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METHODS AND FINDINGS OF STRESS-STRAIN STATE AND STRENGTH ANALYSES OF MULTILAYER THICK-WALLED ANISOTROPIC CYLINDERS UNDER DYNAMIC LOADING (REVIEW). PART 3. PHENOMENOLOGICAL STRENGTH CRITERIA

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The paper addresses the well-known (from available publications) phenomenological strength criteria for anisotropic materials, which are most widely used in application software.

Keywords: phenomenological strength criteria, anisotropy, composite material, limiting fracture surface.

Parts 1 and 2 provided a review of the known (from available publications) methods and findings of theoretical and experimental studies of stress-strain state (SSS) and strength of multilayer thick-walled anisotropic cylinders under dynamic loading.

Here we will review some of the most widely used (in application software) phenomenological strength criteria for elastic (till fracture) composite materials (CM).

To assess CM strength one should know the criteria specifying the allowable stress and strain limits within which the material will be able to work without failure under preset conditions. They are called the limit state criteria. Limit states are the states whereby CM undergoes an elastic–plastic transition or fracture. In such cases, the limit state criterion is usually referred to as the yield criterion and the strength criterion, respectively.

The CM strength can be determined using two main approaches [1]: a structural approach that has a physical basis and allows for the heterogeneous microstructure of composites and a phenomenological one, i.e., macro-structural, that involves generalized failure criteria.

The majority of currently available failure theories are structural (micromechanical) [1]. The elaboration and application of micromechanical strength models present significant difficulties [1]. Furthermore, the structural approaches, unlike phenomenological ones, do not always provide more accurate data as mentioned in [1, 2].

Nowadays, the strength analysis of composite materials under complex stress conditions is most often performed through phenomenological approaches [1]. In a macroscopic approach a heterogeneous reinforced composite is considered as a continuum with a certain symmetry of properties, whose mathematical model is constructed on the basis of experimental data, with no explanation of the mechanisms that dictate the CM behavior.

The phenomenological limit-state criteria describe, to a given confidence level, the macromechanical behavior of CM as a whole, disregarding the micromechanical features that arise during the CM deformation.

The phenomenological approach makes it possible to apply a general strength criterion to the materials that differ in composition and production process but have the same symmetry of properties as well as to the materials with considerable anisotropy where the same stress state can result in physically different limit states if the stress signs or orientation are changed.

If a material is not subjected to any thermal, temporal, chemical, or radiation actions, its fracture is usually related to a limit state of the body; then, the phenomenological strength criterion can be written as [2, 3]

$$f(\sigma_{ij}, F) \le 0, \tag{1}$$

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where σ_{ij} are the stresses and F represents some strength characteristics of the material. Strength is affected when condition (1) is not obeyed. In the case of strict equality in (1) we arrive at a strength surface equation.

For an anisotropic material the quantity F in Eq. (1) shall be a combination of many material parameters which are invariant with respect to any transformation of the coordinate system. For isotropic bodies the material parameter F is a scalar function of three strength characteristics of the material, which in some particular cases can degenerate into a function of two strength parameters or into a scalar constant.

Phenomenological strength criteria are not derived analytically, they are postulated or proposed on the basis of generalization of experimental data. The relative freedom in the statement of strength criteria has resulted in numerous attempts to develop such criteria [1–15, et al.]. The decision on which criterion should be chosen depends on the nature of material, its composition, degree of anisotropy, the calculation concept taken, and available body of experimental data.

One of the most general statements of the strength criterion for anisotropic bodies in a three-dimensional case is written using a matrix notation as follows [11]:

$$(F_i\sigma_i)^{\alpha} + (F_{ij}\sigma_i\sigma_j)^{\beta} + (F_{ijk}\sigma_i\sigma_j\sigma_k)^{\gamma} + \ldots \le 1, \qquad i, j, k, \ldots = 1, 2, \ldots, 6, \tag{2}$$

where σ_i and F_i, F_{ij}, F_{ijk}, ... are the matrix notation of the stress tensor and so-called strength surface tensors (strength tensors) of the second, fourth, sixth, and further even ranks; exponents α, β, γ, etc., are determined based on the best fit to the experimental data. Confining themselves to only two members of relation (2), Gol'denblat and Kopnov [11] proposed the most fully elaborated variant of criterion (2), where $\alpha = 1$ and $\beta = 1/2$.

A variant with exponents in (2) taken to be equal to unity ($\alpha = \beta = \gamma = \ldots = 1$) [2, 12, 15] turned out to be more convenient for practical use. Hence,

$$F_i\sigma_i + F_{ij}\sigma_i\sigma_j + F_{ijk}\sigma_i\sigma_j\sigma_k + \ldots \le 1. \tag{3}$$

The expression (3) was called the tensor-polynomial strength criterion [2, 12, 13].

In applications, the fourth and further summands in expression (3) are usually disregarded because of a large number of material constants required [4, 5, 15, et al.]. Then, the polynomial criterion (3) is brought to Tsai–Wu quadratic criterion or just Tsai–Wu criterion of the form [15]

$$F_i\sigma_i + F_{ij}\sigma_i\sigma_j \le 1. \tag{4}$$

According to the data [12, 15], in the case of strict equality the inequality expression (4) can describe the failure surface (the necessary condition for the existence of this surface is met) if the mixed components F_{ij} $(i \ne j)$ obey the following three necessary stability conditions:

$$F_{ii}F_{jj} - F_{ij}^2 \ge 0 \qquad \text{(no summing up for } i \text{ and } j\text{)}. \tag{5}$$

Generally, the Tsai–Wu criterion allows for the difference between ultimate tensile and compression strengths, the dependence of strength on the direction of tangential stresses in shear, provides the maximum possible flexibility, contains no redundant parameters, permits an easy definition of the principal axes of strength, etc. [12].

If we assume that there exist some failure potential, i.e., that failure is independent of the loading path, the tensor F_{ij} will be symmetric tensor [12]. The tensor F_i will be symmetric too, which follows from the symmetric property of the stress tensor.

Based on the above assumptions, Eq. (4) is written in the expanded form as follows [15]:

$$F_1\sigma_1 + F_2\sigma_2 + F_3\sigma_3 + F_4\sigma_4 + F_5\sigma_5 + F_6\sigma_6$$

$$+ F_{11}\sigma_1^2 + 2F_{12}\sigma_1\sigma_2 + 2F_{13}\sigma_1\sigma_3 + 2F_{14}\sigma_1\sigma_4 + 2F_{15}\sigma_1\sigma_5 + 2F_{16}\sigma_1\sigma_6$$

$$+F_{22}\sigma_2^2+2F_{23}\sigma_2\sigma_3+2F_{24}\sigma_2\sigma_4+2F_{25}\sigma_2\sigma_5+2F_{26}\sigma_2\sigma_6$$

$$+F_{33}\sigma_3^2+2F_{34}\sigma_3\sigma_4+2F_{35}\sigma_3\sigma_5+2F_{36}\sigma_3\sigma_6$$

$$+F_{44}\sigma_4^2+2F_{45}\sigma_4\sigma_5+2F_{46}\sigma_4\sigma_6$$

$$+F_{55}\sigma_5^2+2F_{56}\sigma_5\sigma_6$$

$$+F_{66}\sigma_6^2\le 1. \tag{6}$$

Strength tensor components should be determined from experiment. For particularization of (6), it would take 27 and 17 tests for a material with the general form of anisotropy of properties and a monoclinic material, respectively [3]. This suggests that for such materials it is very difficult to use criterion (6) in applications. It becomes practically usable only for orthotropic or transversely isotropic materials.

According to the data [3], for an orthotropic material Eq. (6), if expressed in the principal axes of orthotropy, takes the form

$$F_1\sigma_1+F_2\sigma_2+F_3\sigma_3+F_{11}\sigma_1^2+F_{22}\sigma_2^2+F_{33}\sigma_3^2+F_{44}\sigma_4^2$$

$$+F_{55}\sigma_5^2+F_{66}\sigma_6^2+2F_{12}\sigma_1\sigma_2+2F_{13}\sigma_1\sigma_3+2F_{23}\sigma_2\sigma_3\le 1. \tag{7}$$

When writing (7) we assume the shear strength to be independent of the direction of tangential stresses, whence it follows that there is no mutual influence between normal and tangential stresses or between different tangential stresses.

In expression (7), the unbound components of strength tensors are denoted by F_1, F_2, F_3, F_{11}, F_{22}, F_{33}, F_{44}, F_{55}, and F_{66}. The values of these components are uniquely determined from uniaxial tension (compression) tests and simple shear tests [3].

The bound (mixed) components of the strength tensor are denoted by F_{12}, F_{13}, and F_{23}. In the case of using the Tsai–Wu criterion the main difficulty is to experimentally find the mixed components of the strength tensor. There are numerous approaches to their determination [3, 5, 7, 12, 15–19, et al.]. For this purpose, one has to perform tests under biaxial stress conditions or to use complex-shaped specimens, which leads to considerable costs [3]. The value of F_{ij} greatly depends on the scatter of experimental data. On the other hand, its slight variations have a noticeable effect on the form of the strength surface [15]. Therefore, much attention is given to the development of an optimal (in terms of selecting a ratio between stresses σ_i and σ_j) experiment to allow a more accurate determination of F_{ij} [17]. Since the optimal stress ratios are, in turn, dependent on the F_{ij} value, the determination of F_{ij} is an iterative process.

In [18], an expression that follows from the stability condition for the failure criterion (5) was proposed, as a first approximation, in the form

$$F_{ij}=c(F_{ii}F_{jj})^{1/2}, \tag{8}$$

where $|c|\le 1$.

In this case, for the determination of F_{ij} there is no need for any tests under biaxial stress conditions, and thus the use of this criterion in applications becomes much simpler. Recently, expression (8) with $c=-1/2$ has been widely employed in modeling for failure of composites [19–24, et al.], the criterion for isotropic materials coinciding with the Mises criterion. Like in [25], the criterion (7), (8) with $c=-1/2$ will be called the generalized Mises criterion.

Based on the data [3, 16], the strength tensor components for the generalized Mises criterion can be determined as follows:

$$\begin{cases} F_1 = \frac{1}{X_t} - \frac{1}{X_c}, \quad F_2 = \frac{1}{Y_t} - \frac{1}{Y_c}, \quad F_3 = \frac{1}{Z_t} - \frac{1}{Z_c}, \\ F_{11} = \frac{1}{X_t X_c}, \quad F_{22} = \frac{1}{Y_t Y_c}, \quad F_{33} = \frac{1}{Z_t Z_c}, \quad F_{44} = \frac{1}{R^2}, \quad F_{55} = \frac{1}{S^2}, \quad F_{66} = \frac{1}{T^2}, \\ F_{12} = -\frac{1}{2\sqrt{X_t X_c Y_t Y_c}}, \quad F_{13} = -\frac{1}{2\sqrt{X_t X_c Z_t Z_c}}, \quad F_{23} = -\frac{1}{2\sqrt{Y_t Y_c Z_t Z_c}}. \end{cases} \tag{9}$$

Hereinafter, X_t, Y_t, and Z_t are the ultimate tensile strength values along the principal directions 1, 2, and 3, respectively, X_c, Y_c, and Z_c are the ultimate compression strength values along the principal directions 1, 2, and 3, respectively, and R, S, and T are the ultimate shear strength values in the principal planes 23, 13, and 12, respectively.

For particularization of the generalized Mises criterion we need to determine 12 coefficients from nine tests: in tension and compression along three principal direction of orthotropy (six tests) and in shear along three principal planes (three tests).

According to [3], for a transversely isotropic material with an isotropy plane 23 we can derive the following expression from Eq. (7):

$$F_1\sigma_1 + F_2(\sigma_2 + \sigma_3) + F_{11}\sigma_1^2 + F_{22}(\sigma_2^2 + \sigma_3^2) + 2(F_{22} - F_{23})\sigma_4^2$$

$$+ F_{66}(\sigma_5^2 + \sigma_6^2) + 2F_{12}(\sigma_1\sigma_2 + \sigma_1\sigma_3) + 2F_{23}\sigma_2\sigma_3 \le 1. \tag{10}$$

The respective seven strength tensor components in Eq. (10) are represented, in view of the data [3], as follows:

$$F_1 = \frac{1}{X_t} - \frac{1}{X_c}, \quad F_2 = \frac{1}{Y_t} - \frac{1}{Y_c}, \quad F_{11} = \frac{1}{X_t X_c}, \quad F_{22} = \frac{1}{Y_t Y_c}, \quad F_{66} = \frac{1}{T^2},$$
$$F_{12} = -\frac{1}{2\sqrt{X_t X_c Y_t Y_c}}, \quad F_{23} = -\frac{1}{2 Y_t Y_c}. \tag{11}$$

For their particularization one has to perform five tests in tension–compression along the directions 1 and 2 and in simple shear in the plane 12. Note that in the case of using the quadratic strength criterion for any transtropic body with an isotropy plane 23 the following equality should be obeyed:

$$F_{44} = F_{22} + F_{33} - 2F_{23}. \tag{12}$$

According to [19], the generalized Mises criterion is in good agreement with experimental data for materials with a high degree of anisotropy.

The issues regarding the mathematically rigorous substantiation of the necessary and sufficient conditions for stability, bulging, simple connectedness, closedness or openness, possible and allowable geometrical shapes of the six-dimensional limiting surface of strength (7) have not been adequately explored so far. In [26] for a general triaxial SSS case we obtained the necessary and sufficient conditions whereby the limiting surface of strength (7) would have a physical meaning: it would be a simply connected or closed surface or would allow existence of maximum one direction (a ray or a straight line) with infinite strength. These conditions are written as

$$A_{12}^2 + A_{23}^2 + A_{31}^2 \le 1 + 2A_{12}A_{23}A_{31} < 3, \tag{13}$$

where $A_{ij} = F_{ij} \big/ \sqrt{F_{ii}F_{jj}}$, if in (13) the first unstrict inequality is replaced with a strict one we will arrive at the necessary and sufficient conditions for the limiting surface to be an ellipsoid. If the equality expression in the left-hand member of (13) is obeyed, we will have elliptic paraboloids or cylinders depending on the coefficients F_i of linear terms in (7). Requirements (13) impose more tight restrictions on the constants F_{ij} than the conventionally used stability conditions (5) do. Thus, the fulfillment of only conditions (5) does not always guarantee the existence of a physically allowable surface of strength.

Also, some other quadratic strength criteria were put forward [27–31, et al.]. As mentioned in [3], the percentage of use of the respective phenomenological criteria is as follows: the maximum strain criterion – 30%, the maximum stress criterion – 22%, the Tsai–Hill criterion – 17%, the Tsai–Wu criterion – 12%, others – 19%. Noteworthy is that the Tsai–Wu criterion provides a more accurate description of failure of composite materials in comparison to the maximum strain criterion, the maximum stress criterion, and the Tsai–Hill criterion [7].

Based on the data [16], we will demonstrate that some strength criteria are particular cases of the polynomial strength criterion (3).

Maximum Stress Criterion. This criterion implies that failure happens if any of the following conditions is not obeyed [16]:

$$-X_c \le \sigma_1 \le X_t, \quad -Y_c \le \sigma_2 \le Y_t, \quad -Z_c \le \sigma_3 \le Z_t, \quad |\sigma_4| \le R, \quad |\sigma_5| \le S, \quad |\sigma_6| \le T. \tag{14}$$

The maximum stress criterion (MSC) can be expressed in terms of the tensor-polynomial criterion (3) as follows [16]:

$$(\sigma_1 - X_t)(\sigma_1 + X_c)(\sigma_2 - Y_t)(\sigma_2 + Y_c)(\sigma_3 - Z_t)(\sigma_3 + Z_c)$$

$$\times (\sigma_4 - R)(\sigma_4 + R)(\sigma_5 - S)(\sigma_5 + S)(\sigma_6 - T)(\sigma_6 + T) \le 0. \tag{15}$$

By comparing Eq. (3) with (15) and ignoring the terms above second order, we arrive at an approximate maximum-stress criterion in the quadratic form for which the nonzero components of the strength tensor are given by

$$\begin{cases} F_1 = \dfrac{1}{X_t} - \dfrac{1}{X_c}, \quad F_2 = \dfrac{1}{Y_t} - \dfrac{1}{Y_c}, \quad F_3 = \dfrac{1}{Z_t} - \dfrac{1}{Z_c}, \\ F_{11} = \dfrac{1}{X_t X_c}, \quad F_{22} = \dfrac{1}{Y_t Y_c}, \quad F_{33} = \dfrac{1}{Z_t Z_c}, \quad F_{44} = \dfrac{1}{R^2}, \quad F_{55} = \dfrac{1}{S^2}, \quad F_{66} = \dfrac{1}{T^2}, \\ F_{12} = -\dfrac{F_1 F_2}{2}, \quad F_{13} = -\dfrac{F_1 F_3}{2}, \quad F_{23} = -\dfrac{F_2 F_3}{2}. \end{cases} \tag{16}$$

The other strength constants are zero.

In view of the data [3], for a transversely isotropic material with an isotropy plane 23 we have

$$F_1 = \frac{1}{X_t} - \frac{1}{X_c}, \quad F_2 = F_3 = \frac{1}{Y_t} - \frac{1}{Y_c}, \quad F_{11} = \frac{1}{X_t X_c}, \quad F_{22} = F_{33} = \frac{1}{Y_t Y_c}, \quad F_{55} = F_{66} = \frac{1}{T^2},$$
$$F_{12} = F_{13} = -\frac{F_1 F_2}{2}, \quad F_{23} = -\frac{F_2^2}{2}. \tag{17}$$

Maximum Strain Criterion. According to the maximum strain criterion (CMS), failure is assumed to occur if any of the following conditions is not obeyed [16]:

$$-X_{\varepsilon c} \le \varepsilon_1 \le X_{\varepsilon t}, \quad -Y_{\varepsilon c} \le \varepsilon_2 \le Y_{\varepsilon t}, \quad -Z_{\varepsilon c} \le \varepsilon_3 \le Z_{\varepsilon t}, \quad |\varepsilon_4| \le R_\varepsilon, \quad |\varepsilon_5| \le S_\varepsilon, \quad |\varepsilon_6| \le T_\varepsilon, \tag{18}$$

where ε_1, ε_2, and ε_3 are tensile strains along the directions 1, 2, 3, respectively, ε_4, ε_5, and ε_6 are the shear strains in the planes 23, 13, 12, respectively, $X_{\varepsilon t}$, $Y_{\varepsilon t}$, and $Z_{\varepsilon t}$ are the ultimate tensile strains along the principal directions 1, 2, 3, respectively, $X_{\varepsilon c}$, $Y_{\varepsilon c}$, and $Z_{\varepsilon c}$ are the ultimate compression strains along the principal directions 1, 2, 3, respectively, and R_{ε}, S_{ε}, and T_{ε} are the ultimate shear strains (strength) in the principal planes 23, 13, 12, respectively.

With the criterion expressed in the tensor-polynomial form, we have [16]

$$(\varepsilon_1 - X_{\varepsilon t})(\varepsilon_1 + X_{\varepsilon c})(\varepsilon_2 - Y_{\varepsilon t})(\varepsilon_2 + Y_{\varepsilon c})(\varepsilon_3 - Z_{\varepsilon t})(\varepsilon_3 + Z_{\varepsilon c})$$

$$\times(\varepsilon_4 - R_{\varepsilon})(\varepsilon_4 + R_{\varepsilon})(\varepsilon_5 - S_{\varepsilon})(\varepsilon_5 + S_{\varepsilon})(\varepsilon_6 - T_{\varepsilon})(\varepsilon_6 + T_{\varepsilon}) \le 0. \tag{19}$$

Based on the use of the flexibility matrix and constitutive expressions of the elasticity theory for an anisotropic body, with terms above second order being disregarded, Eq. (19) can be approximately written in the quadratic form for which the nonzero components of the strength tensors are given by [16]

$$\begin{cases}
F_1 = F_1^A + \dfrac{S_{12}}{S_{22}} F_2^A + \dfrac{S_{13}}{S_{33}} F_3^A, \quad F_2 = \dfrac{S_{12}}{S_{11}} F_1^A + F_2^A + \dfrac{S_{23}}{S_{33}} F_3^A, \\
F_3 = \dfrac{S_{13}}{S_{11}} F_1^A + \dfrac{S_{23}}{S_{22}} F_2^A + F_3^A, \\
F_{11} = \dfrac{1}{X_t X_c} + \left(\dfrac{S_{12}}{S_{22}}\right)^2 \dfrac{1}{Y_t Y_c} + \left(\dfrac{S_{13}}{S_{33}}\right)^2 \dfrac{1}{Z_t Z_c} - \dfrac{S_{13}}{S_{33}} F_1^A F_3^A - \dfrac{S_{12}}{S_{22}} F_1^A F_2^A - \dfrac{S_{12} S_{13}}{S_{22} S_{23}} F_2^A F_3^A, \\
F_{22} = \dfrac{1}{Y_t Y_c} + \left(\dfrac{S_{12}}{S_{11}}\right)^2 \dfrac{1}{X_t X_c} + \left(\dfrac{S_{23}}{S_{33}}\right)^2 \dfrac{1}{Z_t Z_c} - \dfrac{S_{12}}{S_{11}} F_1^A F_2^A - \dfrac{S_{23}}{S_{33}} F_2^A F_3^A - \dfrac{S_{12} S_{23}}{S_{11} S_{33}} F_1^A F_3^A, \\
F_{33} = \dfrac{1}{Z_t Z_c} + \left(\dfrac{S_{13}}{S_{11}}\right)^2 \dfrac{1}{X_t X_c} + \left(\dfrac{S_{23}}{S_{22}}\right)^2 \dfrac{1}{Y_t Y_c} - \dfrac{S_{13}}{S_{11}} F_1^A F_3^A - \dfrac{S_{23}}{S_{22}} F_2^A F_3^A - \dfrac{S_{13} S_{23}}{S_{11} S_{22}} F_1^A F_2^A, \\
F_{44} = \dfrac{1}{R^2}, \quad F_{55} = \dfrac{1}{S^2}, \quad F_{66} = \dfrac{1}{T^2}, \\
F_{12} = \dfrac{S_{12}}{S_{11}} \dfrac{1}{X_t X_c} + \dfrac{S_{12}}{S_{22}} \dfrac{1}{Y_t Y_c} + \dfrac{S_{13} S_{23}}{S_{33}^2} \dfrac{1}{Z_t Z_c} - \dfrac{1}{2}\left(\dfrac{S_{12}^2}{S_{11} S_{22}} + 1\right) F_1^A F_2^A \\
\quad - \dfrac{1}{2}\left(\dfrac{S_{13} S_{12}}{S_{11} S_{33}} + \dfrac{S_{23}}{S_{33}}\right) F_1^A F_3^A - \dfrac{1}{2}\left(\dfrac{S_{12} S_{23}}{S_{22} S_{33}} + \dfrac{S_{13}}{S_{33}}\right) F_2^A F_3^A, \\
F_{13} = \dfrac{S_{13}}{S_{11}} \dfrac{1}{X_t X_c} + \dfrac{S_{13}}{S_{33}} \dfrac{1}{Z_t Z_c} + \dfrac{S_{12} S_{23}}{S_{22}^2} \dfrac{1}{Y_t Y_c} - \dfrac{1}{2}\left(\dfrac{S_{13}^2}{S_{11} S_{33}} + 1\right) F_1^A F_2^A \\
\quad - \dfrac{1}{2}\left(\dfrac{S_{12} S_{13}}{S_{11} S_{22}} + \dfrac{S_{23}}{S_{22}}\right) F_1^A F_2^A - \dfrac{1}{2}\left(\dfrac{S_{13} S_{23}}{S_{22} S_{33}} + \dfrac{S_{12}}{S_{22}}\right) F_2^A F_3^A, \\
F_{23} = \dfrac{S_{23}}{S_{22}} \dfrac{1}{Y_t Y_c} + \dfrac{S_{23}}{S_{33}} \dfrac{1}{Z_t Z_c} + \dfrac{S_{12} S_{13}}{S_{11}^2} \dfrac{1}{X_t X_c} - \dfrac{1}{2}\left(\dfrac{S_{23}^2}{S_{22} S_{33}} + 1\right) F_1^A F_2^A \\
\quad - \dfrac{1}{2}\left(\dfrac{S_{12} S_{23}}{S_{11} S_{22}} + \dfrac{S_{13}}{S_{11}}\right) F_1^A F_2^A - \dfrac{1}{2}\left(\dfrac{S_{23} S_{13}}{S_{11} S_{33}} + \dfrac{S_{12}}{S_{11}}\right) F_2^A F_3^A,
\end{cases} \tag{20}$$

where S_{ij} are the flexibility matrix components, F_1^A, F_2^A, and F_3^A are the expressions provided for F_1, F_2, and F_3 in the maximum stress criterion. The other strength tensor components are zero.

Based on the data [3], for a transversely isotropic material with an isotropy plane 23 we have

$$
\begin{cases}
F_1 = F_1^A + \left(\dfrac{S_{12}}{S_{22}} + \dfrac{S_{13}}{S_{33}}\right) F_2^A, \quad F_2 = F_3 = \dfrac{S_{12}}{S_{11}} F_1^A + \left(1 + \dfrac{S_{23}}{S_{33}}\right) F_2^A, \\
F_{11} = \dfrac{1}{X_t X_c} + \left(\left(\dfrac{S_{12}}{S_{22}}\right)^2 + \left(\dfrac{S_{13}}{S^{33}}\right)^2\right)\dfrac{1}{Y_t Y_c} - \left(\dfrac{S_{13}}{S_{33}} + \dfrac{S_{12}}{S_{22}}\right) F_1^A F_2^A - \dfrac{S_{12} S_{13}}{S_{22} S_{23}} (F_2^A)^2, \\
F_{22} = F_{33} = \left(1 + \left(\dfrac{S_{23}}{S_{33}}\right)^2\right)\dfrac{1}{Y_t Y_c} + \left(\dfrac{S_{12}}{S_{11}}\right)^2 \dfrac{1}{X_t X_c} - \left(\dfrac{S_{12}}{S_{11}} + \dfrac{S_{12} S_{23}}{S_{11} S_{33}}\right) F_1^A F_2^A - \dfrac{S_{23}}{S_{33}} (F_2^A)^2, \\
F_{55} = F_{66} = \dfrac{1}{T^2}, \\
F_{12} = F_{13} = \dfrac{S_{12}}{S_{11}} \dfrac{1}{X_t X_c} + \left(\dfrac{S_{12}}{S_{22}} + \dfrac{S_{13} S_{23}}{S_{33}^2}\right)\dfrac{1}{Y_t Y_c} \\
\quad - \dfrac{1}{2}\left(1 + \dfrac{S_{12}}{S_{11}}\left(\dfrac{S_{12}}{S_{22}} + \dfrac{S_{13}}{S_{33}}\right) + \dfrac{S_{23}}{S_{33}}\right) F_1^A F_2^A - \dfrac{1}{2}\left(\dfrac{S_{12} S_{23}}{S_{22} S_{33}} + \dfrac{S_{13}}{S_{33}}\right)(F_2^A)^2, \\
F_{23} = S_{23}\left(\dfrac{1}{S_{22}} + \dfrac{1}{S_{33}}\right)\dfrac{1}{Y_t Y_c} + \dfrac{S_{12} S_{13}}{S_{11}^2} \dfrac{1}{X_t X_c} \\
\quad - \dfrac{1}{2}\left(1 + \dfrac{S_{23}}{S_{22}}\left(\dfrac{S_{23}}{S_{33}} + \dfrac{S_{12}}{S_{11}}\right) + \dfrac{S_{13}}{S_{11}}\right) F_1^A F_2^A - \dfrac{1}{2}\left(\dfrac{S_{23} S_{13}}{S_{11} S_{33}} + \dfrac{S_{12}}{S_{11}}\right)(F_2^A)^2.
\end{cases}
\tag{21}
$$

The Tsai–Hill Criterion (the Modified Hill Criterion [5]). According to [16], the Tsai–Hill criterion can be written as

$$
\left(\frac{\sigma_1}{X_i}\right)^2 + \left(\frac{\sigma_2}{Y_i}\right)^2 + \left(\frac{\sigma_3}{Z_i}\right)^2 - \left(\frac{1}{X_i^2} + \frac{1}{Y_i^2} - \frac{1}{Z_i^2}\right)\sigma_1\sigma_2 - \left(\frac{1}{Z_i^2} + \frac{1}{X_i^2} - \frac{1}{Y_i^2}\right)\sigma_1\sigma_3
$$

$$
- \left(\frac{1}{Y_i^2} + \frac{1}{Z_i^2} - \frac{1}{X_i^2}\right)\sigma_2\sigma_3 + \left(\frac{\sigma_4}{R}\right)^2 + \left(\frac{\sigma_5}{S}\right)^2 + \left(\frac{\sigma_6}{T}\right)^2 \le 1. \tag{22}
$$

Note that in criterion (22) there no first-order stress components and thus F_1, F_2, and F_3 are zero. The quantities X_i, Y_i, and Z_i become X_t, Y_t, and Z_t or X_c, Y_c, and Z_c depending on the respective sign of normal stresses σ_1, σ_2, and σ_3, respectively.

For this criterion the strength tensor components are as follows:

$$
\begin{cases}
F_i = 0, \quad F_{11} = \dfrac{1}{X_i^2}, \quad F_{22} = \dfrac{1}{Y_i^2}, \quad F_{33} = \dfrac{1}{Z_i^2}, \quad F_{44} = \dfrac{1}{R^2}, \quad F_{55} = \dfrac{1}{S^2}, \quad F_{66} = \dfrac{1}{T^2}, \\
F_{12} = -\dfrac{1}{2}\left(\dfrac{1}{X_i^2} + \dfrac{1}{Y_i^2} - \dfrac{1}{Z_i^2}\right), \quad F_{13} = -\dfrac{1}{2}\left(\dfrac{1}{Z_i^2} + \dfrac{1}{X_i^2} - \dfrac{1}{Y_i^2}\right), \quad F_{23} = -\dfrac{1}{2}\left(\dfrac{1}{Y_i^2} + \dfrac{1}{Z_i^2} - \dfrac{1}{X_i^2}\right).
\end{cases}
\tag{23}
$$

For a transversely isotropic material with an isotropy plane 23 we should also use formulas (23) because F_{ii} and F_{ij} depend not only on the respective ultimate strengths but also on the sign of normal stresses.

The Hoffman Criterion. This criterion can be written as [16, 28]:

$$\frac{1}{2}\left(\frac{1}{Y_cY_t}+\frac{1}{Z_cZ_t}-\frac{1}{X_cX_t}\right)(\sigma_2-\sigma_3)^2+\frac{1}{2}\left(\frac{1}{Z_cZ_t}+\frac{1}{X_cX_t}-\frac{1}{Y_cY_t}\right)(\sigma_3-\sigma_1)^2$$

$$+\frac{1}{2}\left(\frac{1}{X_cX_t}+\frac{1}{Y_cY_t}-\frac{1}{Z_cZ_t}\right)(\sigma_1-\sigma_2)^2+\left(\frac{1}{X_t}-\frac{1}{X_c}\right)\sigma_1+\left(\frac{1}{Y_t}-\frac{1}{Y_c}\right)\sigma_2$$

$$+\left(\frac{1}{Z_t}-\frac{1}{Z_c}\right)\sigma_3+\left(\frac{\sigma_4}{R}\right)^2+\left(\frac{\sigma_5}{S}\right)^2+\left(\frac{\sigma_6}{T}\right)^2\le 1. \quad (24)$$

It is a particular case of the tensor-polynomial criterion with the following selected parameters F_i and F_{ij}:

$$\begin{cases} F_1=\frac{1}{X_t}-\frac{1}{X_c}, \quad F_2=\frac{1}{Y_t}-\frac{1}{Y_c}, \quad F_3=\frac{1}{Z_t}-\frac{1}{Z_c}, \\ F_{11}=\frac{1}{X_tX_c}, \quad F_{22}=\frac{1}{Y_tY_c}, \quad F_{33}=\frac{1}{Z_tZ_c}, \quad F_{44}=\frac{1}{R^2}, \quad F_{55}=\frac{1}{S^2}, \quad F_{66}=\frac{1}{T^2}, \\ F_{12}=-\frac{1}{2}\left(\frac{1}{X_cX_t}+\frac{1}{Y_cY_t}-\frac{1}{Z_cZ_t}\right), \quad F_{13}=-\frac{1}{2}\left(\frac{1}{Z_cZ_t}+\frac{1}{X_cX_t}-\frac{1}{Y_cY_t}\right), \\ F_{23}=-\frac{1}{2}\left(\frac{1}{Y_cY_t}+\frac{1}{Z_cZ_t}-\frac{1}{X_cX_t}\right). \end{cases} \quad (25)$$

The other strength tensor components are zero.

In view of the data [3], for a transversely isotropic material with an isotropy plane 23 we have

$$\begin{cases} F_1=\frac{1}{X_t}-\frac{1}{X_c}, \quad F_2=F_3=\frac{1}{Y_t}-\frac{1}{Y_c}, \\ F_{11}=\frac{1}{X_tX_c}, \quad F_{22}=F_{33}=\frac{1}{Y_tY_c}, \quad F_{55}=F_{66}=\frac{1}{T^2}, \\ F_{12}=F_{13}=-\frac{1}{2}\left(\frac{1}{X_cX_t}\right), \quad F_{23}=-\frac{1}{2}\left(\frac{2}{Y_cY_t}-\frac{1}{X_cX_t}\right). \end{cases} \quad (26)$$

The above-mentioned components of strength tensors for quadratic polynomial failure criteria are given in Table 1 for orthotropic materials and in Table 2 for transversely isotropic ones.

Noteworthy is that in the case of using exact maximum-stress or maximum-strain criteria [in the form of (14), (15) or (18), (19), respectively] for transversely isotropic materials one should carry out six tests instead of five, because the ultimate shear in the plane of isotropy is not related by an expression of the form (12) to the other strength parameters and is generally an independent quantity.

Analysis has demonstrated that all the quadratic criteria (4) have one common drawback: the complete absence of stresses is not the optimum (global minimum) of the strength function, except for the degenerate case where all linear terms (F_i) are zero.

The modified Hill criterion (22), (23) is free from such drawback by definition but has other serious shortcomings:

(i) the strength function experiences discontinuities when passing through the planes $\sigma_i=0$, the resultant limiting surface of strength turns out to be "broken" along these planes;

TABLE 1. Quadratic Polynomial Failure Criteria for an Orthotropic Material

F_i, F_{ij}	Generalized Mises criterion	Tsai–Hill criterion*	Hoffman criterion	MSC (in polynomial form)	CMS (in polynomial form)
F_1	$\frac{1}{X_t}-\frac{1}{X_c}$	0	$\frac{1}{X_t}-\frac{1}{X_c}$	$\frac{1}{X_t}-\frac{1}{X_c}$	$F_1^A+\frac{S_{12}}{S_{22}}F_2^A+\frac{S_{13}}{S_{33}}F_3^A$
F_2	$\frac{1}{Y_t}-\frac{1}{Y_c}$	0	$\frac{1}{Y_t}-\frac{1}{Y_c}$	$\frac{1}{Y_t}-\frac{1}{Y_c}$	$\frac{S_{12}}{S_{11}}F_1^A+F_2^A+\frac{S_{23}}{S_{33}}F_3^A$
F_3	$\frac{1}{Z_t}-\frac{1}{Z_c}$	0	$\frac{1}{Z_t}-\frac{1}{Z_c}$	$\frac{1}{Z_t}-\frac{1}{Z_c}$	$\frac{S_{13}}{S_{11}}F_1^A+\frac{S_{23}}{S_{22}}F_2^A+F_3^A$
F_{12}	$-\frac{1}{2\sqrt{X_tX_cY_tY_c}}$	$-\frac{1}{2}\left(\frac{1}{X_i^2}+\frac{1}{Y_i^2}-\frac{1}{Z_i^2}\right)$	$-\frac{1}{2}\left(\frac{1}{X_tX_c}+\frac{1}{Y_tY_c}-\frac{1}{Z_tZ_c}\right)$	$-\frac{F_1F_2}{2}$	**
F_{13}	$-\frac{1}{2\sqrt{X_tX_cZ_tZ_c}}$	$-\frac{1}{2}\left(\frac{1}{Z_i^2}+\frac{1}{X_i^2}-\frac{1}{Y_i^2}\right)$	$-\frac{1}{2}\left(\frac{1}{Z_tZ_c}+\frac{1}{X_tX_c}-\frac{1}{Y_tY_c}\right)$	$-\frac{F_1F_3}{2}$	**
F_{23}	$-\frac{1}{2\sqrt{Y_tY_cZ_tZ_c}}$	$-\frac{1}{2}\left(\frac{1}{Y_i^2}+\frac{1}{Z_i^2}-\frac{1}{X_i^2}\right)$	$-\frac{1}{2}\left(\frac{1}{Y_tY_c}+\frac{1}{Z_tZ_c}-\frac{1}{X_tX_c}\right)$	$-\frac{F_2F_3}{2}$	**
F_{11}	$\frac{1}{X_tX_c}$	$\frac{1}{X_i^2}$	$\frac{1}{X_tX_c}$	$\frac{1}{X_tX_c}$	**
F_{22}	$\frac{1}{Y_tY_c}$	$\frac{1}{Y_i^2}$	$\frac{1}{Y_tY_c}$	$\frac{1}{Y_tY_c}$	**
F_{33}	$\frac{1}{Z_tZ_c}$	$\frac{1}{Z_i^2}$	$\frac{1}{Z_tZ_c}$	$\frac{1}{Z_tZ_c}$	**
F_{44}	$\frac{1}{R^2}$	$\frac{1}{R^2}$	$\frac{1}{R^2}$	$\frac{1}{R^2}$	$\frac{1}{R^2}$
F_{55}	$\frac{1}{S^2}$	$\frac{1}{S^2}$	$\frac{1}{S^2}$	$\frac{1}{S^2}$	$\frac{1}{S^2}$
F_{66}	$\frac{1}{T^2}$	$\frac{1}{T^2}$	$\frac{1}{T^2}$	$\frac{1}{T^2}$	$\frac{1}{T^2}$

Note. Single asterisk denotes the values of X_i, Y_i, and Z_i equal to X_c, Y_c, and Z_c or X_t, Y_t, and Z_t depending on the sign of normal stresses σ_1, σ_2, and σ_3, respectively; double asterisks indicate the respective expressions from (20).

(ii) for transtropic (and especially isotropic) materials with an isotropy plane 23 the shear strength in it should, for physical reasons, be a well-defined constant quantity. On the other hand, by meeting the requirement for invariance of (12) together with the constitutive relations (23) for this criterion, we will obtain **six** different values for the ultimate shear strength F_{44}.

The publications [12, 32–34] pointed out that for real composites featuring an essential anisotropy and difference between tensile and compressive strength values it is very difficult, if possible, select the parameters F_{ij} $(i \neq j)$ in (7) as constants independent of SSS type, which would adequately describe the material strength under various loading conditions. Therefore, in [32–34] these parameters were proposed to be considered as the following functions rather than constants:

$$F_{ij} = F_{ij}(\operatorname{sign}\sigma_i, \operatorname{sign}\sigma_j). \tag{27}$$

Under this assumption, the particularization of F_{ij} will require, in addition to nine main experiments, 12 more tests for an orthotropic material and six main and four additional experiments for a transtropic material. The resulting surface of failure will still be continuous and simply connected but may become piecewise-smooth, i.e., have jogs when passing through the planes $\sigma_i = 0$. We can demonstrate how for an orthotropic medium F_{ij} can be particularized, in view of (27), on the basis of 12 additional simplest tests, namely: three uniaxial tensile tests, three uniaxial compression tests, and six ones in simple shear along certain directions and in certain planes, which do not

TABLE 2. Quadratic Polynomial Failure Criteria for a Transversely Isotropic Material

F_i, F_{ij}	Generalized Mises criterion	Hoffman criterion	MSC (in polynomial form)	CMS (in polynomial form)
F_1	$\frac{1}{X_t}-\frac{1}{X_c}$	$\frac{1}{X_t}-\frac{1}{X_c}$	$\frac{1}{X_t}-\frac{1}{X_c}$	$F_1^A+\left(\frac{S_{12}}{S_{22}}+\frac{S_{13}}{S_{33}}\right)F_2^A$
F_2	$\frac{1}{Y_t}-\frac{1}{Y_c}$	$\frac{1}{Y_t}-\frac{1}{Y_c}$	$\frac{1}{Y_t}-\frac{1}{Y_c}$	$\frac{S_{12}}{S_{11}}F_1^A+\left(1+\frac{S_{23}}{S_{33}}\right)F_2^A$
F_{12}	$-\frac{1}{2\sqrt{X_tX_cY_tY_c}}$	$-\frac{1}{2}\left(\frac{1}{X_tX_c}\right)$	$-\frac{F_1F_2}{2}$	*
F_{23}	$-\frac{1}{2Y_tY_c}$	$-\frac{1}{2}\left(\frac{2}{Y_tY_c}-\frac{1}{X_tX_c}\right)$	$-\frac{F_2^2}{2}$	*
F_{11}	$\frac{1}{X_tX_c}$	$\frac{1}{X_tX_c}$	$\frac{1}{X_tX_c}$	*
F_{22}	$\frac{1}{Y_tY_c}$	$\frac{1}{Y_tY_c}$	$\frac{1}{Y_tY_c}$	*
F_{66}	$\frac{1}{T^2}$	$\frac{1}{T^2}$	$\frac{1}{T^2}$	$\frac{1}{T^2}$

* Respective expressions from (21).

coincide with the principal directions and planes of anisotropy. For a transtropic material we have only four additional tests: one in uniaxial tension, one in uniaxial compression and two in simple shear.

The quartic tensor-polynomial strength criterion as proposed in [32, 33] (hereinafter, Ashkenazi criterion) describes a limit state of a composite material and includes the influence of hydrostatic pressure (the spherical part of the stress tensor). In the abridged tensor notation the criterion is written as

$$\alpha_{iklm}\sigma_{ik}\sigma_{lm}-\left[\frac{(\sigma_{ik}\delta_{ik})^2+\sigma_{ik}\sigma_{ik}}{2}\right]^{1/2}=0, \qquad i,\ k,\ l,\ m=1,\ 2,\ 3, \tag{28}$$

where α_{iklm} is the material constants which are components of the symmetric fourth-rank tensor – the strength tensor, σ_{ik} and σ_{lm} are the stress tensor components, and δ_{ik} is the Kronecker index.

The augend ($\alpha_{iklm}\sigma_{ik}\sigma_{lm}$) is the common invariant of the stress tensor and strength tensor, the addend represents the dependence of strength of anisotropic bodies on two invariants of the stress tensors. The invariant equation (28) can be considered as generalization of the Mises–Hill “plastic” potential for anisotropic bodies in case their limiting state explicitly depends on the first invariant of the stress tensor (on the hydrostatic pressure). When written in an expanded form, criterion (28) is the quartic polynomial with respect to six components of working stresses.

In the most general case of three-dimensional stress state, with the safety factor k_b taken equal for all orientations, the Ashkenazi criterion in the anisotropy axes x', φ', r for an orthotropic body becomes

$$\begin{cases} \Phi\le 1, \qquad \Phi=\dfrac{c\sigma_{x'}^2+b\sigma_{\varphi'}^2+d\sigma_r^2+p\tau_{x'\varphi'}^2+r\tau_{\varphi' r}^2+s\tau_{rx'}^2+t\sigma_{x'}\sigma_{\varphi'}+f\sigma_{\varphi'}\sigma_r+m\sigma_r\sigma_{x'}}{\sqrt{\sigma_{x'}^2+\sigma_{\varphi'}^2+\sigma_r^2+\tau_{x'\varphi'}^2+\tau_{\varphi' r}^2+\tau_{rx'}^2+\sigma_{x'}\sigma_{\varphi'}+\sigma_{\varphi'}\sigma_r+\sigma_r\sigma_{x'}}}, \\ c=\dfrac{1}{[\sigma_{ux'}]}, \quad b=\dfrac{1}{[\sigma_{u\varphi'}]}, \quad d=\dfrac{1}{[\sigma_{ur}]}, \quad p=\dfrac{1}{[\tau_{ux'\varphi'}]}, \quad r=\dfrac{1}{[\tau_{u\varphi' r}]}, \quad s=\dfrac{1}{[\tau_{urx'}]}, \\ t=\dfrac{4}{[\sigma_{ux'\varphi'}^{(45)}]}-b-p-c, \quad f=\dfrac{4}{[\sigma_{u\varphi' r}^{(45)}]}-b-d-r, \quad m=\dfrac{4}{[\sigma_{urx'}^{(45)}]}-d-s-c, \end{cases} \tag{29}$$

where $[\sigma_{ui'}]=\sigma_{ui'}/k_b$, $[\tau_{uij'}]=\tau_{uij'}/k_b$, $[\sigma^{(45)}_{uij'}]=\sigma^{(45)}_{uij'}/k_b$ $(i',\ j'=x',\ \varphi',\ r,\ i'\neq j')$ are the allowable stresses, $\sigma_{ui'}$ is the ultimate tensile or compression strength, $\tau_{uij'}$ is the ultimate simple-shear strength, $\sigma^{(45)}_{uij'}$ is the ultimate strength along the diagonal direction at $45°$ to the axes of symmetry in the plane that corresponds to the subscripts, and k_b is the safety factor.

It is evident from formulas (29) the use of the Ashkenazi criterion provides no way of including various limit properties of material in tension and compression or in shear along various directions.

The function Φ determines the region at whose border the stresses will be equal to the allowable ones.

According to the Ashkenazi criterion (29), the limiting surface in the general case of three-dimensional stress state for an orthotropic material is plotted on the basis of results of nine independent tests (in tension and compression of the specimens cut out along the axes of symmetry (three tests), at $45°$ to the axes of symmetry in three principal planes (three tests), and in simple shear in three principal planes (three tests). It identically satisfies these experimental (reference) points – the function Φ at these points is identically unity.

In CIS, the Ashkenazi criterion is used for strength analysis of dynamically loaded composites [35–37]; it is known in other countries as well [12 et al.].

In [34], a comparative analysis of the data obtained by the Ashkenazi criterion and the generalized Mises criterion was performed for a particular orthotropic material. They were found to be in a poor agreement; the authors put forward an adjustment of the generalized Mises criterion for the composites featuring equal strength in tension and compression and stated the rules to be followed when choosing an appropriate strength criteria for a composite.

Other phenomenological failure criteria for anisotropic materials are outlined in the above-mentioned publications.

Note that a review similar to that given in Parts 1–3 was undertaken earlier in [38]. Here we provide the essentially revised and supplemented information. Furthermore, we have reviewed many new publications that were not covered in [38].

CONCLUSIONS

1. There have been numerous publications dedicated to the elaboration of phenomenological strength criteria for composite materials. The maximum stress criterion, maximum strain criterion, Tsai–Wu criterion (especially in the form of the generalized Mises criterion), the Tsai–Hill and Hoffman criteria, which are valid for composite materials with different ultimate strength in tension and compression, are most widely accepted in applications. Among these, the Tsai–Wu criteria are the most accurate and general, while the generalized Mises criterion requires no complex tests for particularization.

2. The Ashkenazi criterion has found practical application in strength analysis of dynamically loaded composites in CIS countries. However, it can be used only for composite materials whose ultimate tensile and compression strength are equal.

3. A study of applicability of various phenomenological strength criteria to modeling of failure for multilayer composites has demonstrated that a given criterion can be most accurate in certain cases but lead to significant errors in other cases. None of the criteria widely accepted in the applications can provide a description of all the structural elements considered, under the loading conditions studied. Consequently, it is necessary to check the applicability of various criteria to particular structural elements or a given class of structures under specified loading conditions.

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