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LIMITATIONS OF SOME STRENGTH CRITERIA FOR COMPOSITE MATERIALS

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Based on an analysis of four different types of stress state, the incorrectness of application of the maximum stress strength and maximum strain strength criteria, which are widely used in the world practice and modern commercial application packages, for the strength analysis of isotropic, transtropic and composite materials with at least one plane with low degree of anisotropy is theoretically proved. For the composites with the plane of medium-degree anisotropy, the application of these criteria can also result in significant errors. This incorrectness is due to the non-fulfilment of invariance conditions in the plane of isotropy, if any, and can lead to the fact that the value of the strength function at the same stress-strain state can change by a factor of almost two (and perhaps more) depending on the rotation of the coordinate system. It is pointed out that these two criteria are conceptually constructed so that it is very difficult, if not impossible, to modify them to avoid the above incorrectness. They can be used, with a certain degree of reliability, only for essentially orthotropic composites. It is shown that the quadratic generalized von Mises strength criterion, which is also commonly used in applications, is void of these disadvantages and can be correctly used in the strength analysis of both orthotropic and trans- and isotropic materials if the invariance conditions are met. Other quadratic Tsai–Wu strength criteria, one of whose particular cases is the generalized von Mises criterion, will also be correct if the necessary and sufficient conditions for the existence of a limiting failure surface and invariance conditions are met.

Keywords: orthotropy, transtropy, isotropy, maximum stress criterion, maximum strain criterion, the Tsai–Wu strength criterion, generalized von Mises strength criterion, invariance conditions.

Introduction. Composite materials (CM) are widely used in modern industries. CM is usually a heterogeneous continuum. When calculating the stress-strain state (SSS) and strength of structures and structural elements made using CMs, the heterogeneous continuum is often replaced, in some approximation, by an equivalent homogeneous continuum to simplify the mathematical model and calculations. In many practically important cases, this homogeneous continuum is well described by an orthotropic material model, from which follow, as particular cases, a transtropic body model (e.g., unidirectional fiber-reinforced composites) and an isotropic body model (randomly scattered inclusions in the binding matrix, isotropic fillers, traditional structural metals, etc).

To estimate the strength of CMs, phenomenological strength criteria for orthotropic material are widely used. A survey of publications showed [1, 2] that two criteria: maximum stress criterion and maximum strain criterion, which are written in the principal anisotropy axes x, y, z, are used more often than others. These criteria have the simplest mathematical formulation and are included, in particular, in modern commercial application packages (ABAQUS, ANSYS, LS-DYNA, NASTRAN, etc) for the calculation of the static and dynamic SSS, as well as of the strength of structures and structural elements made using CMs.

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The maximum stress criterion for the general case of triaxial SSS is given by

$$\Phi_\sigma = \max\left(\frac{\sigma_x}{\sigma_x^t}; \frac{-\sigma_x}{\sigma_x^c}; \frac{\sigma_y}{\sigma_y^t}; \frac{-\sigma_y}{\sigma_y^c}; \frac{\sigma_z}{\sigma_z^t}; \frac{-\sigma_z}{\sigma_z^c}; \frac{|\tau_{xy}|}{\tau_{xy}^b}; \frac{|\tau_{yz}|}{\tau_{yz}^b}; \frac{|\tau_{zx}|}{\tau_{zx}^b} \right) < 1, \tag{1}$$

where σ_i and τ_{ij} are stress tensor components, σ_i^t and σ_i^c are the ultimate strengths in uniaxial tension and compression, respectively, in the ith principal direction of anisotropy, and τ_{ij}^b is the ultimate strength in pure shear in the principal plane of anisotropy ij, i, $j = x$, y, z, $i \neq j$.

To specify (1) for an orthotropic CM, nine experiments must be made: three uniaxial tension experiments, three experiments on compression along the corresponding three principal axes of anisotropy, and three experiments on pure shear in the three principal planes of anisotropy. For a transtropic composite, the number of required experiments is reduced to six and for an isotropic one to three.

The limiting surface of strength, $\Phi_\sigma = 1$, in 6D stress space is a closed 6D parallelepiped, inside which (1) holds, strength is provided, and there is no failure; outside it, (1) is violated, strength is not provided, and the material fails. For the composites of the same tensile/compressive strength, the center of the parallelepiped coincides with the origin of coordinates, and for the composites of different strength, it does not coincide.

According to (1), the strength margin S_σ can be estimated thus

$$S_\sigma = (1 - \Phi_\sigma) \cdot 100\% \geq 0 \tag{2}$$

and the overload F_σ can be estimated in a similar way:

$$F_\sigma = (\Phi_\sigma - 1) \cdot 100\% = -S_\sigma \geq 0. \tag{3}$$

If (1) holds, we have a certain positive strength margin (there is no overload, $F_\sigma < 0$). Otherwise we have an overload ($S_\sigma < 0$).

The maximum strain criterion is determined in the same way, only the stress components in (1)–(3) and in what follows must be replaced by the corresponding strains and the ultimate strengths by the corresponding ultimate strains.

The maximum stress criterion, like the maximum strain criterion, is constructed based on the results of nine experiments, has the simplest mathematical formulation and apparently, therefore, is widely used in the strength analysis of CMs. For essentially orthotropic composites, formulas (1)–(3) are, at first sight, not without common sense and look fairly valid and logical. A different situation is observed for transtropic and especially isotropic materials.

Let us consider a transtropic (e.g., unidirectionally reinforced) composite. For definiteness, let z be the principal axis of anisotropy (reinforcement), and xy be the plane of isotropy (perpendicular to the z axis). To specify the maximum stress criterion, six experiments must be made to determine the limiting values:

$$\sigma^t = \sigma_x^t = \sigma_y^t, \qquad \sigma_z^t, \qquad \sigma^c = \sigma_x^c = \sigma_y^c, \qquad \sigma_z^c, \qquad \tau^b = \tau_{xy}^b, \qquad \text{and} \qquad \tau_{zp}^b = \tau_{zx}^b = \tau_{zy}^b.$$

In this case, the two mutually orthogonal x and y axes in the plane of isotropy may be chosen arbitrarily.

Transtropy is a particular case of orthotropy: one pricipal anisotropy axis z is determined unambiguously, and the two others x and y may be chosen arbitrarily perpendicular to z and to each other [3]. Therefore, on the degeneration of orthotropy into transtropy, any mathematically correct strength criterion of an orthotropic body must become invariant with respect to the rotation of the x and y axes about the z axis by any angle, and on the transition of orthotropy to isotropy, it must become invariant with respect to any rotation (choice) of coordinate axes [2, 4].

Unfortunately, this does not occur neither for the maximum stress criterion nor for the maximum strain criterion. Let us illustrate this disadvantage with a concrete example. For simplicity and convenience, consider a plane stress state (PSS) in the isotropy plane xy at $\sigma_z = \tau_{yz} = 0$ for a transtropic (or even isotropic) material. The maximum stress criterion (1) simplifies to

$$\Phi_\sigma = \max\left(\frac{\max(\sigma_x; \sigma_y)}{\sigma^t}; \frac{-\min(\sigma_x; \sigma_y)}{\sigma^c}; \frac{|\tau_{xy}|}{\tau^b}\right) < 1. \quad (4)$$

Consider a material of the same tensile/compressive strength with the following ultimate strengths in the plane of isotropy: $\sigma^t = \sigma^c = \sigma^b = 400$ MPa, $\tau^b = 230$ MPa. Such strength properties are typical of many structural steels [5, 6], e.g., St1, St2, St3, 10, 15K, 20K steels, etc.

Let us examine particular cases of PSS.

1. Uniaxial tension: $\sigma_x = 440$ MPa, $\sigma_y = \tau_{xy} = 0$.

Using maximum stress criteria (3) and (4), we obtain an overload of 10%:

$$\Phi_\sigma = \frac{\sigma_x}{\sigma^b} = \frac{440}{400} = 1.1 > 1, \Rightarrow F_\sigma = 10\%, \ S_\sigma < 0.$$

Without changing the SSS, let us rotate the xy axes by 45°. In the new $x'y'$ axes, we shall have [2, 3, 5, 6]: $\sigma_{x'} = \sigma_{y'} = \tau_{x'y'} = 220$ MPa, and from (2) and (4) we obtain that for the same SSS, the strength margin in the rotated axes is over 4%:

$$\Phi_\sigma = \frac{\tau_{x'y'}}{\tau^b} = \frac{220}{230} = 0.9565 < 1, \Rightarrow S_\sigma = 4.35\%, \ F_\sigma < 0.$$

2. A more illustrative example is pure shear: $\tau_{xy} = 300$ MPa, $\sigma_x = \sigma_y = 0$. From (3) and (4) we obtain an overload of over 30%:

$$\Phi_\sigma = \frac{\tau_{xy}}{\tau^b} = \frac{300}{230} = 1.30435 > 1, \Rightarrow F_\sigma = 30.435\%, \ S_\sigma < 0.$$

By rotating the xy axes by 45°, we obtain: $\sigma_{x'} = -\sigma_{y'} = 300$ MPa, $\tau_{x'y'} = 0$, and from (2) and (4) we have, instead of overload, a strength margin of 25%:

$$\Phi_\sigma = \frac{\sigma_{x'}}{\sigma^b} = \frac{300}{400} = 0.75 < 1, \Rightarrow S_\sigma = 25\%, \ F_\sigma < 0.$$

3. Similar pure shear for a material of different tensile/compressive strength: $\sigma^t = 320$ MPa, $\sigma^c = 500$ MPa, $\tau^b = 230$ MPa.

In the "old" xy coordinates, we have the same overload $F_\sigma = 30.435\%$ and in the rotated $x'y'$ coordinates the strength margin:

$$\Phi_\sigma = \frac{\sigma_{x'}}{\sigma^t} = \frac{300}{320} = 0.9375 < 1, \Rightarrow S_\sigma = 6.25\%.$$

4. Consider one more example of triaxial SSS of the following form: $\tau_{xy} = \tau_{yz} = \tau = 220$ MPa; $\sigma_x = \sigma_y = \sigma_z = 0$ for an isotropic material of the same tensile/compressive strength (such as steel) with the strength characteristics as in the first two examples.

For such materials, the strength criterion (1) takes the form

$$\Phi_\sigma = \max\left(\frac{\max(\sigma_x;\, \sigma_y;\, \sigma_z)}{\sigma^b};\ \frac{-\min(\sigma_x;\, \sigma_y;\, \sigma_z)}{\sigma^b};\ \frac{\max(|\tau_{xy}|;\, |\tau_{yz}|;\, |\tau_{zx}|)}{\tau^b} \right) < 1, \tag{5}$$

and in the x, y, z coordinate axes, we have: $\Phi_\sigma = \frac{\tau_{xy}}{\tau^b} = \frac{220}{230} = 0.9567 < 1, \Rightarrow S_\sigma = 4.35\%,\ F_\sigma < 0$, i.e., the strength margin is over 4%.

We determine the principal stresses from a cubic equation corresponding to the given SSS [6]: $\sigma^3 - 3\tau^2\sigma = 2\tau^3$, by solving which we find: $\sigma_1 = 2\tau = 440$ MPa, $\sigma_2 = \sigma_3 = -\tau = -220$ MPa. Thus, in the principal axes of stresses we have: $\Phi_\sigma = \frac{\sigma_1}{\sigma_2} = \frac{440}{400} = 1.1, \Rightarrow S_\sigma < 0,\ F_\sigma = 10\% \Rightarrow$ the overload reaches 10%.

Moreover, there are also $x'y'z'$ axes where $\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{440 + 220}{2} = 330$ MPa will act, and in these axes we obtain: $\Phi_\sigma = \frac{\tau_{\max}}{\tau^b} = \frac{330}{230} = 1.435 > 1, \Rightarrow S_\sigma < 0,\ F_\sigma = 43.5\% \Rightarrow$ overload of over 43%, i.e., without changing the SSS and only rotating the coordinate axes, we obtained in this example either a strength margin of $> 4\%$ or an overload of $> 43\%$.

There are rather many paradoxical examples similar to these. They prove the fact that both the maximum stress criterion and the maximum strain criterion are mathematically constructed incorrectly. One of the basic requirements to the strength criteria: invariance with respect to any rotation in the plane of isotropy, if any, is not satisfied. The maximum stress and strain criteria cannot be used to calculate the strength of transtropic CMs, not to mention that they are unsuitable for isotropic materials, when all the three x, y, and z axes may be chosen arbitrarily, and the function Φ_σ for the maximum stress criterion or Φ_ε for the maximum strain criterion must not change in this case.

Even for orthotropic CMs, for which these criteria were proposed in the form of (1), they can be used only after a preliminary analysis of the degree of anisotropy of the composite and the type of SSS. If the composite is orthotropic, but has a plane of weak anisotropy, i.e., the material in it is close to isotropic one, these criteria cannot be used either. Indeed, in the plane of weak anisotropy, the strength function, which determines the strength margin or overload, must weakly depend on the rotation of the corresponding coordinate axes and show close results independent of how they are rotated, and in the limit (weak anisotropy changes into isotropy), it must not depend at all on the rotation of the axes; in fact, this is not the case, which follows from the most illustrative examples: depending on the rotation of the xy axes, we have situations from the strength margin of 25% to the overload of $> 30\%$ in example 2 and from $S_\sigma > 4\%$ to $F_\sigma > 43\%$ in example 4.

From the above examples it is seen that on rotating the axes in the plane of isotropy, the strength function Φ_σ (and similarly Φ_ε) can change by a factor of almost two and for other CMs and types of SSS (are not considered here), perhaps by a factor of more than two. Therefore, the use of the maximum stress and strain criteria for the orthotropic materials having at least one plane even with a medium degree of isotropy, when the ultimate strengths change by a factor of no more than two or three on rotation by 90°, becomes rather problematic.

These criteria can be used, with a certain degree of reliability, only for essentially orthotropic CMs. In these composites, three principal axes and hence three planes of anisotropy are unambiguously determined, in which case in each of the three principal axes of anisotropy, the maximum and minimum strength characteristics and, as a rule, elastic characteristics differ greatly under rotation by 90°, and the use of the maximum stress criterion in the form of (1), which is written only in the principal axes of anisotropy and in no others (and similarly the maximum strain criterion), becomes at least logically justified.

The incorrectness of the maximum stress and strain criteria is a direct consequence of the ideology of their construction: both criteria are written in the principal axes of anisotropy of CMs and have the simplest mathematical formulation. Whereas this is somehow logically justified for essentially orthotropic materials, this is illogical for weakly anisotropic, transtropic and especially isotropic materials. If the CM is transtropic, then there are no principal directions of anisotropy in the plane of isotropy (more precisely, there is an infinite set of them), and in this plane, maybe the principal stresses σ_1 and σ_2, and $\tau_{\max} = (\sigma_1 - \sigma_2)/2$ corresponding to them can give some information on strength. If the material is isotropic, then there are no axes of anisotropy at all (or they may be chosen arbitrarily), and maybe the three principal stresses σ_1, σ_2, and σ_3, and $\tau_{\max} = (\sigma_1 - \sigma_3)/2$ corresponding to them can give an information on strength (at least in some reasonable approximation). Unfortunately, the maximum stress and strain criteria are conceptually constructed so that it is very difficult, if not impossible, to correctly modify them in order that the degeneration from orthotropy to transtropy and especially isotropy may be continuous.

It should be noted that the quadratic strength criteria for orthotropic material (the so-called Tsai–Wu criteria) are also written in the principal axes of anisotropy, and nevertheless they are correct if, firstly, the necessary and sufficient conditions for the existence of limiting surface (NSCELS) are met and, secondly, if in the presence of a plane of isotropy, a certain relation (invariance condition) between their coefficients is satisfied [2, 4]. The Tsai–Wu criteria are given by

$$\Phi_{TW} = \sigma_x\left(\frac{1}{\sigma_x^t} - \frac{1}{\sigma_x^c}\right) + \sigma_y\left(\frac{1}{\sigma_y^t} - \frac{1}{\sigma_y^c}\right) + \sigma_z\left(\frac{1}{\sigma_z^t} - \frac{1}{\sigma_z^c}\right) + \frac{\sigma_x^2}{\sigma_x^t\sigma_x^c} + \frac{\sigma_y^2}{\sigma_y^t\sigma_y^c} + \frac{\sigma_z^2}{\sigma_z^t\sigma_z^c}$$

$$+\left(\frac{\tau_{xy}}{\tau_{xy}^b}\right)^2 + \left(\frac{\tau_{yz}}{\tau_{yz}^b}\right)^2 + \left(\frac{\tau_{zx}}{\tau_{zx}^b}\right)^2 + 2a_{xy}\sigma_x\sigma_y + 2a_{yz}\sigma_y\sigma_z + 2a_{zx}\sigma_z\sigma_x < 1. \tag{6}$$

By choosing a_{ij} constants by one or another method, which satisfies the NSCELS and for transtropic (isotropic) CMs the condition (conditions) of invariance in the plane (planes) of isotropy [2, 4], we obtain different particular cases of the quadratic strength criteria (generalized von Mises criterion, Hoffman criterion, etc) of an orthotropic body, and in particular of a transtropic and an isotropic body.

According to [2], the NSCELS are of the form

$$A_{xy}^2 + A_{yz}^2 + A_{zx}^2 \le 1 + 2A_{xy}A_{yz}A_{zx} < 3, \qquad A_{ij} = a_{ij}\sqrt{\sigma_i^c\sigma_i^t\sigma_j^c\sigma_j^t}, \tag{7}$$

and the condition of invariance in the isotropy plane ij [2, 4] is

$$a_{ij} = (\sigma_i^c\sigma_i^t)^{-1} - (\tau_{ij}^b)^{-2}/2, \qquad \sigma_i^c = \sigma_j^c, \qquad \sigma_i^t = \sigma_j^t. \tag{8}$$

For example, the well-known generalized von Mises criterion [1, 2, 4], one of the particular cases of the Tsai–Wu criteria, is obtained from (6) at $a_{ij} = -(\sigma_i^t\sigma_j^t\sigma_i^c\sigma_j^c)^{-1/2}/2$. It is also widely used in applied strength analyses, the NSCELS for it are met automatically, invariance with respect to the rotation of the coordinate axes (8) in the plane of isotropy is provided if the equality $3(\tau^b)^2 = \sigma^c\sigma^t$ in this plane holds [2, 4], and in this case, it will be correct. For an isotropic continuum of different tensile/compressive strength, it transforms, if these requirements are met, into a combination of the spherical part of stress and stress intensity and for a continuum of the same strength, only into stress intensity and becomes the classical fourth theory of strength [5].

CONCLUSIONS

1. The maximum stress and strain criteria are non-applicable for orthotropic CMs: the requirement of invariance with respect to the rotation of the coordinate axes in the plane of isotropy, if any, is not met.

2. The above non-applicability of these criteria leads to the fact that, depending on the rotation of the coordinate axes in the plane of isotropy, the strength function can change by a factor of almost two (and perhaps more).

3. In the general case of SSS, these criteria cannot be used for isotropic and transtropic composites, as well as for materials having at least one plane of weak anisotropy. As applied to CMs having a plane with the medium degree of anisotropy, they can give significant errors and can be used only after a preliminary analysis of the types of SSS and the degree of anisotropy. In particular, these criteria cannot be used in the strength analysis of such a wide class of composites as unidirectionally reinforced composites, which are modeled by a homogeneous transtropic continuum.

4. The range of CMs, to which the maximum stress and strain criteria can be applied with one or another degree of confidence, narrows to essentially orthotropic materials, namely to those whose maximum and minimum strength characteristics in each of the principal planes of anisotropy differ greatly under rotation by 90° (cross-reinforced CMs or CMs reinforced in three mutually perpendicular directions at the same time). For some particular types of SSS, the range of the CMs, to which the maximum stress and strain criteria can be applied, can be somewhat widened after a preliminary joint analysis of the type of a particular SSS, anisotropy, and its degree.

5. It has been found [2] that in the presence of a plane (planes) of isotropy, the Tsai–Wu quadratic strength criteria, in particular the generalized von Mises criterion, are void of this disadvantage, it the NSCELS and invariance conditions are met, and can be correctly used in the strength analysis of any orthotropic (in the general case) CMs: of both essentially orthotropic and weakly anisotropic, trans- and isotropic CMs.

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