Struct Chem (2017) 28:57–62
DOI 10.1007/s11224-016-0793-y

ORIGINAL RESEARCH

Tiling approach for the description of the sevenfold symmetry in quasicrystals

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Received: 28 May 2016 / Accepted: 17 June 2016 / Published online: 29 June 2016

Abstract An example of substitution rules for the construction of heptagonal rhombic tilings is proposed. Rigorous inflation/deflation rules make it possible to expand the tiling up to infinity without additional ad hoc rearrangements. The derived tilings are self-similar and consist of characteristic patterns with seven-pointed stars surrounded by similar seven-pointed stars.

Keywords Sevenfold symmetry · Quasicrystal · Self-similarity · Substitution tiling · Heptagonal tiling

He from the centre of the realm upraised a Behistun, from which that of Farhad fled.
In such a Behistun, which seven columns had, he raised up to the heavens seven domes.
And in those walls, which touched upon the sky, he saw a rampart round the lofty spheres.
He saw seven domes within those walls built up after the nature of the planets seven.

Nizami Ganjavi, "The Seven Beauties".

Dedicated to Professor Alan L. Mackay on the occasion of his 90th birthday.

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Introduction

John D. Bernal was probably one of the first who drew the attention of researchers to the origin of the sevenfold symmetry in molecular and atomic systems. According to his suggestion, the possible reason for unusual symmetries in living and non-living matter is due to that the complex systems are made up of subunits, which were almost but not absolutely identical. In particular, he wrote that "the new crystallography, generalized crystallography, has been growing up to include structures and polymers where some of the conditions of crystallography are relaxed to include polymers with a helical structure or the presence of fivefold and sevenfold axes" [1]. Alan L. Mackay laid down the foundations of generalized crystallography (see, e.g., [2–5]), especially emphasizing the importance of self-similarity, curvature, hierarchy, and information, as well as the need in higher dimensional and non-Euclidean approaches instead of the "orthodox" crystallography, and the decision-tree algorithms and cellular automata instead of simple face-to-face packing of single unit cells. The sevenfold symmetry is also among his multifarious interests [6, 7]. Recently, Schoen [8] derived a variety of fascinating quasi-recursive rhombic sevenfold tilings and made them available on his website, as well as the calculated by Mackay diffraction pattern from the central core of one of the most representative sevenfold samples. Since we continuously drew inspiration from his works [9, 10], we deemed appropriate to make a small contribution to the problem of heptagonal symmetry as a humble tribute for his influence.

The search for novel materials exhibiting the sevenfold symmetry continues. At this time, only a few examples of relating quasicrystals have been reported [11–13]. Thus, theoretical models of heptagonal tilings could be very useful for these purposes.

The quasicrystalline tilings of the plane are rather diverse. According to Danzer [14], there exist several infinite series of such tilings, as well as quite a few sporadic tilings with exotic properties. First, the analogues of the Penrose tiling [15, 16] exist for every natural $n \geq 7$ (except for $n = 8$, 10, 12) when the de Bruijn method is used [17, 18]. These tilings seem to lack simple inflation rules in many cases. Alternatively, the tilings that are defined by inflation rules proposed by Nischke and Danzer [19] exist for every odd $n \geq 7$ not divisible by 3, but these tilings can hardly be constructed by the de Bruijn method. There exist several substitution tilings of the Ammann–Beenker type, which can be also compatible with the n-fold rotational symmetry [20–22].

Next, a series of tilings of congruent rectangular triangles can be derived similarly to the pinwheel tiling, which was introduced by Conway and Radin first [23]. These tilings can neither be produced by a strip projection method, nor be defined by some local matching rules. Generally, in the global pinwheel tiling, the triangles occur in infinitely many orientations. So, it is hard to imagine that such tilings may be compatible with repeatedly appearing n-fold rotational symmetry, though the similar construction being applied to isosceles triangles could probably give rise to the sevenfold symmetry.

A great number of heptagonal tilings have been published as both triangular and rhombic. Both strip projection and substitution algorithms have been used for the pattern generation. Different inflation factors have been found to be compatible with the sevenfold symmetry. The resulting tilings were with and without repeatedly appearing heptagonal patterns [24–35]. For example, the rhombic tiling proposed by Harriss [27] was characterized by very clear inflation rules but lacked the repetitive sevenfold patches, even if seven or fourteen rhombuses were initially arranged into a star. Kari and Rissanen [34] discussed two types of local environments, which they referred as *roses*, and offered an original and very attractive solution, though the assumption that all vertices of initial rhombuses had equivalent local environments led to an enormously large inflation factor.

To make the general problem more clear and understandable, let us discuss some special features of the great family of heptagonal quasi-recursive rhombic tilings derived by Schoen [8]. He highlighted that, in contrast to the Penrose tiling, the multiple reflections of the initial heptagon create overlapping regions in each newly generated heptagonal annulus. Overlapping regions must be replaced by an orderly ad hoc arrangement of tiles called a wedge, in order to connect the adjacent tiled regions seamlessly. After the second recursion stage, the wedge design becomes challenging already because of the large number of tiles involved. Schoen intended to assume—without proof—that it was possible to construct a wedge at every stage of recursion [8].

The goal of this paper is to develop a rigorous procedure to construct a sevenfold tiling that satisfies a number of strict requirements. First, the inflation/deflation rules should allow the expansion of the tiling up to infinity without additional ad hoc rearrangements. The enlarged rhombuses should either contain the smaller rhombuses wholly within themselves or their edges should bisect the smaller tiles. Second, the tiling should be self-similar and contain repeatedly appearing patches with sevenfold rotational symmetry. In other words, it should consist of seven-pointed stars surrounded by seven-pointed stars, each of which is surrounded by seven-pointed stars, and so forth. Third, the inflation/deflation rules must ensure that after several iterations, the local environment of every vertex transforms into one of the standard sevenfold patterns from a certain atlas. Actually, we expect only two different vertex types in corresponding infinitely fragmented fractal [36] and, consequently, only two different types of sevenfold patterns are expected to exist in the whole two-page atlas. Thus, we are looking for a pair of *complementary* heptagonal tilings constructed according to the common rules. Finally, the inflation/deflation procedure should not move the origin of the tiling and should not permute the vertices of both types (compare with the Penrose tiling, the substitution rules of which permanently interchange the "sun" and "star" patterns).

Heptagonal tiling: an example

We used the basic concepts of the fractal approach that have already successfully proven themselves for explanation of structural peculiarities of icosahedral quasicrystals [36–38]. We presumed that a rhombic tiling of the Euclidean plane E^2 has a corresponding fractal "parent" in the extended complex plane or the Riemann sphere PC^1, the symmetry operations of which—linear fractional transformations or the Möbius homography—predefine the inflation/deflation rules for its "daughter". We were not able to avoid completely some trial-and-error procedures and ad hoc steps. So, let us report the final result and provide some comments for clarification.

The substitution rules use the set of rhombuses with angles $2\pi k/14$ and with integer $k \in \{1\ldots6\}$. There are only two types of vertices. They are referred by us as the A and B types and denoted as open and solid circles, respectively. There are only two types of edges marked by the single and double arrows. The arrows always begin at the vertices of the first type and point to those of the second type. In any rhombus, two opposite vertices belong to the first type, whereas two other vertices belong to the second type. The local environment of any vertex in the tiling is a subgraph of fourteen alternating edges, both as single- and double-arrowed (Fig. 1).

There are exactly six different rhombuses, which can be designated as T_1^7, T_2^7, T_3^7, T_4^7, T_5^7, and T_6^7. The pairs of tiles—T_1^7 and T_6^7, T_2^7 and T_5^7, T_3^7 and T_4^7—seem to be equal, but they differ by the types of opposite vertices. Quasicrystalline tilings can have a special kind of defects referred to as phason shifts [39]. The heptagonal tilings also have the similar type of disorder. For example, the cluster of two narrow rhombuses T_6^7 and one medium T_2^7 may be always replaced by the complementary cluster of two T_1^7 and one T_5^7 without affecting the adjacent tiles. A lot of such rearrangement rules can be found for a specific tiling. This means that, once derived, the substitution rule cannot be considered as a unique solution. The local matching rules that are considered as alternating order of smaller rhombuses along the edges of inflated ones also are not unique. For example, when the initial global heptagonal tiling has had mirror planes, it can lose this kind of symmetry after several subsequent rearrangements of indistinguishable clusters. The rearrangement of indistinguishable clusters can affect the global symmetry of tiling! The rearrangement can also reduce the number of tiles in the fundamental basic set. Clearly, we have entered into a virtually unexplored area of geometry, relating to the polymorphism of aperiodic tilings. After some trial-and-error steps, we have succeeded to exclude the last narrow rhombus T_6^7 from the basic set of tiles. On one the hand, our motivation was to simplify the rules, and on the other hand, to reproduce at least the central core of the known tiling [8]. Thus, the basic set of prototiles is formed by only five different rhombuses (compare with [8, 27, 28, 33, 34, 40]).

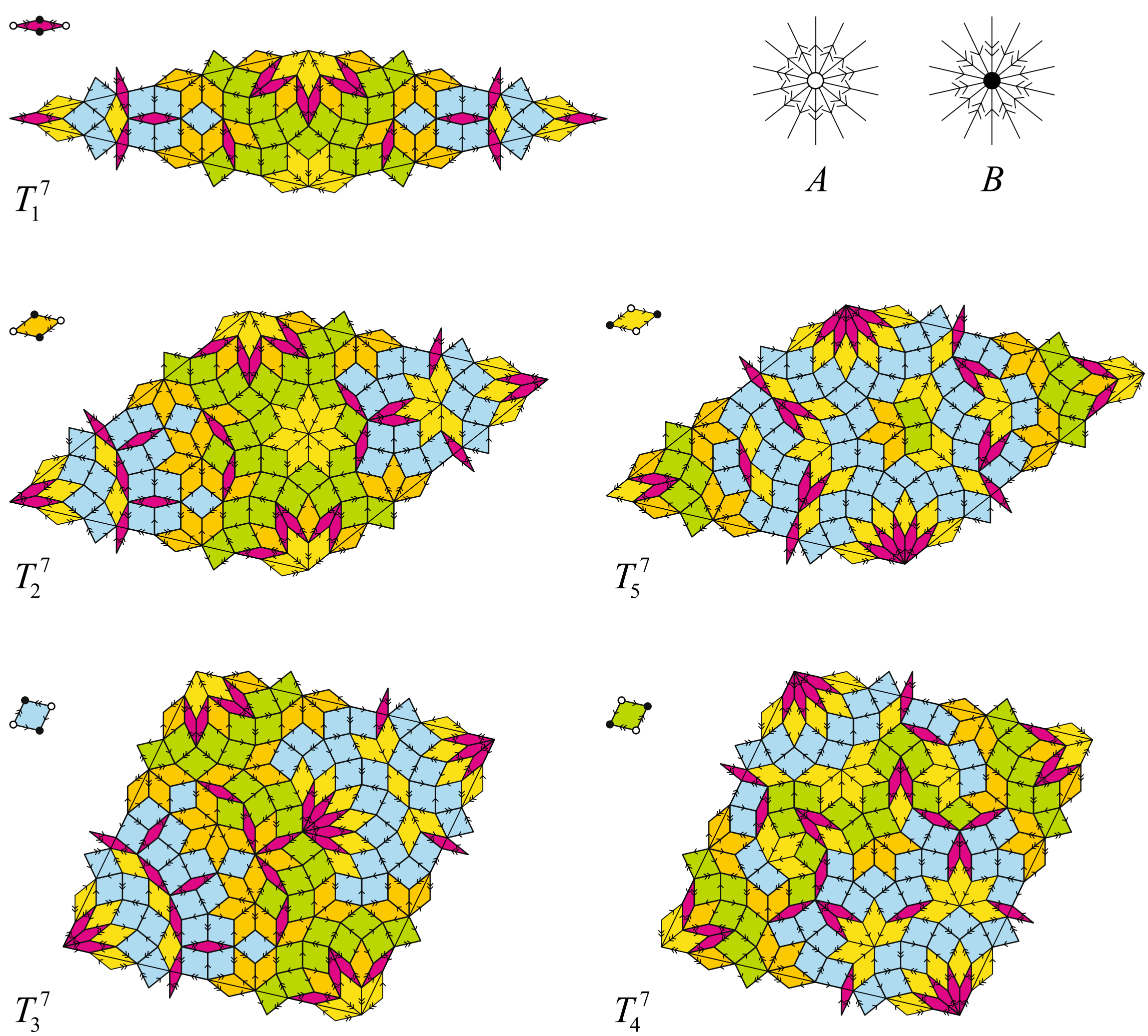

Fig. 1 Inflation/deflation rules for the sevenfold rhombic tiling. There exist two types of vertices (A and B), two types of edges (*single* and *double*-arrowed), and five types of inequivalent rhombuses (T_1^7, T_2^7, T_3^7, T_4^7, and T_5^7)

The inflation/deflation rules are presented in Fig. 1. The inflation factor is $m = 3 + 3\alpha + 2\beta + \gamma$, where $\alpha = 2\cos(2\pi/14)$, $\beta = 2\cos(2 \times 2\pi/14)$, and $\gamma = 2\cos(3 \times 2\pi/14)$. These rules can be applied iteratively till the entire plane is covered by rhombuses. Figures 2 and 3 show the results of the inflation/deflation procedure applied to the local environment of the A- and B-type vertices, respectively.

Now, let us give some comments on how we got these rules. We started with highly symmetrical rosettes to ensure the repetitive appearance of seven-pointed stars. Next, we selected the appropriate value of the inflation factor, such as to guaranty that the opposite vertices of the narrowest tile T_1^7 coincide with the existing vertices of the proper type. Further, we took into account that every vertex should, in turn, act as a centre of the sevenfold symmetry after inflation. We thus placed the corresponding rosettes at every vertex of the inflated tiles—and here we came to a peculiar problem—the rhombuses T_5^7 overlap at the acute angles of the inflated T_4^7, as well as those at the obtuse angles of the T_2^7. One of the possible solutions is to assume that two stars of seven rhombuses T_5^7 generated by rotations around opposite B-type vertices should be considered as indistinguishable in the fractal parent and differ only by rotation on the angle $2\pi/14$. By other words, the single- and double-arrowed edges always alternate around any vertex, though some of them are missing in the finite tiling. As a consequence, the stars of seven T_5^7 at the opposite obtuse angles of the enlarged T_1^7 should be placed in such a manner that the enlarged single-arrowed edges bisect the star-forming rhombuses, whereas the double-arrowed

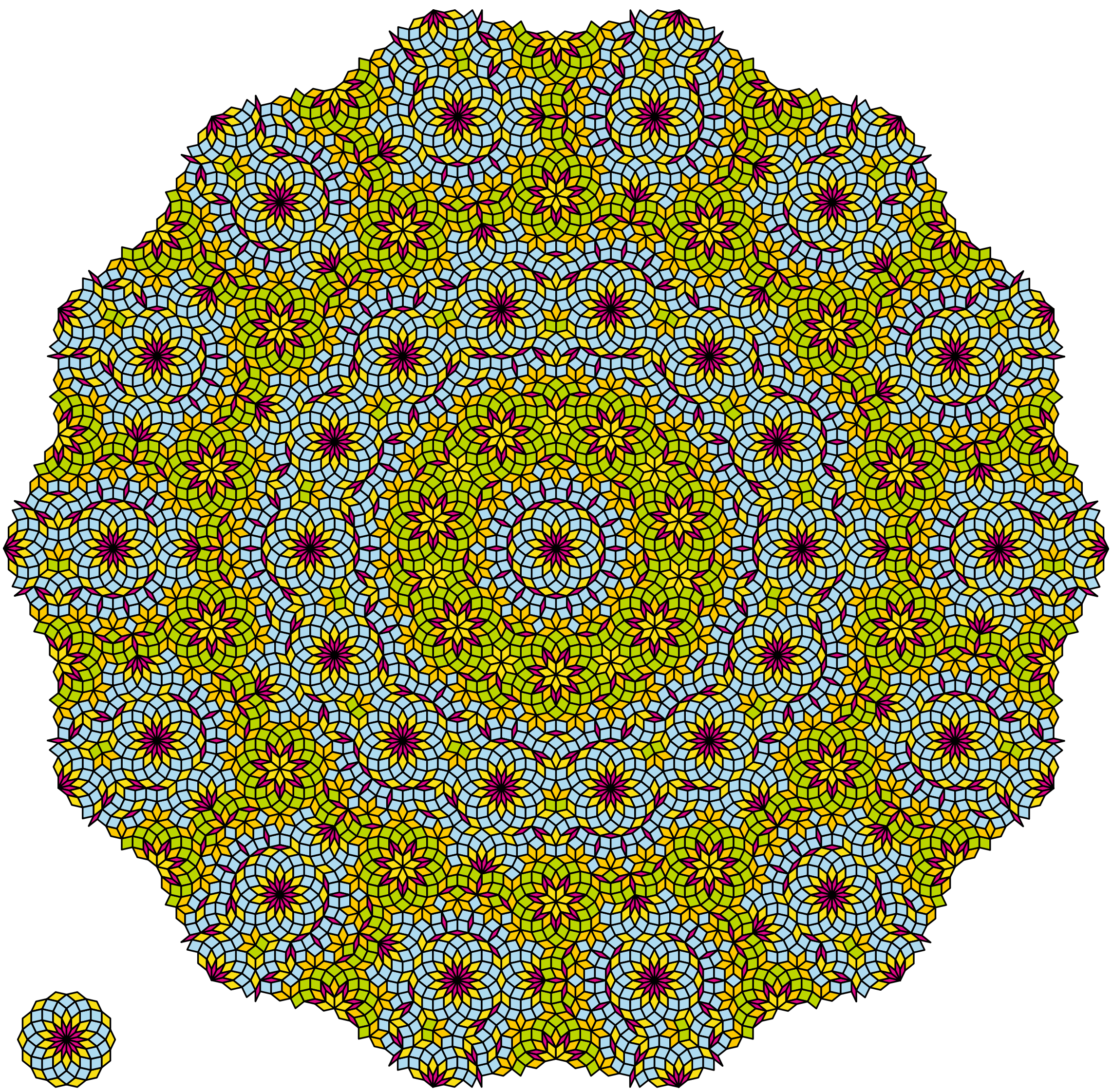

Fig. 2 First of two complementary heptagonal tilings (with *A*-type vertex at the origin). Initial region and the result of the first iteration are presented

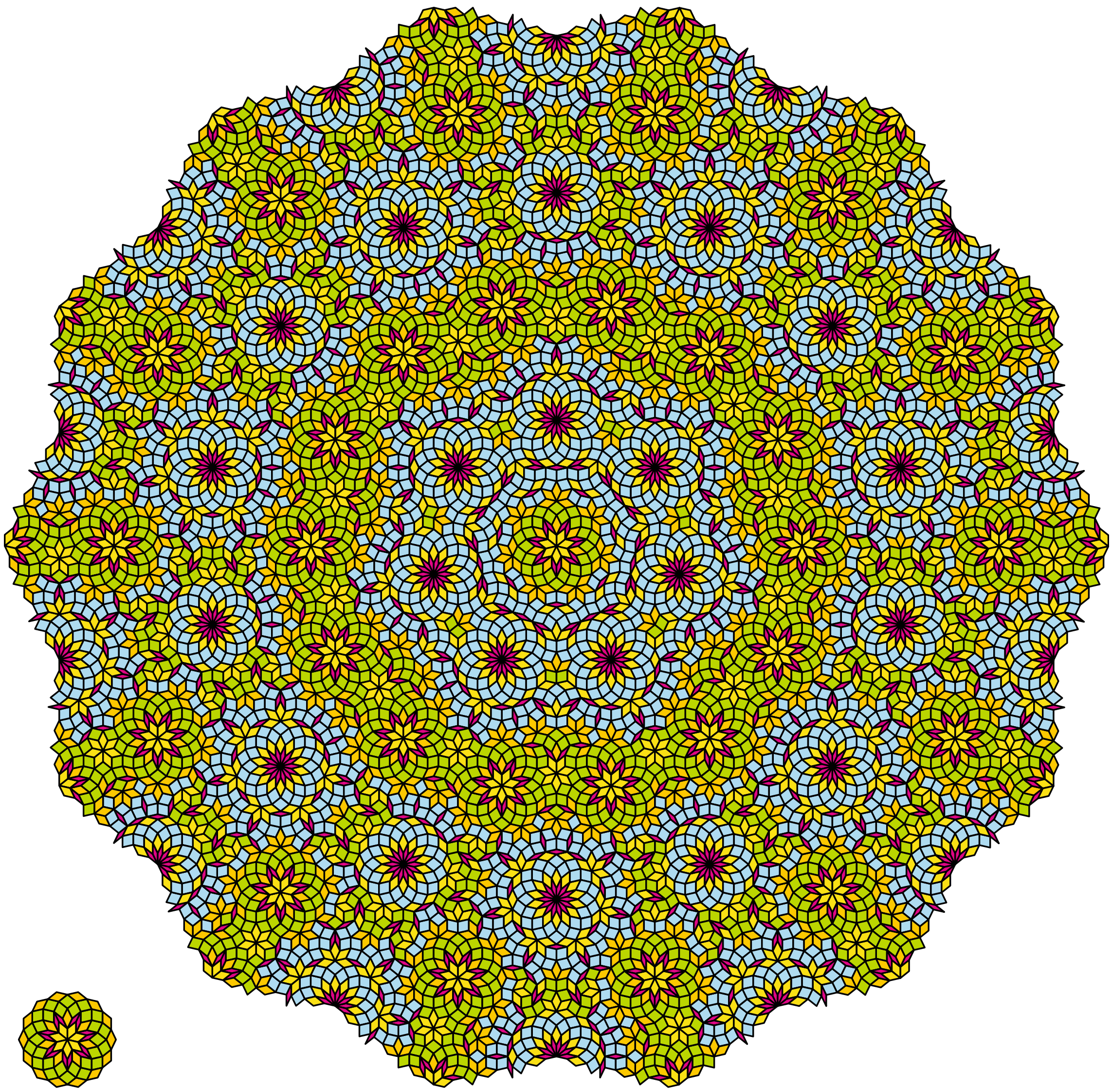

Fig. 3 Second of two complementary heptagonal tilings (with *B*-type vertex at the origin). Initial region and the result of the first iteration are presented

edges do not bisect those next to the opposite angle. That is why two types of edges exist. Such arrangement of tiles imposes additional constraints on the inflation factor value. Several trial-and-error attempts were necessary to make the correct decision. After that, finally, we placed missing tiles in the middles of the inflated edges, replicated the as-obtained patterns on the equivalent edges, and filled the rest. There is a relative freedom in placing the last tiles and filling the large areas inside the inflated tiles. The smaller is the inflation factor, the less is the uncertainty. We were focused on the aesthetic appeal and gaining the highest possible local symmetry.

We have to admit that another solution also exists. It may be referred to as the "entangled" substitution rules, for which some highly symmetrical patterns cyclically morph each into another and finally turn back into original ones after several iterations. In this case, the self-similarity factor of the entire tiling is an integer power of the inflation factor. By using such rules, we were able to construct an original 14/7 tiling, which aesthetically is even more beautiful and will be published elsewhere soon.

Discussion and conclusions

So, we offered example of substitution rules that make it possible to construct the heptagonal rhombic tilings. The tilings exhibit polymorphism. Due to polymorphism, the difference between derived and existing heptagonal tilings is a complex issue to discuss. Probably, our tiling is not the only one that can be constructed according to similar rules, including rules with different inflation factors. The question then arises what are the general algebraic restrictions on the permissible values of the inflation factor? What is the lowest possible value of the scaling factor compatible with the aforesaid requirements of self-similarity? Can essentially different tilings with equal scaling factors exist? Is it possible to transform the heptagonal rhombic tilings into triangular and vice versa; in the same manner as the Penrose rhombic tiling was possible to transform into the kite-and-darts tiling? All these questions await further investigations.

The described rules can be generalized for infinite tilings with higher order axes (with $n = 9$, 11, 13, and so forth). Indeed, one can start with "roses" [12, 34] resembling Buddhist mandalas, inflate them with a suitable factor to ensure that two pairs of opposite vertices of the inflated T_1^n rhombuses coincide with the vertices of two alternative types, and find the way to fill the rest. However, even though it sounds very easy, it actually requires significant efforts.

Our results may stimulate further research in various fields of generalized crystallography including scale and superspace crystallography [41–45]. We expect that our mathematical models will help researchers to better understand the ordering phenomena in soft matter and relating materials, to explain the appearance of anomalous symmetries in colloidal layers, and to design new types of photonic crystals, artificial solids, metamaterials, and so forth [46–53]. We believe that the nature of the sevenfold symmetry will become a little less puzzling in both science and art [54–57].

Acknowledgments I thank Jelena R. Kambak for proofreading and writing assistance.

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