Strength of Materials, Vol. 46, No. 1, January, 2014

ON SIX-DIMENSIONAL QUADRIC HYPERSPACE OF STRENGTH FOR ORTHOTROPIC MATERIALS

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For the general case of a stress state, the necessary and sufficient conditions have been strictly derived. These conditions should be satisfied by undetermined coefficients of quadratic strength criteria for orthotropic materials, so that the limiting quadric hypersurface of strength in the six-dimensional space of stresses has a physical meaning.

Keywords: orthotropy, strength criteria, limiting quadric hypersurface of strength, quadratic form, six-dimensional space of stresses, stability conditions.

Composite materials (CM) are widely used in various state-of-the-art technologies. Usually, these materials feature a structural irregularity (heterogeneity) and, in many cases, a considerable anisotropy of physical-mechanical properties. To assess strength of CMs under a complex stress state one should know the limit state criteria (the strength criteria) that specify the allowable stress limits within which the material will be able to work without failure. The limit states are the states whereby CM undergoes a transition from the elastic (or elastoplastic) state to fracture. More often than not, strength of CMs is analyzed through the phenomenological macrostructural approach which considers a heterogeneous composite as a continuous, generally anisotropic medium. A mathematical model of this medium is constructed on the basis of experimental findings. The majority of modern CMs are quite adequately described by a model of a homogeneous orthotropic body or by some particular cases of this model.

Phenomenological strength criteria are not derived analytically, they are postulated or put forward based on generalization of experimental data. Recently, quadratic strength criteria for an orthotropic body have been widely accepted [1–8]; in available publications they are called the Tsai–Wu criteria [8]. They take into account the difference between tensile and compression strengths, offers the maximum possible flexibility, contain no redundant parameters, make it possible to easily determine the principal axes of strength, etc.

It would be natural to assume that the principal axes of anisotropy x, y, z of an orthotropic material are simultaneously the principal axes for both the elastic and strength characteristics. Moreover, in the most cases the shear strength of a composite is independent of the tangential stress sign. Thus, the linear-quadratic strength criterion in the coordinates x, y, z is written as [1–8]

$$\Phi=\sigma_x\left(\frac{1}{S_x^+}-\frac{1}{S_x^-}\right)+\sigma_y\left(\frac{1}{S_y^+}-\frac{1}{S_y^-}\right)+\sigma_z\left(\frac{1}{S_z^+}-\frac{1}{S_z^-}\right)+\frac{\sigma_x^2}{S_x^+S_x^-}+\frac{\sigma_y^2}{S_y^+S_y^-}+\frac{\sigma_z^2}{S_z^+S_z^-}+\frac{\tau_{xy}^2}{T_{xy}^2}+\frac{\tau_{yz}^2}{T_{yz}^2}+\frac{\tau_{zx}^2}{T_{zx}^2}$$

$$+2a_{xy}\sigma_x\sigma_y+2a_{yz}\sigma_y\sigma_z+2a_{zx}\sigma_z\sigma_x\le 1, \tag{1}$$

where σ_i and τ_{ij} are the stress tensor components, S_i^+ and S_i^- are the ultimate strengths in tension (+) and compression (–) along the ith principal direction of anisotropy, and T_{ij} are the ultimate strengths in pure shear in the respective principal planes of anisotropy ij (i, $j=x$, y, z, $i\ne j$).

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Criterion (1) has been defined concretely on the basis of nine simplest experiments: three in uniaxial tension, three in uniaxial compression, and three in pure shear. It identically satisfies (becomes equal to unity) these nine experimental (reference) points in the six-dimensional space of stresses at any values of three parameters a_{ij}. These parameters, in turn, must meet certain conditions so that the limiting hypersurface of strength $\Phi = 1$ has a physical meaning: it should be a simply connected and closed surface or, in extreme case, should allow existence of maximum one direction (a ray or a straight line) with infinite strength. The issues regarding the mathematically rigorous substantiation of the necessary and sufficient conditions (the so-called stability conditions) as well as conditions for bulging, simple connectedness, closedness or openness, possible and allowable geometrical shapes of the six-dimensional limiting hypersurface of strength for an orthotropic material (1) have not been adequately explored so far. In particular, in the well-known publications [1–7], based on the biaxial plane stress analysis for principal planes of anisotropy the authors derive only the necessary stability conditions which, as shown below, do not always ensure the existence of physically allowable hypersurface of strength in the six-dimensional stress space for the general case of triaxial stress state.

From the multitude of six-dimensional quadric surfaces there are only three surfaces which satisfy the above-mentioned requirements: ellipsoids (closed and convex), elliptic paraboloids (one-sidedly open and convex), or elliptic cylinders (double-sidedly open and almost convex). The objective of the present work is to strictly define the necessary and sufficient conditions which should be satisfied by undetermined coefficients a_{ij} in (1) so that the limiting hypersurface $\Phi = 1$ should belong to one of the three above-mentioned physically substantiated types.

Let us introduce the dimensionless variables

$$\Sigma_i = \frac{\sigma_i}{\sqrt{S_i^+ S_i^-}}, \qquad \Theta_{ij} = \frac{\tau_{ij}}{T_{ij}}, \qquad i,\ j = x,\ y,\ z, \quad i \neq j \tag{2}$$

and represent the limiting hypersurface $\Phi = 1$ as

$$\Phi = \Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2 + 2A_{xy}\Sigma_x\Sigma_y + 2A_{yz}\Sigma_y\Sigma_z + 2A_{zx}\Sigma_z\Sigma_x + \Theta_{xy}^2 + \Theta_{yz}^2 + \Theta_{zx}^2 + 2B_x\Sigma_x + 2B_y\Sigma_y + 2B_z\Sigma_z = 1, \tag{3}$$

where

$$A_{ij} = a_{ij}\sqrt{S_i^+ S_i^- S_j^+ S_j^-}, \qquad B_i = \frac{1}{2}\left(\sqrt{S_i^- / S_i^+} - \sqrt{S_i^+ / S_i^-}\right). \tag{4}$$

1. Consider a case of no tangential stresses:

$$\Theta_{xy} = \Theta_{yz} = \Theta_{zx} = 0. \tag{5}$$

In view of (5), the limiting hypersurface (3) takes on the classical quadratic form – the quadric surface in the three-dimensional space of normal stresses in rectangular coordinates Σ_x, Σ_y, and Σ_z:

$$\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2 + 2A_{xy}\Sigma_x\Sigma_y + 2A_{yz}\Sigma_y\Sigma_z + 2A_{zx}\Sigma_z\Sigma_x + 2B_x\Sigma_x + 2B_y\Sigma_y + 2B_z\Sigma_z = 1, \tag{6}$$

and it can be analyzed through the use of the well-elaborated theory of quadric surfaces [9, 10].

Considering the surface $\Sigma_x = 0$ we can easily derive the necessary condition $|A_{yz}| \leq 1$. When it is violated, the surface (6) will a fortiori belong to none of the three above-mentioned physically allowable types [1–8]. Following the same procedure for the other two planes $\Sigma_y = 0$ and $\Sigma_z = 0$, we arrive at the necessary conditions for three coefficients:

$$|A_{ij}| \leq 1. \tag{7}$$

Due to (4), the coefficients B_i are subject to no limitations on physical grounds and can be take any values. Hereinafter, this statement will be proved through exact mathematical reasoning.

In the well-known publications [1–7], the quadratic strength criteria under consideration are bounded by the conditions (7) (they are called the stability conditions). However, as shown here below, they are not sufficient for the limiting failure surface to belong to the three above-mentioned types.

Following the procedure [9, 10], we write four invariants of Eq. (6):

$$\begin{cases} I_1 = 1+1+1=3, \\ I_2 = \begin{vmatrix} 1 & A_{xy} \\ A_{xy} & 1 \end{vmatrix} + \begin{vmatrix} 1 & A_{yz} \\ A_{yz} & 1 \end{vmatrix} + \begin{vmatrix} 1 & A_{zx} \\ A_{zx} & 1 \end{vmatrix} = 3 - A_{xy}^2 - A_{yz}^2 - A_{zx}^2, \\ I_3 = \begin{vmatrix} 1 & A_{xy} & A_{zx} \\ A_{xy} & 1 & A_{yz} \\ A_{zx} & A_{yz} & 1 \end{vmatrix} = 1 + 2A_{xy}A_{yz}A_{zx} - A_{xy}^2 - A_{yz}^2 - A_{zx}^2, \\ I_4 = \begin{vmatrix} 1 & A_{xy} & A_{zx} & B_x \\ A_{xy} & 1 & A_{yz} & B_y \\ A_{zx} & A_{yz} & 1 & B_z \\ B_x & B_y & B_z & -1 \end{vmatrix}. \end{cases} \tag{8}$$

It can be shown that $I_1 > 0$ and, due to (7) we have $I_2 \geq 0$; the invariant I_3 should meet the requirement

$$I_3 \geq 0. \tag{9}$$

Otherwise, as follows from [9, 10], the surface $\Phi = 1$ would have a shape of a hyperboloid or a cone, which is inadmissible.

1.1. Consider the case

$$I_3 > 0. \tag{10}$$

It can be easily shown that to satisfy (10) and (7) simultaneously the following condition should be met:

$$|A_{ij}| < 1. \tag{11}$$

We introduce the function

$$f(X, Y, Z) = X^2 + Y^2 + Z^2 + 2A_{xy}XY + 2A_{yz}YZ + 2A_{zx}ZX. \tag{12}$$

Let us prove that the quadratic form (12) subject to conditions (10) and (11) is a positively definite one. For this form, the first three invariants of type (8) will be strictly positive; the fourth one for the equation

$$f(X, Y, Z) = 1 \tag{13}$$

will be written as $I_4 = -I_3 < 0$.

According to [9, 10] the surface (13) is ellipsoid-shaped and thus

$$f(X, Y, Z) \geq 0. \tag{14}$$

We will demonstrate that the surface

$$f(X, Y, Z) + 2B_xX + 2B_yY + 2B_zZ = 1 \tag{15}$$

is also ellipsoid-shaped at any B_i values.

Since condition (10) is fulfilled, the surface (15) will be central and its center (X_0, Y_0, Z_0) will be unambiguously defined from the set of equations

$$\begin{cases} X_0 + A_{xy}Y_0 + A_{zx}Z_0 = -B_x, \\ A_{xy}X_0 + Y_0 + A_{yz}Z_0 = -B_y, \\ A_{zx}X_0 + A_{yz}Y_0 + Z_0 = -B_z. \end{cases} \tag{16}$$

The origin of coordinates is placed at the center of symmetry of the surface

$$X_N = X - X_0, \qquad Y_N = Y - Y_0, \qquad Z_N = Z - Z_0. \tag{17}$$

It can be easily shown that in new coordinates (17) the surface (15) is rearranged, in view of (16), to the form

$$f(X_N, Y_N, Z_N) = 1 + f(X_0, Y_0, Z_0). \tag{18}$$

Due to (14) the quantity in the right-hand part of (18) is not less than unity. For the surface (18), like for (13), the first three invariants of type (8) will be strictly positive, the fourth one will be given by $I_4 = -I_3[1 + f(X_0, Y_0, Z_0)] < 0$, therefore, the surface (18), and hence the surface (15) too, are ellipsoid-shaped.

Thus, the situation (10) and (11) is allowable; in this case, the strength surface (6) has the shape of an ellipsoid.

1.2. Consider the case

$$I_3 = 0 \tag{19}$$

with condition (7) satisfied. For the first three invariants from (8) we have

$$I_1 > 0, \qquad I_2 \ge 0, \qquad I_3 = 0, \tag{20}$$

while for the fourth one in view of (19) we can easily arrive at the following expression:

$$I_4 = -[B_x\sqrt{1 - A_{yz}^2} \pm B_y\sqrt{1 - A_{zx}^2} \pm (\pm B_z\sqrt{1 - A_{xy}^2})]^2 \le 0. \tag{21}$$

1.2.1. The absolute value of one of A_{ij} is 1. Assume that for the sake of definiteness we have

$$A_{xy} = \pm 1. \tag{22}$$

Then, it follows from (19) that

$$A_{yz} = \pm A_{zx}. \tag{23}$$

In this case, the situation

$$|A_{xy}| = |A_{yz}| = |A_{zx}| = 1 \tag{24}$$

is inadmissible because we will have

$$I_1 > 0, \qquad I_2 = 0, \qquad I_3 = 0, \qquad I_4 = 0, \tag{25}$$

and according to [9, 10] the surface will be either a parabolic cylinder or a pair of parallel planes.

Thus, if (22) has been fulfilled, condition (23) is satisfied too. Moreover, we have

$$|A_{yz}| = |A_{zx}| < 1, \tag{26}$$

and the invariants have the following signs:

$$I_1 > 0, \qquad I_2 > 0, \qquad I_3 = 0, \qquad I_4 \le 0. \tag{27}$$

If, in this case, the condition

$$I_4 < 0 \tag{28}$$

is met, we have a quite allowable type of surface – an elliptic paraboloid. If

$$I_4 = 0, \tag{29}$$

we will also have an allowable type of the strength surface.

Let us prove this statement by analyzing the sign of the semi-invariant [9, 10]:

$$K_3 = \begin{vmatrix} 1 & A_{yz} & B_y \\ A_{yz} & 1 & B_z \\ B_y & B_z & -1 \end{vmatrix} + \begin{vmatrix} 1 & A_{zx} & B_x \\ A_{zx} & 1 & B_z \\ B_x & B_z & -1 \end{vmatrix} + \begin{vmatrix} 1 & A_{xy} & B_x \\ A_{xy} & 1 & B_y \\ B_x & B_y & -1 \end{vmatrix}$$

$$= -I_2 - 2(B_x^2 + B_y^2 + B_z^2 - A_{xy}B_xB_y - A_{yz}B_yB_z - A_{zx}B_zB_x). \tag{30}$$

It can be demonstrated that

$$K_3 \le -I_2 - (|B_x| - |B_y|)^2 - (|B_y| - |B_z|)^2 - (|B_z| - |B_x|)^2 \le -I_2 < 0, \tag{31}$$

and thus the surface at hand will have the shape of an elliptical cylinder [9, 10].

So, the situation (19), (22), (23), and (26) is permissible. Through permutation of subscripts x, y, z we will have two more allowable situations.

1.2.2. The absolute values of all A_{ij} are less than 1:

$$|A_{xy}| < 1, \qquad |A_{yz}| < 1, \qquad |A_{zx}| < 1. \tag{32}$$

In this case, we have the allowable types of surfaces – elliptic paraboloids or cylinders. This can be proved in the same way as in Subsection 1.2.1 above, i.e., the situation (19) and (32) is permissible.

Upon a simple generalizing analysis of the data given in Section 1, the following statement can be made. The sought-for necessary and sufficient conditions, which should be satisfied by the coefficients of the quadratic form (6) so that the limiting surface of strength for an orthotropic material in the three-dimensional space of normal stresses has a physical meaning, are as follows:

$$A_{xy}^2 + A_{yz}^2 + A_{zx}^2 \le 1 + 2A_{xy}A_{yz}A_{zx} < 3, \tag{33}$$

if in (33) the first unstrict inequality is replaced with a strict one we will have the necessary and sufficient conditions for the limiting surface $\Phi = 1$ to be an ellipsoid. If the equality expression in the left-hand member of (33) is obeyed, we will have elliptic paraboloids or cylinders depending on the coefficients of linear terms in (6).

2. The general case of stress state: all the six stress tensor components can take nonzero values, the equation for the limiting six-dimensional hypersurface of strength has the form (3).

It is evident that Eq. (3) has already been reduced to the canonical form in terms of variables Θ_{xy}, Θ_{yz}, and Θ_{zx}. Accordingly, let us transform the coordinates in the space of normal stresses Σ_x, Σ_y, and Σ_z, namely: turn the coordinate system, translate its origin and multiply by a positive normalization factor; then, we write Eq. (3) in the canonical form. If conditions (33) are met, we may arrive at only three forms [9, 10]:

1) six-dimensional ellipsoid:

$$\frac{X_e^2}{C_{xe}^2}+\frac{Y_e^2}{C_{ye}^2}+\frac{Z_e^2}{C_{ze}^2}+\frac{\Theta_{xy}^2+\Theta_{yz}^2+\Theta_{zx}^2}{D_e^2}=1; \tag{34}$$

2) six-dimensional elliptic paraboloid:

$$\frac{X_p^2}{C_{xp}^2}+\frac{Y_p^2}{C_{yp}^2}+\frac{\Theta_{xy}^2+\Theta_{yz}^2+\Theta_{zx}^2}{D_p^2}=2Z_p; \tag{35}$$

3) six-dimensional elliptic cylinder:

$$\frac{X_c^2}{C_{xc}^2}+\frac{Y_c^2}{C_{yc}^2}+\frac{\Theta_{xy}^2+\Theta_{yz}^2+\Theta_{zx}^2}{D_c^2}=1. \tag{36}$$

All the three types [(34)–(36)] of the hypersurface of strength are physically allowable. Therefore, the conclusion made for the case of no tangential stresses is also valid for the general case of stress state.

Sections 1 and 2 have covered all the possible stress state cases for a composite material. Let us state the final outcome.

Inequality expressions (33) represent the necessary and sufficient stability conditions which should be satisfied by the coefficients of the quadratic form (3) or (1) so that the limiting six-dimensional hypersurface of strength for an orthotropic material has a physical meaning. If in (33) the first unstrict inequality is replaced with a strict one we will have the necessary and sufficient conditions for the limiting surface $\Phi=1$ to be a six-dimensional ellipsoid. If the equality expression in the left-hand member of (33) is satisfied, we will have elliptic paraboloids or cylinders depending on the coefficients B_i or b_i of linear terms in (3) or (1), respectively. Requirements (33) are independent of B_i (and, hence, of b_i either); therefore, if they are met we will arrive at physically allowable hypersurfaces of strength for any values of the coefficients of linear terms in (3) or (1).

In conclusion, note that conditions (33) are essentially stronger than the conventionally used stability conditions (7) or, when enhanced, (11). For instance, the case

$$A_{xy}=A_{yz}=A_{zx}=-2/3 \tag{37}$$

satisfies both (7) and (11), but the requirements (33) are not met in this case. Looking at the situation (5) and analyzing the signs of the first three invariants in (8), it can be easily demonstrated [9, 10] that equality expressions (37) will lead to physically absurd surfaces of strength – hyperboloids or cones – which allow for the existence of an unlimited set (sheaf) of lines with infinite in the three-dimensional space of normal stresses. This pertains to the six-dimensional limiting hypersurface because (5) represents a particular case (a null measure subset) of six-dimensional space (set) of normal and tangential stresses. The latter statement can be easily proved in a way similar to that in Section 2.

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