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# SPATIAL STRESS-STRAIN STATE OF TRIBOFATIGUE SYSTEM IN ROLL–SHAFT CONTACT ZONE

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*We consider the mechanical-and-mathematical model of stress-strain state system subjected to action of contact and non-contact loads. The stressed state is derived by superposition of stress fields induced by normal and tangential contact loads with elliptic distribution, as well as by non-contact bending. A significant change of the system stress-strain state, as compared to purely contact problem solution, is demonstrated.*

***Keywords***: tribofatigue system, spatial stress-strained state, contact interaction, non-contact loads.

**Tribofatigue System: Combined Stress-Strained State.** Elaboration of mechanical-mathematical model of stressed state of tribofatigue systems within framework of the theory of elasticity requires, in the general case, to take into account the distributed normal $p(x, y)$ and tangential $q(x, y)$ contact forces together with non-contact loads $M_i$, $N_i$, and $Q_i$, as well as temperature gradient $\Delta T_\Sigma$ from all heat sources (Fig. 1) [1, 2].

Further tribofatigue system analysis is based on the following substantive provisions.

1. At least, one of the loads applied to the system induces in it both local contact deformations, and bulk deformation of at least one element of the system.

2. Stresses generated by contact and non-contact loads act simultaneously and in the same zone.

3. Contact shape and area are additionally controlled by curvature surface variation of that element of the system, which is subjected to bulk deformation.

4. Friction force and friction coefficient depend on additional boundary conditions in the contact zone imposed by the non-contact load action.

Thus, with reference to a tribofatigue system the following two boundary problems have to be solved: the contact problem for interaction of the system elements:

$$\sigma_{nn}^{(c)}\Big|_S = p(F_c, S), \qquad \sigma_{n\tau}^{(c)}\Big|_S = fp(F_c, S), \qquad \sigma_{ij}^{(c)}\Big|_{\rho\to\infty} \to 0 \tag{1}$$

and the elastic theory problem:

$$Q\Big|_{S_Q} = Q(F_b), \qquad N\Big|_{S_N} = N(F_b), \qquad M\Big|_{S_M} = M(F_b), \qquad u_i\Big|_{S_u} = u_b, \qquad T_\Sigma\Big|_{S_T} = T_\Sigma^*, \tag{2}$$

where $S(x, y)$ is contact surface, $F_c$ and $F_b$ are contact and bending forces, respectively, $f$ is friction coefficient, $\rho$ is distance from the contact center, $\sigma_{ij}^{(c)}$ is the stressed state in contact, $n \perp S$, $\tau || S$, $Q$, $N$, and $M$ are internal lateral and longitudinal forces, and the internal moment, respectively, $u$ is displacement, $T_\Sigma$ is temperature resulting from all heat sources, $T_\Sigma^*$ is temperature at the body surface, and $S_Q$, $S_N$, $S_M$, $S_u$, and $S_T$ are sets of points of a solid body, to which the external loads corresponding to internal stresses, displacements and temperature are applied.

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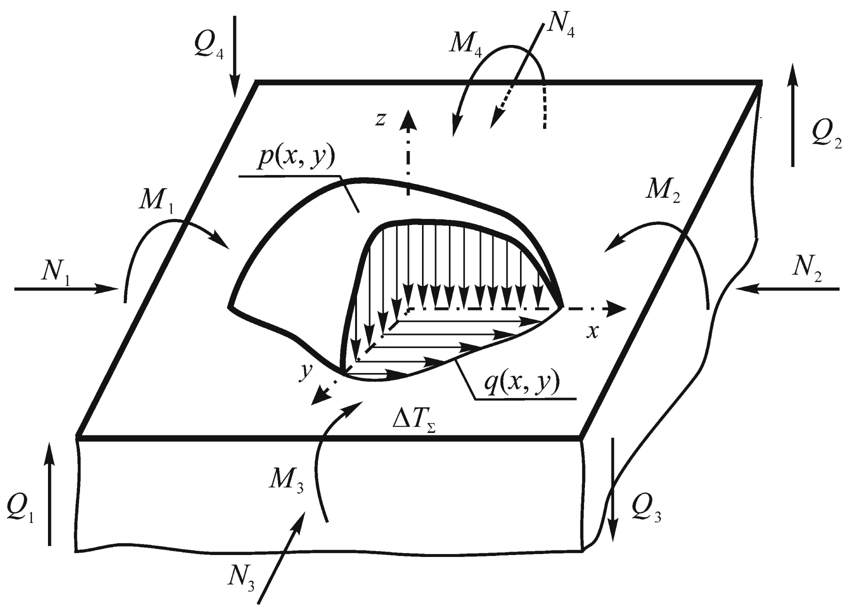



Fig. 1. General loading scheme of a tribofatigue system.

Noteworthy is that determination of stresses and strains for the boundary condition of type $T_\Sigma\big|_{S_T} = T_\Sigma^*$ requires to provide preliminary derivation of the respective heat conductivity problem solution. Insofar as within framework of the elastic problem statement the dependence between stresses and strains is linear, in order to assess the stress-strain state of tribofatigue system elements, it would be convenient represent the combined stressed state in the form of superposition of the stressed states corresponding to particular boundary conditions:

$$\sigma_{ij} = \sum_k \sigma_{ij}^{(k)}, \tag{3}$$

where index $k$ can attain values $c$, $b$, and $T_\Sigma$, which would designate contact, non-contact and temperature boundary conditions, respectively.

We consider a bulk deformation as load action, which occurs according to Saint-Venant's principle, i.e., in contrast to the contact interaction, no account is taken of stress peculiarities in points of its application. The stressed state in any arbitrary point $M(x, y, z)$ of a tribofatigue system is controlled by the following general relation [1–5]:

$$\sigma_{ij} = \sigma_{ij}^{(n)} + \sigma_{ij}^{(\tau)} + \sigma_{ij}^{(b)}, \qquad i, j = x, y, z, \tag{4}$$

where $\sigma_{ij}^{(n)}$, $\sigma_{ij}^{(\tau)}$, and $\sigma_{ij}^{(b)}$ are stresses induced by the normal and tangential contact loads, and the non-contact loads, respectively. The stressed state analysis in a contact zone within framework of the exact problem statement is usually reduced to derivation of stress components in points of $z$-axis and in some points of the contact surface [6–8]. Determination of all stress component in any arbitrary point of a half-space in the exact problem statement is cumbersome due to complexity of functions being integrated.

In tribofatigue systems, where friction process is realized, unmatched movable contact between elements occurs, which implies that, in the general case, normal $p(x, y)$ and tangential $q(x, y)$ are distributed along the contact surface. The respective stressed state in case of a contact interaction is described by superposition of stresses $\sigma_{ij}^{(n)}$ and $\sigma_{ij}^{(\tau)}$:

$$\sigma_{ij} = \sigma_{ij}^{(n)} + \sigma_{ij}^{(\tau)}. \tag{5}$$

Calculation of stresses $\sigma_{ij}^{(hs)}$ in any arbitrary point $M(x,\ y,\ z)$ at $z<0$ of the half-space, in case of normal forces $p(x,\ y)$ acting at the surface is provided by numerical methods with application of the dominant functions $G_{ij}^{(B)}$ from the fundamental solution of the Boussinesq problem on a point normal force applied to a half-space [7]:

$$\sigma_{ij}^{(hs)}(x,\ y,\ z) = \iint\limits_{S(\xi,\eta)} p(\xi,\ \eta) G_{ij}^{(B)}(\xi - x,\ \eta - y,\ z) d\xi d\eta, \tag{6}$$

where

$$G_{xx}^{(B,\sigma)} = \frac{1}{2\pi}\left\{\frac{(1-2\nu)}{r^2}\left[\left(1-\frac{z}{\rho}\right)\frac{x^2-y^2}{r^2}+\frac{zy^2}{\rho^3}\right]-\frac{3zx^2}{\rho^5}\right\},\quad G_{yy}^{(B,\sigma)} = \frac{1}{2\pi}\left\{\frac{(1-2\nu)}{r^2}\left[\left(1-\frac{z}{\rho}\right)\frac{y^2-x^2}{r^2}+\frac{zx^2}{\rho^3}\right]-\frac{3zy^2}{\rho^5}\right\},$$

$$G_{zz}^{(B,\sigma)} = -\frac{3}{2\pi}\frac{z^3}{\rho^5},\quad G_{xy}^{(B,\sigma)} = \frac{1}{2\pi}\left\{\frac{(1-2\nu)}{r^2}\left[\left(1-\frac{z}{\rho}\right)\frac{xy}{r^2}-\frac{xyz}{\rho^3}\right]-\frac{3xyz}{\rho^5}\right\},\quad G_{xz}^{(B,\sigma)} = -\frac{3}{2\pi}\frac{xz^2}{\rho^5},\quad G_{yz}^{(B,\sigma)} = -\frac{3}{2\pi}\frac{yz^2}{\rho^5},$$

$$r^2 = x^2 + y^2,\qquad \rho^2 = x^2 + y^2 + z^2.$$

One should take into account that calculation of stresses $\sigma_{ij}^{(hs)}$ at the half-space surface by direct integration according to (6) is quite intricate. This is stipulated by that the results of calculation via numerical methods show a very slow convergence in points of the half-space surface with $z=0$ due to the singularity in the unit point force application point. Moreover, the calculated results exhibit a poor correlation with known solutions for some subdomains of the half-space surface.

Calculation of stresses in any arbitrary point $M(x,\ y,\ 0)$ of half-space surface in case of action of normally distributed forces $p(x,\ y)$ is performed using the following formula [1, 9, 10]:

$$\sigma_{ij}^{(surf)}(x,\ y,\ 0) = \sigma_{ij}^{(S)}(x,\ y), \tag{7}$$

where $\sigma_{ij}^{(S)}(x,\ y)$ are stresses at the half-space surface induced by action of pressure distributed along the surface $S(x,\ y)$. In the explicit form, expressions (7) in view of solutions [1, 2, 5] are as follows:

$$\frac{\sigma_{xx}^{(surf)}}{p_0} = \begin{cases} -\dfrac{b+2\nu a}{a+b}\sqrt{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}} & \text{for } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}<1, \\ 0 & \text{for } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}>1, \end{cases} \tag{8a}$$

$$\frac{\sigma_{yy}^{(surf)}}{p_0} = \begin{cases} -\dfrac{a+2\nu b}{a+b}\sqrt{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}} & \text{for } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}<1, \\ 0 & \text{for } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}>1, \end{cases} \tag{8b}$$

$$\frac{\sigma_{zz}^{(surf)}}{p_0} = \begin{cases} \sqrt{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}} & \text{for } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}<1, \\ 0 & \text{for } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}>1, \end{cases} \tag{8c}$$

$$\frac{\sigma_{xy}^{(surf)}}{p_0} = \begin{cases} -(1-2\nu)\dfrac{b}{ae^2}\left[\dfrac{y}{ae}\arctan\left(\dfrac{ex}{a}\right) - \dfrac{x}{ae}\arctan\left(\dfrac{aey}{b^2}\right)\right] = H(x,\ y) & \text{for} \quad H(x,\ y) < 0, \\ 0 & \text{for} \quad H(x,\ y) > 0, \end{cases} \tag{8d}$$

$$\frac{\sigma_{xz}^{(surf)}}{p_0} = 0, \qquad \frac{\sigma_{yx}^{(surf)}}{p_0} = 0. \tag{8e}$$

Thus, values $\sigma_{ij}^{(n)}$ in formula (5) with account of (6)–(8) can be presented in the following form:

$$\sigma_{ij}^{(n)} = \begin{cases} \sigma_{ij}^{(hs)} & \text{for} \quad z < 0, \\ \sigma_{ij}^{(surf)} & \text{for} \quad z = 0. \end{cases} \tag{9}$$

Calculation of stressed state $\sigma_{ij}^{(\tau)}$ induced by action of friction force, which is simulated by distribution of tangential forces $q(x,\ y)$ is also provided by numerical methods with application of domination functions $G_{ij}^{(C)}$ from the Cerruti problem solution in case of a point tangential force acting at a half-space [7]:

$$\sigma_{ij}^{(\tau)}(x,\ y,\ z) = \iint\limits_{S(\xi,\eta)} q(\xi,\ \eta) G_{ij}^{(C)}(\xi - x,\ \eta - y,\ z)\, d\xi d\eta, \tag{10}$$

where

$$G_{xx}^{(C,\sigma)} = \frac{1}{2\pi}\left\{-\frac{3x^3}{\rho^5} + (1-2\nu)\left[\frac{x}{\rho^3} - \frac{3x}{\rho(\rho+z)^2} + \frac{x^3}{\rho^3(\rho+z)^2} + \frac{2x^3}{\rho^2(\rho+z)^3}\right]\right\},$$

$$G_{yy}^{(C,\sigma)} = \frac{1}{2\pi}\left\{-\frac{3xy^2}{\rho^5} + (1-2\nu)\left[\frac{x}{\rho^3} - \frac{x}{\rho(\rho+z)^2} + \frac{xy^2}{\rho^3(\rho+z)^2} + \frac{2xy^2}{\rho^2(\rho+z)^3}\right]\right\},$$

$$G_{zz}^{(C,\sigma)} = -\frac{3}{2\pi}\frac{xz^2}{\rho^5}, \qquad G_{xy}^{(C,\sigma)} = \frac{1}{2\pi}\left\{-\frac{3x^2 y}{\rho^5} + (1-2\nu)\left[-\frac{y}{\rho(\rho+z)^2} + \frac{x^2 y}{\rho^3(\rho+z)^2} + \frac{2x^2 y}{\rho^2(\rho+z)^3}\right]\right\},$$

$$G_{xz}^{(C,\sigma)} = -\frac{3}{2\pi}\frac{x^2 z}{\rho^5}, \qquad G_{yz}^{(C,\sigma)} = -\frac{3}{2\pi}\frac{xyz}{\rho^5}.$$

With account of Eq. (8) formula (5) takes the following form:

$$\sigma_{ij} = [\sigma_{ij}^{(hs)} \underset{z}{\vee} \sigma_{ij}^{(surf)}] + \sigma_{ij}^{(\tau)}. \tag{11}$$

Figures 2 and 3 illustrate distributions of normal and tangential stresses in case of action of elliptically distributed normal $p(x,\ y) = p_0\sqrt{1 - x^2/a^2 - y^2/b^2}$ and tangential $q(x,\ y) = fp_0(x,\ y)$ contact forces. The parameters of such distribution are $a$, $b$, and $p_0$ is major and minor semi-axes of the contact surface and pressure in the contact center (friction coefficient $f = 0.5$, $b/a = 0.5$) [7]. As is seen from Figs. 2 and 3, distributions $\sigma_{xx}^{(n)} + \sigma_{xx}^{(\tau)}$ and $\sigma_{xz}^{(n)} + \sigma_{xz}^{(\tau)}$ exhibit significant difference from distributions $\sigma_{xx}^{(n)}$ and $\sigma_{xz}^{(n)}$ due to action of a frictional force, which induces generation of stresses $\sigma_{xx}^{(\tau)}$ and $\sigma_{xz}^{(\tau)}$.

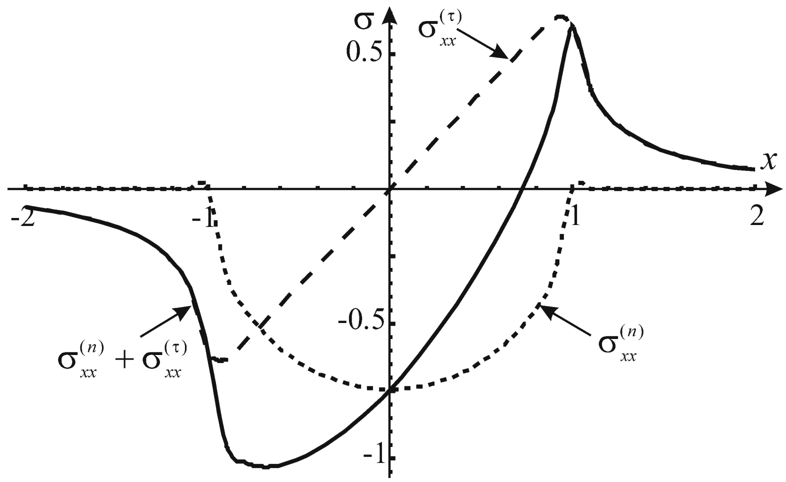



Fig. 2. Distribution of stresses $\sigma_{xx}$ normalized by $p_0$ along the contact surface at $z=0$ and $y=0$.

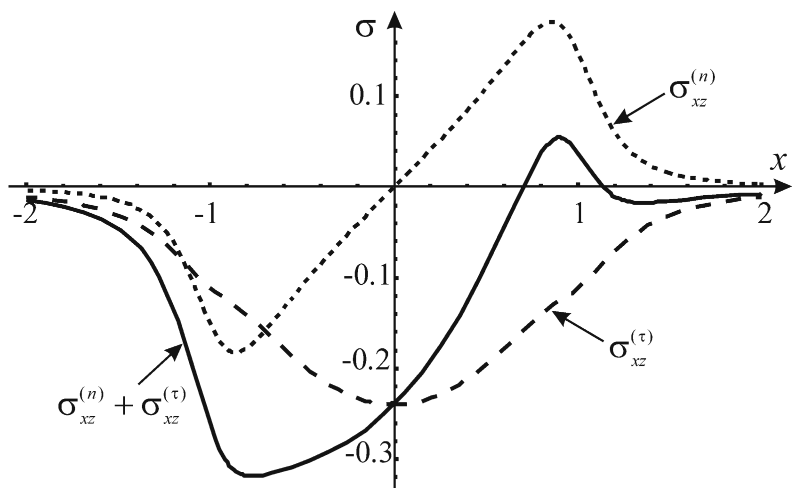



Fig. 3. Distribution of stresses $\sigma_{xz}$ normalized by $p_0$ below the contact surface at $z=-0.3a$ and $y=0$.

We'll derive stresses induced by non-contact loads using a theory, which is the most appropriate for the particular geometry and boundary conditions of interacting bodies [11]:

$$\sigma_{ij}^{(b)}=\sigma_{ij}^{(M)}+\sigma_{ij}^{(N)}+\sigma_{ij}^{(Q)}, \tag{12}$$

where $M$, $N$, and $Q$ are the internal moment, the longitudinal and lateral forces, respectively.

The combined stressed state controlled by expression (4), in view of (6)–(12) takes the following form

$$\sigma_{ij}=\sigma_{ij}^{(n)}+\sigma_{ij}^{(\tau)}+\sigma_{ij}^{(b)}=[\sigma_{ij}^{(hs)}\underset{z}{\vee}\sigma_{ij}^{(surf)}]+\sigma_{ij}^{(\tau)}+\sigma_{ij}^{(b)}$$

$$=\left[\iint\limits_{S(\xi,\eta)} p(\xi,\ \eta)\sigma_{ij}^{(B)}(\xi-x,\ \eta-y,\ z)d\xi d\eta\underset{z}{\vee}\sigma_{ij}^{(S)}(x,\ y)\right]$$

$$+\iint\limits_{S(\xi,\eta)} q(\xi,\ \eta)\sigma_{ij}^{(C)}(\xi-x,\ \eta-y,\ z)d\xi d\eta+\sigma_{ij}^{(M)}(x,\ y,\ z)+\sigma_{ij}^{(N)}(x,\ y,\ z)+\sigma_{ij}^{(Q)}(x,\ y,\ z). \tag{13}$$

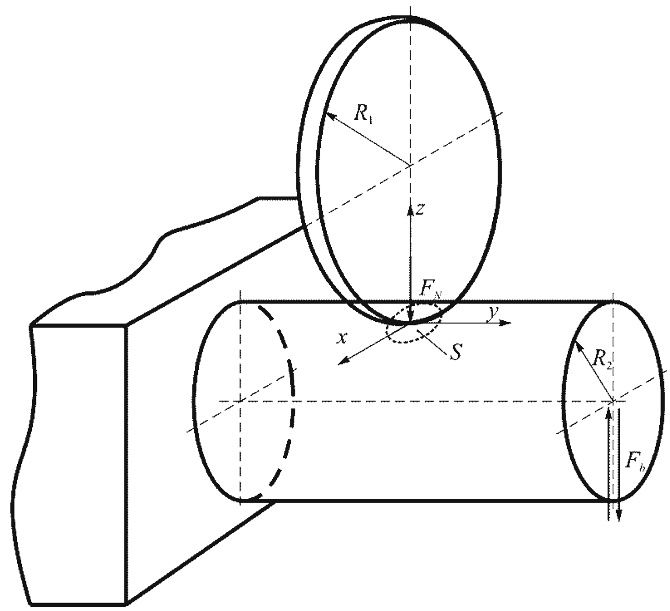



Fig. 4. Roller–shaft tribofatigue system scheme.

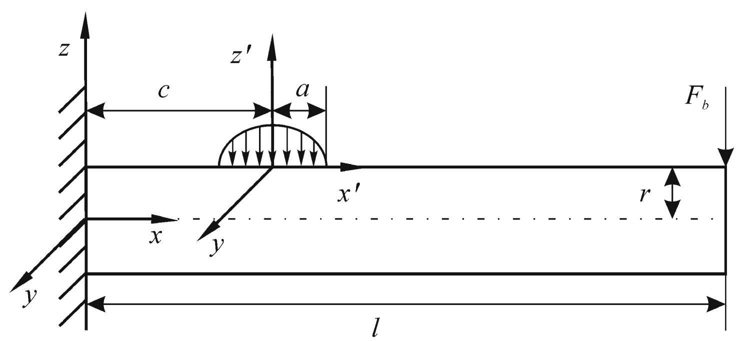



Fig. 5. The shaft loading scheme for the roller–shaft system.

**Roller–Shaft System.** Consider implementation of model (13) with reference to roller–shaft tribofatigue system, which simulates such critical objects as gearwheel, wheel–rail interaction, etc. This model combines roller–shaft contact interaction with shaft non-contact bending (Fig. 4) [11, 12].

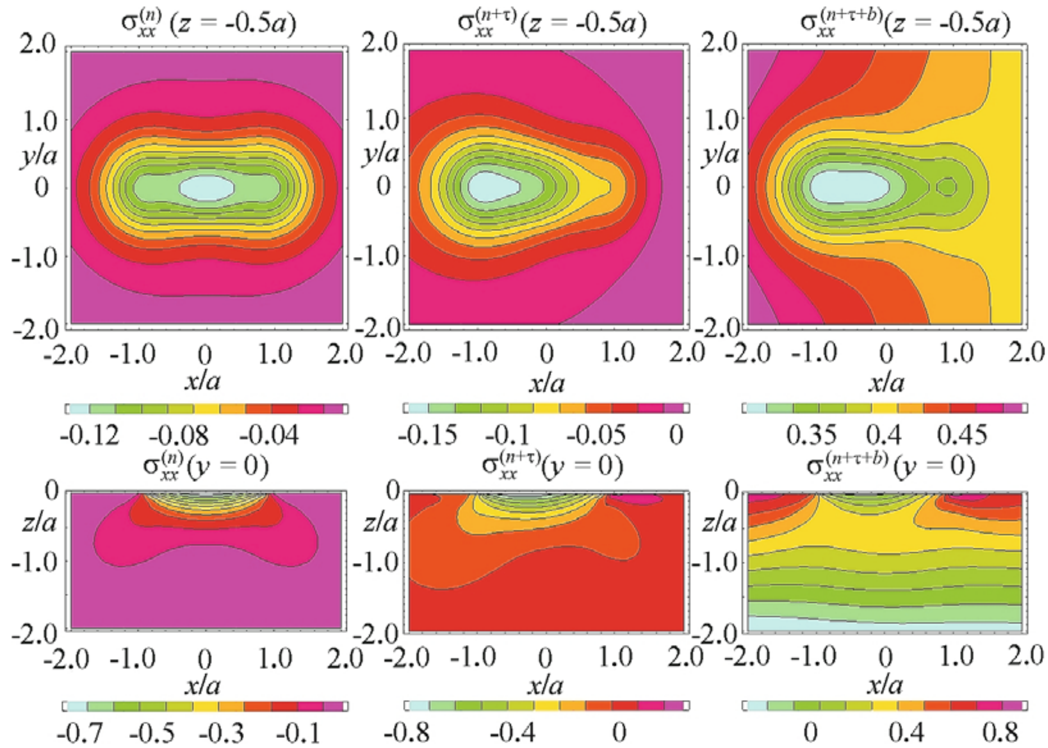
The stressed state in any arbitrary point $M(x, y, z)$ of this system is determined from the general relationship (4). While the roller stressed state is purely contact one, the shaft has stresses induced by non-contact bending load, in addition to contact stresses.

Insofar as the roller and the shaft are restricted by surfaces of the second order, according to the theory of unmatched Hertz contact [7, 13], the roller–shaft interaction can be reduced to elliptic distribution of normal and tangential forces along the half-space.

Distributions of all independent contact stress components $\sigma_{ij}^{(n)}$ and $\sigma_{ij}^{(\tau)}$, as well as of the principal stresses, are described elsewhere [2]. In the roller–shaft system analysis, we will consider stresses $\sigma_{xx}^{(n)}$, $\sigma_{yy}^{(n)}$, $\sigma_{zz}^{(n)}$, and $\sigma_{xz}^{(n)}$, which mainly control the stress-strain state peculiarities of the system, as compared to a contact couple.

The stress-strain state induced by action of force $F_b$ in plane $y = 0$ (Fig. 5) is assessed according to known dependences [11]:

$$\sigma_{xx}^{(b)} = \frac{F_b (l - x) z}{4(1+\nu)J}, \qquad \sigma_{xz}^{(b)} = \frac{(3+2\nu)F_b}{8(1+\nu)J}(r^2 - z^2), \qquad \sigma_{xx}^{(b)} = \sigma_{zz}^{(b)} = \sigma_{xz}^{(b)} = \sigma_{xy}^{(b)} = 0, \tag{14}$$




a b c

Fig. 6. Fields of stresses $\sigma_{xx}$ normalized by $p_0$ for the contact problem $\sigma_{xx}^{(n)}$ (a), friction couple $\sigma_{xx}^{(n+\tau)}$ (b), and tribofatigue system $\sigma_{xx}^{(n+\tau+b)}$ (c).

where

$$J = \frac{\pi r^4}{64}, \qquad F_b = 0.22\, p_0 \frac{4(1+\nu)J}{lr}.$$

As an example, we'll compare one solution of a conventional contact problem (11) for the roller–roller contact couple having elliptic distribution of contact pressure with a similar solution (13) for a tribofatigue system, in which a shaft with cantilever-type fixation is also loaded by a non-contact bending force $F_b$. The bending force is applied in such a way that a zone of tensile stresses is formed in the contact vicinity.

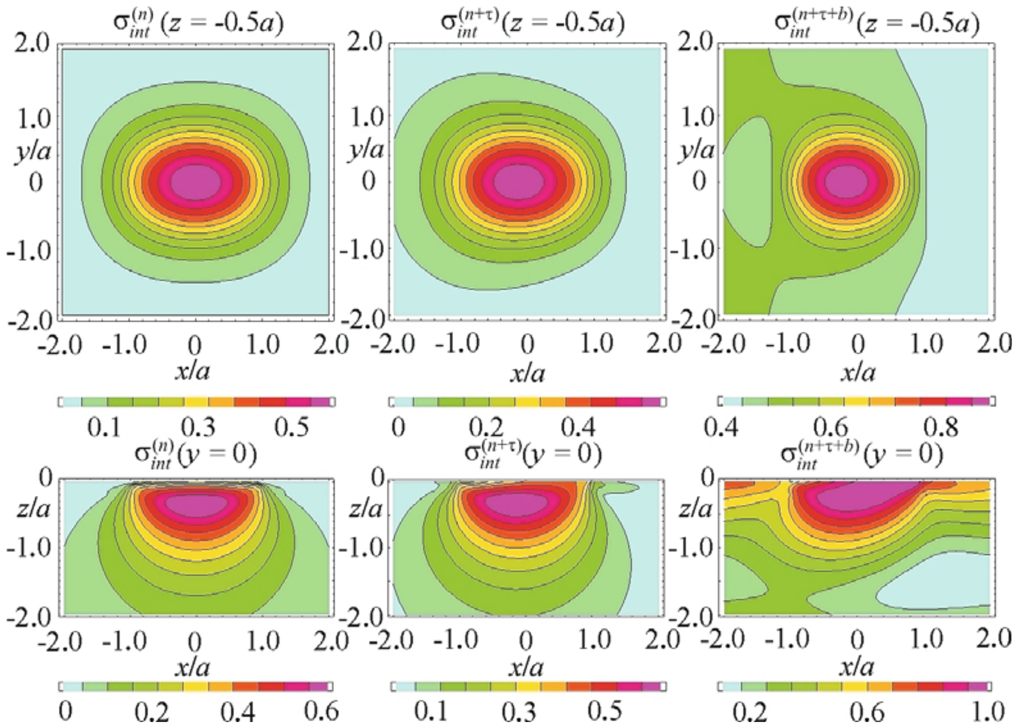
In the performed stressed state analysis, calculation has been conducted for the area $z \in [0;\, 1.5a]$, $x \in [-1.5a;\, 1.5a]$, and $y \in [-1.5a;\, 1.5a]$ for $20 \times 39 \times 39$ points and non-contact load $F_b = 2.7 p_0 \dfrac{(1+\nu)J}{12r^2}$, where $r$ is shaft radius, $r = 1.5a$, $J$ is shaft moment of inertia, and shaft length is equal to $12r$. The following values for properties of materials and geometrical characteristics have been taken: the Young moduli $E_1 = E_2 = 2.01 \cdot 10^{11}$ Pa, Poisson's ratios $\nu_1 = \nu_2 = 0.3$, friction coefficient $f = 0.2$, and the ratio of minor and major semi-axes of the elliptic contact zone $b/a = 0.574$. Results of calculations are depicted in Figs. 6 and 7.

As is seen from Fig. 6, the effect of non-contact stresses on the stressed state variation in the contact zone consists in the following:

a) distribution $\sigma_{xx}$ exhibits non-uniform shift in plane $y = 0$, in particular: according to the magnitude of acting bending stresses, which vary linearly change along $z$-coordinate;

b) the field of compressive stresses $\sigma_{xx}^{(n)}$ is partially converted into the field of tensile stresses $\sigma_{xx}^{(n+\tau+b)}$;

c) the maximal by absolute value compressive stresses $\sigma_{xx}$ decrease approximately twice due to shaft bending.




a b c

Fig. 7. Fields of stress intensity $\sigma_{int}$, normalized by $p_0$, for a contact problem $\sigma^{(n)}_{int}$ (a), friction couple $\sigma^{(n+\tau)}_{int}$ (b), and tribofatigue system $\sigma^{(n+\tau+b)}_{int}$ (c).

Figure 7 depicts a considerable variation of stress intensity [in comparison with $\sigma^{(n)}_{int}$] in the contact vicinity, as a result of friction force $\sigma^{(n+\tau)}_{int}$ and non-contact bending $\sigma^{(n+\tau+b)}_{int}$ action.

**Conclusions.** The combined stressed state defined by boundary conditions (1) and (2) is described by Eqs. (3), (4), and (13) as superposition of stresses induced by both contact, and non-contact loads. Therefore, from the tribofatigue standpoint the analysis via (13) is applicable, since, on the one hand, it allows one to investigate how the stress field is perturbed due to bulk deformation in the local zone, where a field of contact stresses is induced simultaneously. Such analysis is used when the direct effect in a tribofatigue system is realized. On the other hand, such analysis allows one to investigate variation of the local field of contact stresses when a stress field generated by bulk deformation is superimposed on it. The proposed analysis is useful, when the reverse effect in a tribofatigue system is realized. Application of Eq. (13) makes it possible to conduct both quantitative and qualitative analyses for practically any arbitrary loading conditions.

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