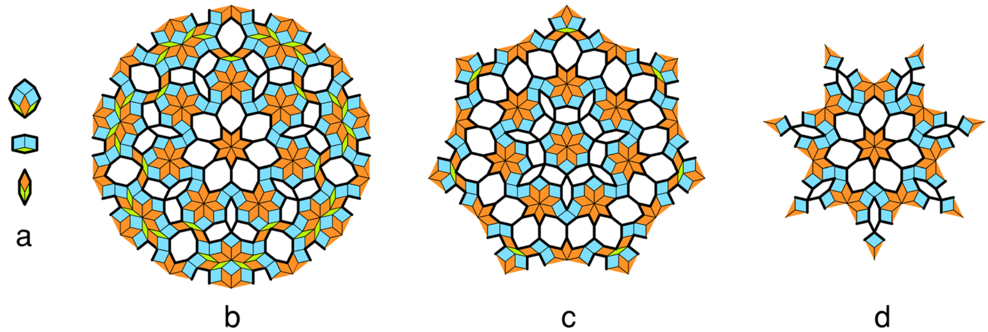

Fig. 3 The connecting formations provides an extra degree of local freedom, which allows a rearrangements to happen on the local level without affecting the global order. **a** The main connection formations. **b**–**d** The main connecting fragments can be rotated or flipped in place without dislocating the points of the high-symmetry or changing the relative distances or orientations between the *seeds* of high symmetry in the three clusters of the first hierarchy

(Figs. 4a, 5a, and 6a), and a limited number of connecting clusters or fragments (Figs. 4b, 5b, and 6b). All *seed* clusters of the second order are distributed according to the new generation framework of the nested tetradecagrams (Figs. 4c, 5c, and 6c). The locations of the high-symmetry seed clusters are determined entirely by the intersection points generated by the higher generation framework of nested tetradecagrams. Please note that for

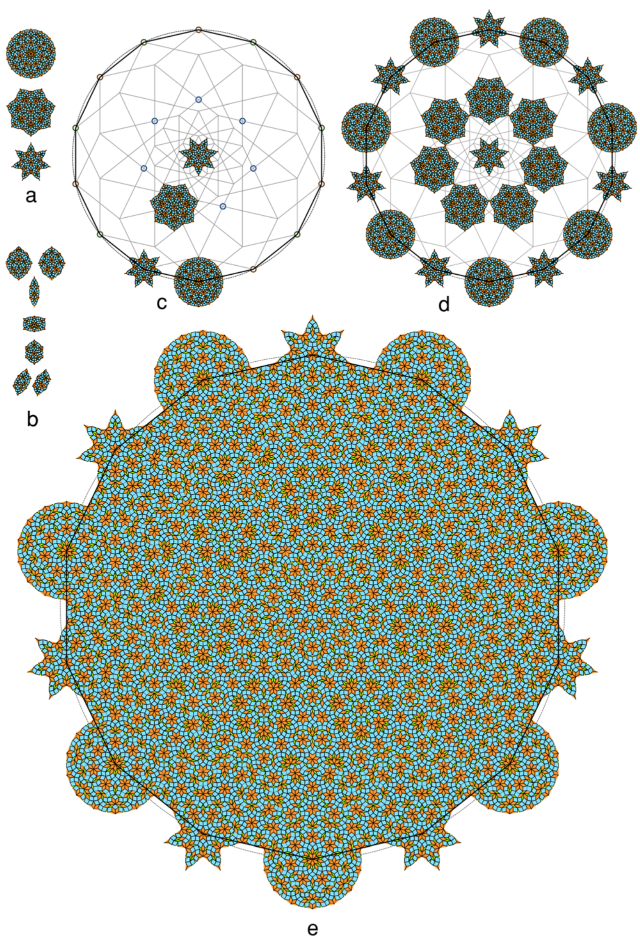

Fig. 4 The process of constructing the first high-symmetry cluster for the second hierarchy of the 7-fold quasilattice. **a** The three main *seed* clusters with high symmetry. **b** The connecting formations. **c**, **d** The locations of the high-symmetry *seed* clusters are determined entirely by the intersection points generated by the higher generation framework of nested tetradecagrams. **e** The final generated 7-fold clusters of the second hierarchy

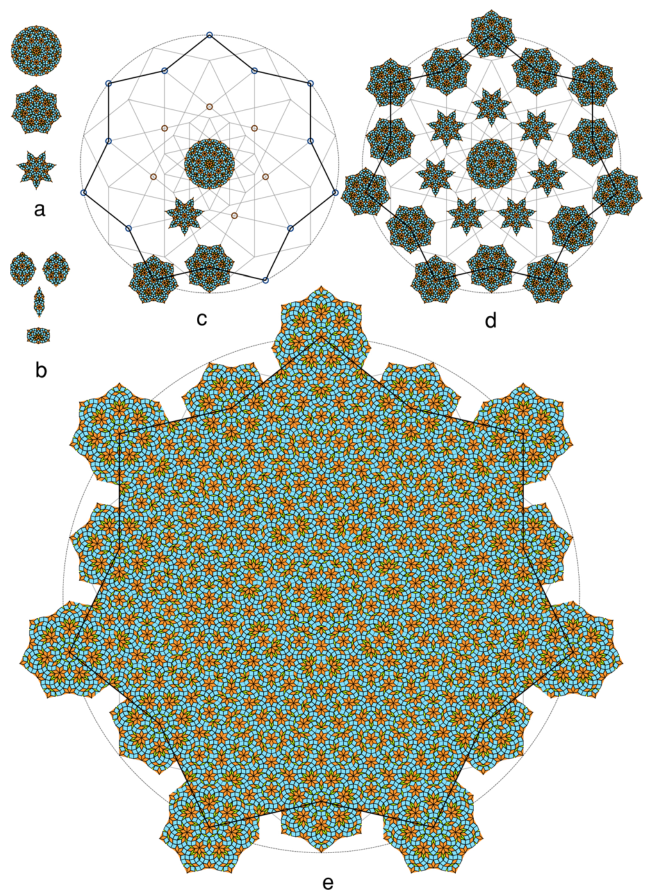

Fig. 5 The process of constructing the second high-symmetry cluster for the second hierarchy of the 7-fold quasilattice. **a** The three main *seed* clusters with high symmetry. **b** The connecting formations. **c**, **d** The locations of the high-symmetry *seed* clusters are determined entirely by the intersection points generated by the higher generation framework of nested tetradecagrams. **e** The final generated 7-fold clusters of the second hierarchy

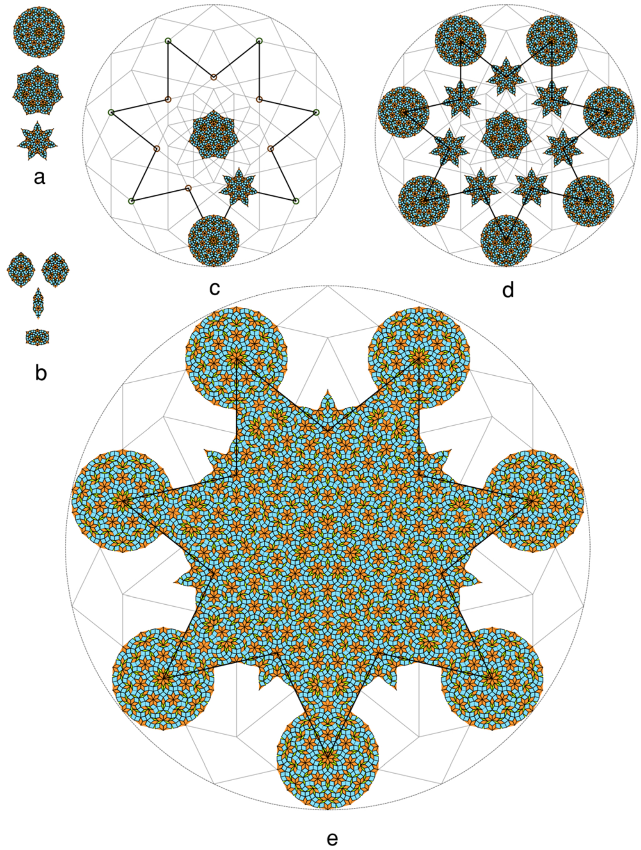

Fig. 6 The process of constructing the third high-symmetry cluster for the second hierarchy of the 7-fold quasilattice. **a** The three main *seed* clusters with high symmetry. **b** The connecting formations. **c**, **d** The locations of the high-symmetry *seed* clusters are determined entirely by the intersection points generated by the higher generation framework of nested tetradecagrams. **e** The final generated 7-fold clusters of the second hierarchy

each new generation cluster, the center of the rotational symmetry is occupied by one of the three main *seed* clusters (Figs. 4d, 5d, and 6d). Similarly to the first hierarchy, the gaps between the three *seed* clusters are filled with unique but flexible connecting formations. The three final generated 7-fold clusters of the second hierarchy are shown in Figs. 4e, 5e, and 6e. It is also important to point out that the connecting formations shown in Figs. 4b, 5b, and 6b can be rotated and locally rearranged without disturbing the global order. This flexibility of local rearrangement extends beyond the first hierarchy and provides an extra degree of freedom in every hierarchy. However, more investigation is still needed to define the exact rules for such rearrangement at every level.

Inflation/deflation rules

An important characteristic of this multilevel proportional system is that the same quasilattice formations recur at different scales. This quality is often described as "self-similarity" principle, which is the key principle in the structure of quasicrystals. A powerful tool associated with this property is the ability to derive deflation/inflation (substitution) rules to construct the infinite hierarchical tilings [44]. A close examination of all the generated clusters of the second hierarchy reveals simple but flexible deflation/inflation rules (Figs. 7, 8, and 9). Figures 7a, 8a, and 9a show the flexible connecting formations of the first hierarchy which can be used to derive the generalized substitution rules of the second hierarchy based on the ratio of $(1 + 2\cos(2\pi/7))^2$ (Figs. 7b, c, 8b, c, and 9b, c). In principle, for any substitution tiling, it is possible to construct local matching rules that can force the global hierarchical structure to emerge [45]. This suggests that matching rules for 7-fold tiling do exist. However, because of the built-in local flexibility, this may require the distinction of many tiles of equal shapes, resulting in a large total number of tiles and their associated matching rules.

It is worth noting that the imbedded localized flexibility can be found in the connecting formations in every level in this hierarchical system. This degree of local flexibility can be seen as a generalization of the classical "deterministic" inflation/deflation rules of ideal self-similar quasiperiodic tilings (i.e., Penrose tiling, Ammann octagonal tiling, pinwheel tiling, etc.). This approach has similarities with random substitution rules, which allow variations based on a finite set of possible arrangements at every hierarchy [46]. Frank and Sadun [47] introduced a related approach for generating hierarchical tilings, in which flexible "fusion rules" allow for rearrangements or change in geometry at different levels. However, it is still not clear to why certain arrangements are favorable over the others. In this

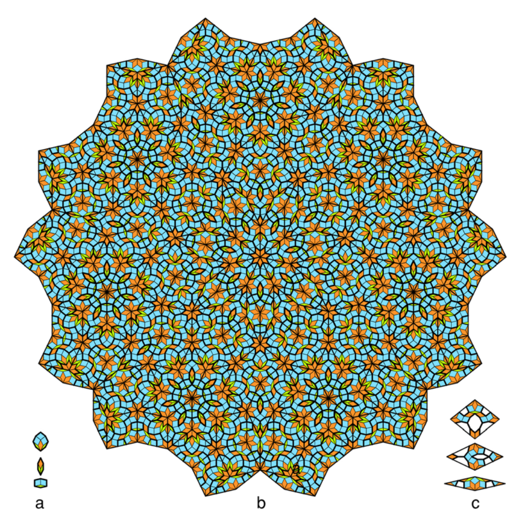

Fig. 7 The deflation/inflation rules of the first high-symmetry cluster of the second hierarchy. **a** The connecting formations of the first hierarchy. **b** A close-up view of the rhombic deflation/inflation rules of the second hierarchy. **c** The generalized inflation/deflation rules of the second hierarchy

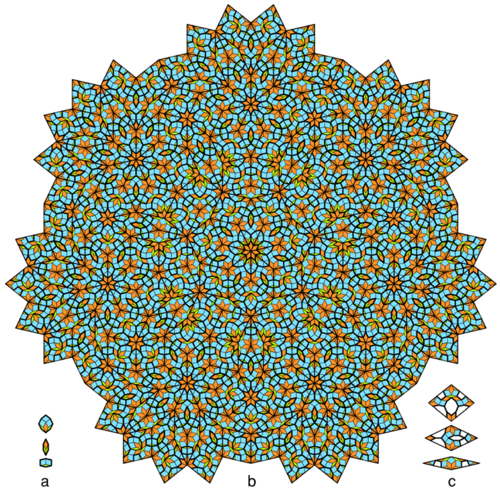

Fig. 8 The deflation/inflation rules of the second high-symmetry cluster of the second hierarchy. **a** The connecting formations of the first hierarchy. **b** A close-up view of the rhombic deflation/inflation rules of the second hierarchy. **c** The generalized inflation/deflation rules of the second hierarchy

context, both the deterministic inflation/deflation approach and the flexible/random substitutions have been used to describe these arrangements within the structure of real quasicrystals [48, 49]. On one hand, deterministic substitution tilings (i.e., Penrose, etc.) assumes that energetic interactions favor quasiperiodicity [48]. On the other hand, undeterministic rules (i.e., random tilings) assume that entropy is responsible for stabilizing the quasicrystal structure [50, 51]. The proposed seed-based model provides a hybrid approach. By utilizing a local degree of freedom, associated only with connecting formations, it is possible to generate the 7-fold quasiperiodic lattice without changing the arrangement or location of high-symmetry clusters. However, the question of which approach is best suited to explain the structure and formation of quasicrystals is still an open question.

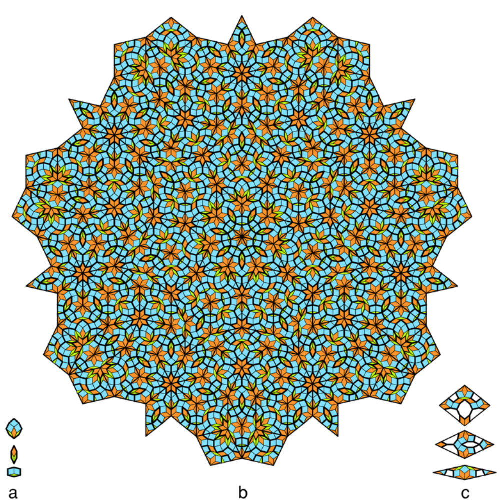

Fig. 9 The deflation/inflation rules of the third high-symmetry cluster of the second hierarchy. **a** The connecting formations of the first hierarchy. **b** A close-up view of the rhombic deflation/inflation rules of the second hierarchy. **c** The generalized inflation/deflation rules of the second hierarchy

Correlation with real growth of quasicrystals

The proposed hierarchical model offers a fresh new approach to the structural investigations of quasicrystals and suggests that the quasilattice formations are constructed from an arrangement of several types of clusters, which has proven to be valuable for describing the real atomic structure of quasicrystals [41–43]. More importantly, the proposed seed-based construction approach works in concert with the growth models of periodic and aperiodic crystals in which a number of initial *seed* nucleus are used to initiate the crystal growth via nucleation or self-assembly [18, 52, 53]. In this self-organizing growth process, a number of neighboring high-symmetry nuclei (seed clusters) generate growing crystallites around them until their boundaries merge to generate the larger quasiperiodic structure. Recent investigations have shown that a dislocation-free growth of 12-fold quasicrystals can be achieved from two seeds [18]. However, to grow perfect quasicrystalline structures using multiple nucleation centers, the boundary condition between these crystallites needs to be flexible to alleviate the resulted stress. Schmiedeberg and his colleagues found that the availability of the extra degree of freedom or phasonic flips in quasicrystals are essential to mitigating the boundary stress by allowing for local rearrangements to happen without dislocating the nucleation centers [18]. Martinsons and his colleagues have also found that phasonic flips facilitated the formation of 2D light-induced quasicrystals with 14-fold symmetry, which allowed the particles to rearrange in the system [17]. The problem of incorporation this local degree of freedom or phasonic fluctuations in the structural investigation of quasicrystalline systems remains a challenge [36]. Therefore, structural models with built-in ability to locally reorganize without affecting the locations of the high symmetry could provide an advantage over existing models and can be of critical value to the wider quasicrystalline scientific community.

Conclusion

In this paper, we presented a new seed-based hierarchical model for constructing and growing rhombic quasilattice formations with 7-fold rotational symmetry. This structural model does not appeal to any of the existing theoretical approaches (i.e., inflation, deflation, substitution, matching overlapping, projecting, etc.) and suggests that the position of high-symmetry motifs, locally and globally, is defined by one long-range framework and not based on local rules. The presented structural model allows the quasilattice to expand infinitely without any gaps, overlaps, or mismatches. The constructed quasilattice is self-similar, self-consistent, and not singular. The different hierarchical levels contain repeatedly appearing patches with the same original motifs. We also showed that a clear generalized inflation/deflation rules can be derived using three basic rhombi. Most importantly, the proposed model has a built-in local degree of freedom, which allows for a correlated lattice rearrangement to happen without affecting the location or orientation of the high-symmetry seed units. This is in concert with the seed initiated nucleation growth models of quasicrystals and could provide new insight into understanding the atomic fluctuations and the phasonic flips that are unique to quasicrystals.

Most intriguingly, three main different quasilattice high-symmetry formations morph into each other cyclically with changing hierarchies, triggering a continuous chain of rearrangements to happen at every level. This suggests that local flexibility is essential for maintaining the global odder. This might explain the difficulties in devising inflation rules for high-symmetry systems using the tiling inflation/deflation approach, which does not account for the chain of continuous change in arrangements. The proposed model could possibly provide a deeper understanding of the structure of quasicrystals at an atomic scale and hopefully will help scientists describe how the atoms in quasicrystals interact to form its complicated visual formations. Future research will be dedicated to investigating such quality as it relates to quasilattice with high rotational symmetries.

The study of quasiperiodic ordering systems and structures of unique high rotational symmetries is not only important for crystallography but also of interest to many other scientific, applied, and creative fields and has the potential to open up many new scientific and creative opportunities. The use of global relational logic provides scientists, artists, and teachers with a simple method to create and study a wide variety of complicated hierarchical quasilattice formations without the need for any specialized software or complicated mathematics. By manipulating the two main components, the hierarchical framework and the internal decoration of the seed units, an unlimited range of design possibilities can emerge. Future efforts should be directed toward investigating the use of the hierarchical approach to construct other unique quasicrystalline formations with high rotational symmetries (i.e., 9-, 11-, 13-, 14-fold, etc.) as well as how these abstract geometric models can actually correlate with real atomic structure and growth of quasicrystals.

Compliance with ethical standards

Conflict of interest The author declares that she has no conflict of interest.

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