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POWER LAW OF CRACK LENGTH DISTRIBUTION IN THE MULTIPLE DAMAGE PROCESS

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Multiple fatigue damage, which is characterized by crack initiation and propagation processes, is considered. We proposed two models of multiple damage, which imply random crack initiation and further propagation, with the exponential dependence between their length on the number of loading cycles. Crack initiation is modeled by the stationary Poisson flow with a constant intensity, while crack propagation is characterized by the rate parameter controlling the dependence of crack propagation rate and its length. The first model describes the deterministic case of multiple crack propagation at a fixed value of the above rate parameter, while the second one predicts their propagation by random trajectories according to distribution of the rate parameter. In the former case, crack length distribution is shown to be the Pareto power law with the exponent, which is defined by the ratio of kinetic parameters of initiation and propagation of defects. In the latter case, the rate parameter is uniformly distributed, in accordance with experimental data, so that the power-law distribution of crack length is close to the Pareto distribution. The respective distribution exponent also depends on the ratio of kinetic parameters of multiple damages and tends to drop during damage accumulation to the threshold level (namely, reaches the value of 2). This finding complies with experimental data on multiple damages of various classes of materials, including metals and rock masses. We also substantiated the range of ratios of kinetic parameters of defect initiation and propagation, which ensure the Pareto law of cracks length distribution and can be used to estimate the critical state of cracked bodies.

Keywords: multiple damages, initiation and propagation of fatigue cracks, Pareto distribution.

Introduction. When assessing the strength and durability of structures, dimensional indicators of defects are crucial. While the critical state of a solid body with a single crack is successfully assessed by multiple mature criteria, the presence of a large number of defects, which have usually random sizes, makes prediction of the bearing capacity of real structures quite problematic.

In the case of multiple damage, which is characterized by the processes of initiation and propagation of scattered defects [1, 2], the size a distribution of defects of different nature and size level (from micro- to mesoscale) was generally assumed to described by the power function in the form $a^{-\gamma}$, where γ is the distribution parameter. Moreover, it was proved that there is a scale invariance (scaling) of the distribution curves: regardless of the type of material, loading conditions, size of defects, and damage degree, these distributions can be reduced to the following single dependence with the general index γ [1, 2].

The power distribution of a hyperbolic type $f(x) = A/x^{\gamma}$, where $f(x)$ is the frequency of occurrence of the random variable x, and A and γ are constants, is usually referred to as the Zipf or Zipf–Mandelbrot law. This is a kind of non-Gaussian distributions, which is used to describe a wide range of complex systems, including the Bradford law in scientometrics, the Lotka law in bibliometrics, the Pareto law in economics, the Yule law in natural

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history; the Auerbach law in social geography, etc. [3]. The application of hyperbolic distributions is also quite lucrative for the description of physical phenomena, in particular, phase transformations [4]. For example, in the multiple fracture mechanics, the application of such distributions to the statistical description of a set of defects is expedient for the following reasons.

Power distributions is the only class of scale-independent distributions, which remain unchanged regardless of the scale, in which the random variable is described. Therefore, they are also referred to as scale-invariant distributions [4], which can describe the above-mentioned scale invariance of the curves of defect size empirical distributions in multiple damage processes.

The characteristic feature of power-law distributions is the presence of the so-called heavy/fat tails. This means that the number of events with an extremely large random variable x decreases slowly at $x \to \infty$. When analyzing Gaussian distributions, the largest values of x located in the rapidly decreasing tail are usually neglected. In the case of the power-law distribution, it is not recommended to neglect large, but rare events. Moreover, in some cases, only the tail should be considered, neglecting the distribution behavior at small x. This finding necessitated the application of power-law distributions to the damage description in catastrophic events and risk assessment [4, 5]. With regard to multiple fractures, a heavy tail of the distribution makes it possible to predict the appearance of defects with maximum sizes.

The universality of power-law distributions when describing statistics for a wide range of phenomena (social and natural) and their large-scale invariance allows one to assume that they can be related to fundamental manifestations of similarity at the evolution of complex systems, namely, with their fractal indicators.

The main parameter of the power distribution is the exponent γ. For many natural phenomena, the following inequality is applicable $2 < \gamma < 3$ [4]. However, there are distributions, for which this parameter can vary within a wider interval ($1 < \gamma < 10$). It is quite obvious that with decreasing γ, the distribution of the random variable x shifts to the field of larger values; the frequency of occurrence of such events $f(x)$ increases. In relation to the extent of defects in the loaded body, this implies strength reduction. Therefore, γ can be used as an indicator of the critical state, in the case of multiple failures [6, 7]. It is necessary to establish a physical dependence between this parameter and fracture characteristics, which can be defined analytically or experimentally.

In study [8], on the basis of the proposed model of multiple fracture and experimental data on the initiation and distribution of fatigue cracks, it was shown that the defect size distribution can be described by the Pareto power law.

The purpose of this study is to justify the power law distribution of the defects' size, in case of multiple damages of materials, to determine the parameters of such distribution, and to establish their relation with kinetic indicators of damage.

Models of Dimensional Stochasticity of Defects. The basic premise for modeling the dimensional stochasticity of defects during multiple failure process is to consider two main damage factors, such as initiation and propagation of defects which are realized in a stressed body in the loading process. In [7], the discrete model was proposed based on the Yule* process, in which the exponent γ was described by the following dependence:

$$\gamma = 1 + \frac{n}{m}, \tag{1}$$

where n and m are the numbers of acoustic emission pulses accumulated during the damage processes at initiation and propagation stages of defects, respectively.

The parameter n/m is interpreted as the energy ratio of new defects' initiation and their distribution. According to this discrete model, the variation of indicator γ in (1) is defined by the ratio of these processes.

Let us consider the initiation model of probabilistic distribution of the fatigue cracks length, taking into account their continuous initiation and propagation in time.

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* The Yule process that models the formation of genus from species is the fundamental law substantiating the nature of power distribution laws [4].

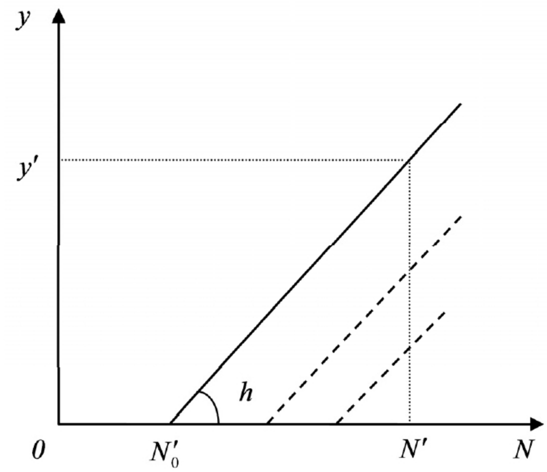

Fig. 1. The scheme of crack propagation with the constant value of the velocity parameter h.

The Deterministic Crack Propagation Model. The deterministic crack propagation is defined by the general and constant trajectory of their distribution. The stochasticity of cracks' sizes is caused by the random value of the cyclic loading time before their initiation.

To elaborate the above model, we made the following assumptions.

1. The dependence of the crack length a on the number of loading cycles N is exponential:

$$a = a_0 \exp[h(N - N_0)], \tag{2}$$

where a_0 is the initial size of the crack during cyclic loading N_0 and h is the parameter of crack propagation rate.

It should be noted that the exponential propagation of fatigue cracks at the initial distribution stage is typical for a constant level of cyclic stress [9], as well as for the cyclic loadingal (block) load spectrum [10].

2. Propagation of all cracks occurs with the same value of h.

3. The initiation of defects with an initial crack size a_0 is random and represents the stationary Poisson flow with a constant intensity λ. Then, the probability of occurrence of at least one crack in the interval of the cyclic loading time $N_i \ldots N_j$ of cycles will have the following form:

$$\Pr\{N_0 \in (N_i,\ N_f]\} = 1 - \exp[-\lambda(N_j - N_i)], \tag{3}$$

where $\Pr\{\cdot\}$ is the event probability.

Using Eq. (2), we introduce the relative size parameter y related to the number of cycles N by the linear relation:

$$y = \ln\frac{a}{a_0} = -hN_0 + hN, \tag{4}$$

where hN_0 is a constant value for the particular crack.

We define the distribution function of the parameter y at the fixed value h. Obviously, this function is determined by the event probability $Y < y$, where Y is the random parameter value y.

According to the accepted assumptions, for arbitrary values N' and y' and the constant value h (solid line in Fig. 1), the parameter of any crack size y at the time of cyclic loading N' will be less than y' if it is formed in the cyclic loading time interval $(N'_0,\ N')$ (dashed lines in Fig. 1). Therefore, for any crack, the probability of an event $Y < y'$ is determined by the probability of its initiation in the cyclic loading time interval

$$N' - N'_0 = \frac{y'}{h}. \tag{5}$$

Based on formula (3) and taking into account relation (5), the distribution function for arbitrary y' and N' can be derived as

$$G(y) = \Pr\{Y < y\} = 1 - \exp\left(-\frac{\lambda y}{h}\right), \tag{6}$$

Representing parameter y in (6) by the logarithmic ratio of crack sizes (4), we obtain the crack length distribution function, which corresponds to the Pareto power law:

$$P(a) = 1 - \left(\frac{a_0}{a}\right)^{\gamma - 1}. \tag{7}$$

The distribution density (7) will be equal to

$$p(a) = \frac{\gamma - 1}{a_0}\left(\frac{a_0}{a}\right)^{\gamma}, \tag{8}$$

where

$$\gamma = 1 + \frac{\lambda}{h}. \tag{9}$$

The structure of Eq. (9) is similar to Eq. (1) obtained on the basis of the multiple fracture model by the Yule process [7]. Comparing Eqs. (1) and (9), we can get direct relations between the parameters of crack initiation λ and propagation h and the corresponding indicators of the number of acoustic emission pulses, i.e., $\lambda \sim n$ and $h \sim m$. In Eq. (9), parameter h is considered to be constant for all cracks, while in Eq. (1), there is no such restriction on its value. Next, we apply the power distribution law to various crack propagation trajectories specified by a random value of h.

Crack Propagation Model by Random Trajectories. As in the previous model, the exponential propagation of cracks (2) with a random value h is considered. Additionally, we use the following assumptions.

1. The initiation of cracks with the initial size a_0 is random and continuous in the cyclic loading interval $[0, N]$. The function $F_i(N)$ and density $f_i(N)$ of the distribution of cyclic loading to crack initiation are known.

2. The distribution of cracks is stationary. Stationary is understood as the constant random value h for each crack during the entire period of its propagation. The distribution function of this parameter $F_h(h)$ is known.

When $h = 0$ the crack a_0 in length is non-distributing.

Let the crack be formed in a sufficiently small time interval $[N_0', N_0' + dN]$, where N_0' is the random variable. The probability of such an event will be equal to $\mu(N_0')dN$, where $\mu(N)$ is the function of the crack initiation intensity, which depends on the cyclic loading and is defined by the relation [11]

$$\mu(N) = \frac{f_i(N)}{1 - F_i(N)}. \tag{10}$$

The arisen crack grows at the random value h, and while cyclic loading N' $(N' > N_0')$ its size will be equal to a random value Y (Fig. 2). Let us determine for such a crack the conditional probability of the event $Y < y'$, where y' is the arbitrarily chosen parameter of the crack size while cyclic loading N' (Fig. 2). Obviously, this event will be carried out at the crack distribution with the velocity parameter

$$h < h' = \frac{y'}{N' - N_0'}, \tag{11}$$

In this case, the probability of such an event is determined by the parameter distribution function $h - F_h(h')$.

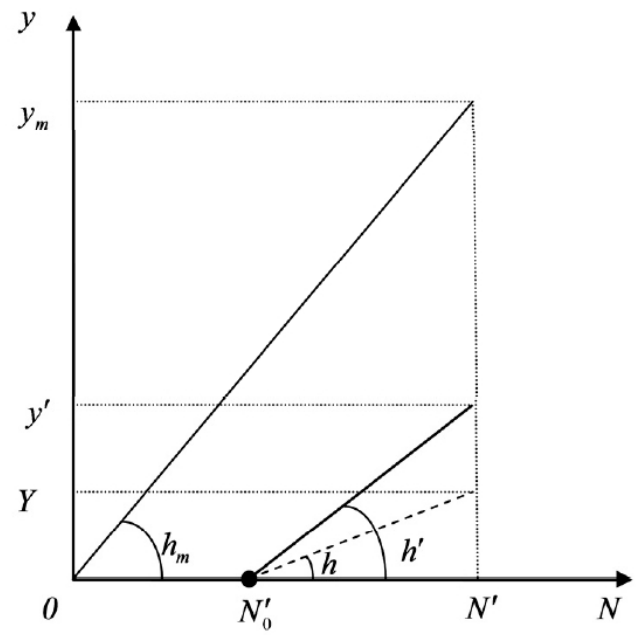

Fig. 2. The scheme of crack propagation at the random value.

Then, the conditional distribution function of the parameter y for the crack formed at the moment of cyclic loading N'_0 and growing with the velocity parameter $h < h'$ is defined by the relation

$$G(y;\ N|N'_0) = \Pr\{Y < y;\ N|N'_0\} = \mu(N'_0)dN\, F_h\left(\frac{y}{N - N'_0}\right). \tag{12}$$

Since cracks can form within entire considered cyclic loading interval $N_0 \in [0,\ N]$, for the unconditional distribution function of the parameter y, based on (12), we get

$$G(y;\ N) = \int_0^N \mu(N_0) F_h\left(\frac{y}{N - N_0}\right) dN_0. \tag{13}$$

Let us consider a special case of distribution (13), when the cyclic loading time to crack initiation corresponds to the exponential distribution law $F_i(N) = 1 - \exp(-\lambda N)$, where λ is the intensity of crack initiation. In this case $f_i(N) = \lambda \exp(-\lambda N)$, and from formula (10) we get $\mu = \lambda$.

In Eq. (13), we replace the variable N_0 on h using the relation (11). Then, expression (13) takes the form

$$G(y;\ N) = \lambda y \int_{y/N}^{\infty} h^{-2} F_h(h) dh. \tag{14}$$

The function $F_h(h)$ can be specified explicitly not for all distributions of random variables. Therefore, in (14), it is expedient to replace function $F_h(h)$ by the distribution density $f_h(h)$.

The partial integration of Eq. (14), yields

$$G(y; N) = \lambda N F_h\left(\frac{y}{N}\right) + \lambda y \int_{y/N}^{\infty} \frac{f_h(h)}{h} dh. \tag{15}$$

Differentiating (15) by y, we obtain the expression for the distribution density of the crack length parameter:

$$g(y;\ N) = \lambda \int_{y/N}^{\infty} \frac{f_h(h)}{h} dh. \tag{16}$$

Let us consider the parameter range h in relation to propagation of fatigue cracks. If it is limited on the left by the value $h = 0$, the arisen crack with an initial size a_0 does not extend. When $h \to \infty$, it is unstable that leads to instant destruction. Therefore, for the actual propagation of the fatigue crack, the range of parameter h should be limited on the right to the maximum possible value h_m.

To justify h_m, we should use Eq. (2), which was obtained by solving the equation $da/dN = ha$. We write this equation in the form

$$\frac{da}{a} = hdN. \tag{17}$$

From (17) follows that the parameter h represents the relative propagation (lengthening) of the crack da/a per one loading cycle. For the elastoplastic propagation of fatigue cracks, the maximum possible value h can be limited by the size of the plastic deformation zone s at the crack tip:

$$h_s = \frac{s}{a} = \xi \left(\frac{\Delta\sigma}{\sigma_Y} \right)^2, \tag{18}$$

where $\Delta\sigma$ is the stress range per cycle, σ_Y is the yield strength of material, and ξ is coefficient ($\xi < 1$).

When the crack grows in each loading cycle, the coefficient ξ is equal to the constant used to estimate the plastic deformation zone at the crack tip. For example $\xi = 1/4\alpha$, where $\alpha \cong 2$ for plane-stress and $\alpha \cong 6$ plane-strain states [12]. With intermittent crack propagation, the value ξ has the order $1/\Delta N$, where ΔN is the duration of crack arrest, which precedes its further jumplike increment.

According to the experimental data, during the exponential propagation of fatigue cracks, the uniform distribution of the parameter h can be accepted [8]. We use the maximum interval of possible values h from $h_{\min} = 0$ to $h_m = \beta h_s$, where the coefficient β is defined by the ratio between the distance over which the crack moves forward in one loading cycle and the size of the plastic deformation zone s ($\beta \le 1$). Obviously, for any N the maximum possible crack size in the sample, it will be limited by the value of parameter y_m provided that it is formed at the moment of cyclic loading $N = 0$ (Fig. 2):

$$y_m = h_m N. \tag{19}$$

The distribution density of the parameter h for the chosen interval is written as

$$f_h(h) = \frac{1}{h_m}. \tag{20}$$

Substituting $f_h(h)$ from (20) into Eq. (16), in which the upper limit of integration will be equal to h_m, we obtain

$$g(y;\ N) = \frac{\lambda}{h_m} \ln \frac{y_m}{y}, \qquad y \in [0,\ y_m], \tag{21}$$

where y_m varies with the cyclic loading N due to dependence (19).

Taking into account the logarithmic dependence (4), we pass from the distribution of the parameter y in (21) to the distribution of the crack length a:

$$p(a)=\frac{\lambda}{ah_m}\ln\frac{\ln(a_m/a_0)}{\ln(a/a_0)}, \qquad a\in[a_0,\ a_{th}], \tag{22}$$

where a_m is the maximum possible crack length at the number of cycles N that can be hypothetically reached at the loading start (at $N=0$) and grow at the maximum propagation rate corresponding to h_m.

According to relations (4) and (19), we get

$$a_m=a_0\exp(h_m N). \tag{23}$$

From the normalization condition for the distribution density (22), we get

$$\frac{\lambda}{h_m}\int_{a_0}^{a_m}\frac{1}{a}\ln\frac{\ln(a_m/a_0)}{\ln(a/a_0)}\,da=1, \tag{24}$$

where the replacement $z=\dfrac{\ln(a/a_0)}{\ln(a_m/a_0)}$ reduces the integral to the tabulated form [13]:

$$\int_0^1\left(\ln\frac{1}{z}\right)^{k-1}dz=\Gamma(k),$$

$\Gamma(\cdot)$ is the gamma function.

After integration, from (24) when $k=2$ follows:

$$\frac{h_m}{\lambda}=\ln\frac{a_m}{a_0}. \tag{25}$$

Substituting relation (25) into formula (22), we obtain

$$p(a)=\frac{\varphi(a/a_0)}{a_0}, \tag{26}$$

where the function $\varphi(a/a_0)$ is given by

$$\varphi(a/a_0)=\frac{1}{(a/a_0)\ln(a_m/a_0)}\ln\frac{\ln(a_m/a_0)}{\ln(a/a_0)}. \tag{27}$$

The calculations by formula (27) for the various values a_m/a_0 show that at $a<a_m$ for a sufficiently wide range of crack length changes the dependence of the function $\varphi(a/a_0)$ on a/a_0 in double logarithmic coordinates is close to linear (Fig. 3). Deviation from the linear dependence takes place at large values of the crack length (in the distribution tail) at $a\to a_m$. Therefore, when $a<a_m$ the function $\varphi(a/a_0)$ can be represented as exponential one:

$$\varphi(a/a_0)=\frac{A}{(a/a_0)^{\gamma}}, \tag{28}$$

where coefficients A and γ are determined from the approximation of the curves calculated for different a_m/a_0 (Table 1).

The regression dependence between the coefficients A and γ given in Table 1 has the form $A=1.0013\gamma-0.9766$ (Fig. 4). Taking into account the natural errors at approximation, we can write

$$A=\gamma-1. \tag{29}$$

TABLE 1. The Values of the Coefficients in the Formula (28) for Various a_m/a_0

a_m/a_0	A	γ	Correlation coefficient
1.93	4.8771	5.8528	0.9941
2.33	3.8325	4.7990	0.9944
3.01	2.8785	3.8493	0.9965
3.5	2.5985	3.5719	0.9957
4.5	2.1634	3.1260	0.9968
5	1.9856	2.9523	0.9976
10	1.4134	2.3867	0.9977
15	1.1869	2.1658	0.9980
20	1.0795	2.0610	0.9980

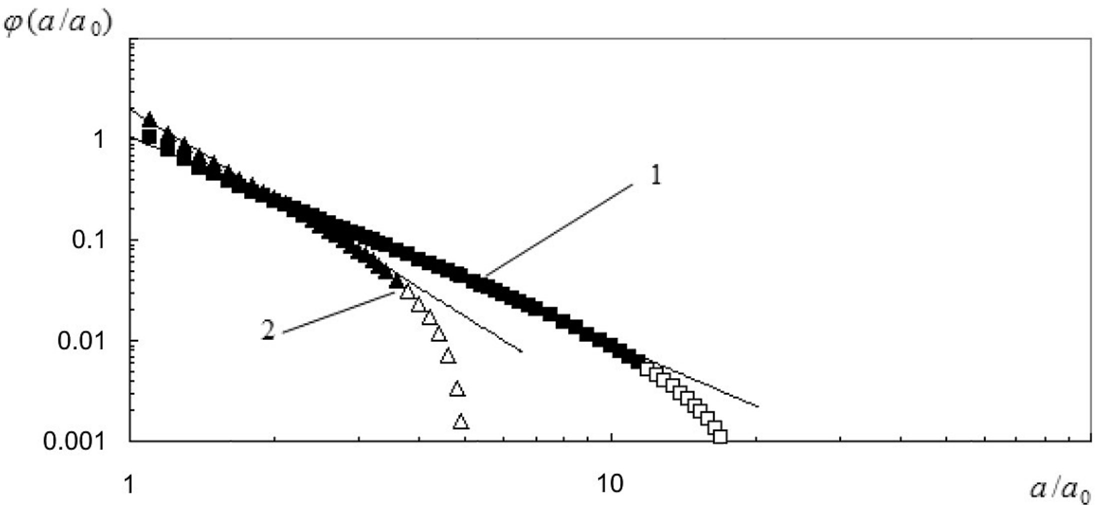

Fig. 3. Calculation (symbols) and approximation (lines) of the function (27): (*1*) $a_m/a_0 = 15$; (*2*) $a_m/a_0 = 5$. (The approximation was carried out by the solid symbols.)

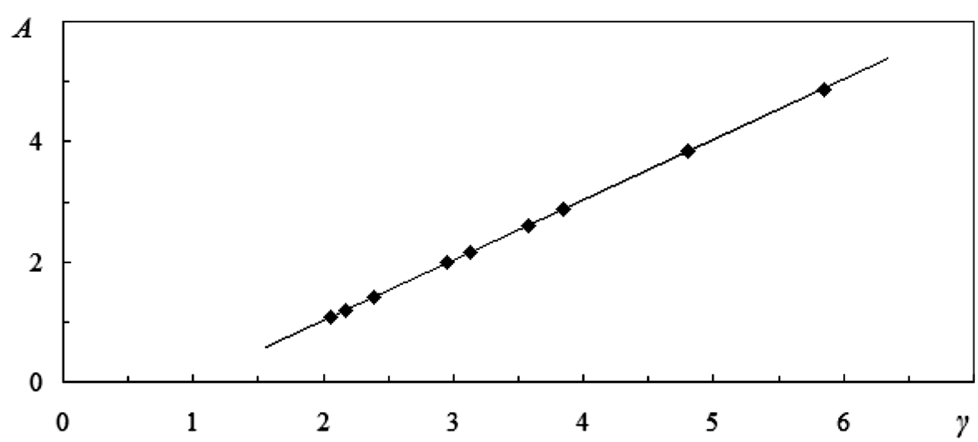

Fig. 4. The relation between the coefficients in the formula (28). (Correlation coefficient is $R^2 = 1$.)

When substituting relations (28) and (29) in formula (26), we obtain the distribution density of the crack length which corresponds to the Pareto law (8).

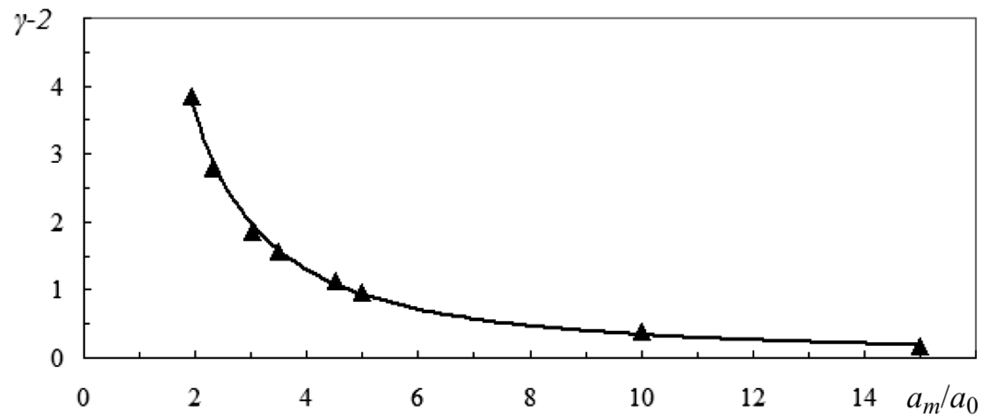

Fig. 5. The dependence of the Pareto distribution degree on the parameter a_m/a_0. (Correlation coefficient is $R^2 = 0.995$.)

Discussion of the Results. Limiting the speed parameter h to the value h_m limits the maximum possible crack length a_m. In this case, the tail of the distribution of the random variable a in the field of large values $(a \to a_m)$ does not correspond to the Pareto distribution, which is typical for the limited range of defect sizes at $a < a_m$ (Fig. 3). If we assume that the power distribution is carried out for all cracks in the range $[a_0, a_m]$, then the error in the probabilistic prediction of the presence of large length cracks $(a \to a_m)$ will go to the safety margin, i.e., the predicted size of the defect will exceed actual value. For example, according to formula (27), at $a_m/a_0 = 15$ the value $\varphi(a/a_0) = 10^{-3}$ corresponds to the relative length of the maximum crack $a_{\max}/a_0 = 13.5$. If the Pareto distribution is taken for the entire range of crack sizes, then the predicted value $a_{\max}/a_0 = 26.3$ follows from Eq. (28) for $\varphi(a/a_0) = 10^{-3}$.

In accordance with the results given in Table 1, the exponent distribution index γ (8) decreases with increasing the parameter a_m/a_0 and asymptotically approaches to $\gamma = 2$ (Fig. 5). The obtained dependence is approximated by the equation

$$\gamma \cong 2 + 10\left(\frac{a_m}{a_0}\right)^{-1.5}. \tag{30}$$

Taking into account formula (23), it follows from (30):

$$\gamma \cong 2 + 10\exp(-1.5 h_m N). \tag{31}$$

According to Eqs. (30) and (31), the minimum threshold value $\gamma = 2$ is inherent for the Pareto distribution exponent. This finding has a specific mathematical justification.

For the Pareto distribution, there are special values γ, for which it does not have the mean and variance. For example, the mathematical expectation of the cracks length for the distribution density (8) is defined at $\gamma > 2$, and the dispersion – at $\gamma > 3$ [8]. This means that if the samples of the random variable, which are limited by volume, distributed due to the power law can have finite mean and variance values for the indicated γ, then for the general population these numerical characteristics are absent (varying), i.e., for one limited sample from the general population, the mean and dispersion values can be very small, while for another they can be quite large [4]. Such a feature of the power distribution is caused by the role of the tail with increasing of γ [5], which excludes the possibility of an adequate prediction of extreme values of the random variable, in particular, cracks which are dangerous in size.

The fatigue damage degree of structures with cracks depends on the loading duration. According to the formula (31), an increase in the number of loading cycles N, the Pareto distribution exponent γ decreases, which is

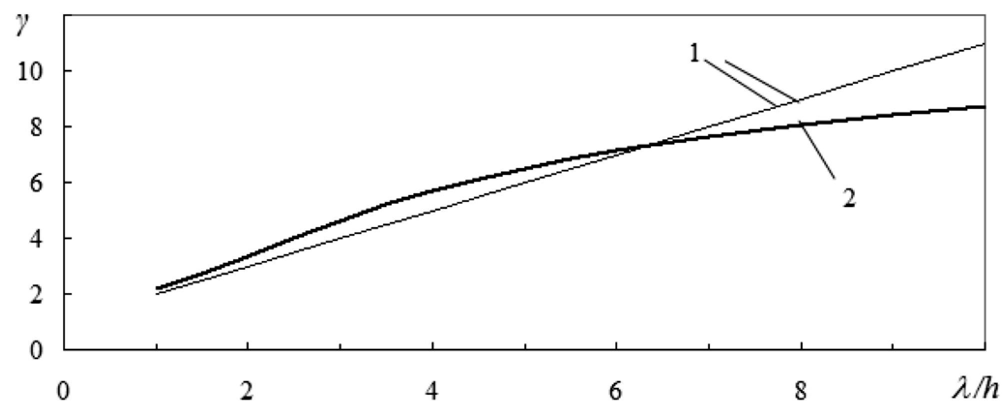

Fig. 6. The dependence of the Pareto distribution degree on the parameter λ/h: (*1*) calculation by the formula (9); (*2*) calculation by the formula (32) for the average value $\langle h \rangle$.

consistent with experimental data on multiple damages at the microstructural dimensional level for metallic materials, as well as for rocks during tectonic events. In [6–8] was noted that on the basis of this phenomenon, the parameter γ is proposed to be interpreted as the limiting state indicator at multiple damages of materials.

The dependence of the indicator γ on the ratio of the basic kinetic parameters of multiple damages, namely, the intensity of the crack initiation and the parameter of their propagation rate, follows from the formulas (30) and (25):

$$\gamma \cong 2 + 10\exp\left(-1.5\frac{h_m}{\lambda}\right). \tag{32}$$

The dependence (32), in contrast to Eq. (9), which was obtained for the deterministic value of the parameter *h*, is not linear. However, expressing h_m the crack velocity parameters $\langle h \rangle = h_m/2$ in (32) via the mathematical expectation, we obtain the nonlinear dependence, which is quite close to Eq. (9), as seen in Fig. 6.

Therefore, at deterministic and random values h, the crack size distribution is well-described by the Pareto law, the exponent of which is defined by the ratio of kinetic parameters of multiple damages. In this case, the value of ratio between the intensity of crack initiation and the parameter of their propagation rate has the order of $1 \le \lambda/h < 10$.

CONCLUSIONS

1. In the case of multiple fatigue damage, which implies a random crack initiation with the constant intensity and the dependence of cracks' size on the number of loading cycles described by the exponential rule, the distribution of cracks' size is described by the Pareto law. The distribution exponent is defined by the ratio of parameters characterizing the kinetics of the defects' initiation and propagation.

2. In case of exponential propagation of fatigue cracks, the Pareto distribution is valid for the constant values of the velocity parameter for all defects and for the crack propagation by random trajectories.

3. The power distribution of crack lengths can be used to predict the ultimate state of the particular structure with a large number of cracks (riveted joints for aircraft skin), as well as the parts with one or more cracks that represent damage of the general population of such components installed and operating in multiple similar machines.

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