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STRENGTH ASSESSMENT FOR COMPOSITE AND METAL-COMPOSITE CYLINDERS UNDER PULSE LOADING. PART 1. RULES OF CHOOSING VARIOUS STRENGTH CRITERIA FOR ANISOTROPIC MATERIAL AND COMPARATIVE ANALYSIS OF SUCH CRITERIA

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The author formulates the rules one should follow when choosing strength criteria for a composite. A comparative analysis of Ashkenazi and Tsai–Wu strength criteria is performed for a specific orthotropic material, and the criteria are shown to agree with each other. An update of the Tsai–Wu criterion is put forward for composites that have uniform tension and compression strength.

Keywords: orthotropy, strength criteria, limiting surface, stability and invariance conditions, comparative analysis.

During the design of explosionproof chambers (EPC) special attention is given to choosing appropriate structural materials for load-bearing shells. For that purpose, along with metals (steel, titanium alloys, etc.) it is sometimes recommended to use oriented fiber-reinforced composites [1–5] which often feature a pronounced anisotropy. These materials are usually orthotropic or, if unidirectionally reinforced or uniformly reinforced in plane, transtropic ones [4]. Single- or multilayer composite or metal-composite cylindrical shells are used as load-bearing elements of EPC. In this case, "layer" is hereinafter used to mean a part of the structural element (even if such part has been technologically fabricated through multi-ply winding), within which the elastic characteristics and directions of principal anisotropy axes can be assumed to be constant.

The composite layers which are usually produced by winding, represent spirally orthotropic (often spirally transtropic) bodies [2–5]. The internal isotropic metallic layer is an auxiliary one: it sets a shape for the shell during its fabrication, partially insulates the composite layer to protect it against thermal overloading, blocks its dynamic instability (by taking off its elastic energy) and absorbs a portion of explosion energy to plastic deformation. This improves the structural reliability – such a multilayer metal-composite shell is more energy-intensive with a smaller mass in comparison to an all-metal one. In this case, it is the first half-period of vibrations, which presents the greatest danger from the standpoint of strength [2]. Therefore, to numerically predict strength of such structural elements under dynamic loading is currently quite an important task.

The publication [5] outlines numerical methods and results of investigation of axisymmetrical stress-strain state (SSS) and strength of single- and two-layer composite cylinders with various spiral reinforcement configurations under internal pulse loading. The Ashkenazi criterion [4] was applied as a strength criterion for an orthotropic body. However, this criterion is seldom used nowadays and only by researchers in CIS countries. In most cases, contemporary researchers prefer using the Tsai–Wu failure criterion [6–9]. In this connection, it would be important to compare the Ashkenazi and Tsai–Wu failure criteria for an anisotropic material, update the numerical method and an application software package (ASP) [5] for the use of the Tsai–Wu criterion, which would enable one to assess SSS and strength of composite and metal-composite cylinders under internal pulse loading. These are the tasks the present work is intended to accomplish.

This investigation dealt with multilayer finite-length cylinders with free ends, which were subjected to internal loading with an axisymmetrical pressure pulse. We will use cylindrical coordinates x, φ, r. The origin of coordinates is placed at the center of symmetry of the shell. In the case of a spirally orthotropic layer, one of the

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principal axes of anisotropy coincides with the direction of the radial coordinate r and two others (x' and φ') can be rotated with respect to the global coordinate axes x and φ through the reinforcement angle α which is constant within the layer under consideration. The extreme cases $\alpha = 0$ and $\alpha = \pm 90°$ correspond to the axial and circumferential reinforcement, respectively, with the spiral orthotropy degenerating into a cylindrical one. Isotropic layers are considered to be a particular case of spirally orthotropic ones, provided that the appropriate identical equality expressions are met for a series of elastic, plastic, and strength characteristics of the material, the reinforcement angle being arbitrary.

The thick-walled cylindrical shells have the following geometrical dimensions: inner radius $R_1 = 0.1$ m, outer radius $R_2 = 0.12$ m, cylinder length $L = 0.4$ m. The shells were loaded using a spherical charge of trotyl-hexogen explosive, which was located along their axis of symmetry. According to data [5, 10, 11], the pressure at the cylinder's inner surface was specified as follows:

$$P(x,\ t) = P_0 (R_1/l)^3 H(l/a_0 - t), \tag{1}$$

$$P_0 = 0.35\, qM_{ch} / R_1^3, \qquad a_0 = \sqrt{q}/0.35,$$

where t is time, a_0 is the characteristic velocity, $a_0 = 6310$ m/s, P_0 is the nominal pressure at a distance R_1 from the charge center, l is the distance from the charge center to a point under consideration, $l = \sqrt{R_1^2 + (x - x_0)^2}$, x_0 is the axial coordinate of the explosive charge center, and $H(t)$ is the Heaviside function. The value $a_0 = 6310$ m/s corresponds to the explosive's calorific value $q = 4.877$ MJ/kg. The nominal pressure P_0 is directly proportional to the charge mass M_{ch}, for $M_{ch} = 0.1$ kg it is 170.7 MPa [5, 10, 11].

The mathematical statement of a boundary-value problem of the determination an axisymmetrical SSS and the numerical finite-difference method for solving thereof were detailed in [5]. Therefore, we will dwell on the strength aspects. Note that the dynamic SSS in the problems to be considered will be quite diverse: depending on time and spatial coordinates it can be uniaxial, biaxial, and triaxial (where all of the six stress tensor components $\sigma_{x'}$, $\sigma_{\varphi'}$, σ_r, $\tau_{x'\varphi'}$, $\tau_{\varphi' r}$, and $\tau_{rx'}$ are simultaneously nonzero and of the same order of magnitude). Therefore, when choosing strength criteria we will use those designated to describe the most general SSS type, i.e., the triaxial one.

The strength of a thick-walled cylinder in the process of variation of its dynamic triaxial SSS was assessed by the Ashkenazi or Tsai–Wu strength criteria. The Ashkenazi criterion is valid only for composites that have uniform strength in tension and compression, and in the principal anisotropy axes x', φ', r for a general case of triaxial SSS it is written as [4, 5]

$$\Phi \le 1, \tag{2}$$

$$\Phi = \frac{a\sigma_{x'}^2 + b\sigma_{\varphi'}^2 + c\sigma_r^2 + d\tau_{x'\varphi'}^2 + e\tau_{\varphi' r}^2 + f\tau_{rx'}^2 + g\sigma_{x'}\sigma_{\varphi'} + h\sigma_{\varphi'}\sigma_r + m\sigma_r\sigma_{x'}}{\sqrt{\sigma_{x'}^2 + \sigma_{\varphi'}^2 + \sigma_r^2 + \tau_{x'\varphi'}^2 + \tau_{\varphi' r}^2 + \tau_{rx'}^2 + \sigma_{x'}\sigma_{\varphi'} + \sigma_{\varphi'}\sigma_r + \sigma_r\sigma_{x'}}},$$

where

$$a = \frac{1}{\sigma_{ux'}}, \qquad b = \frac{1}{\sigma_{u\varphi'}}, \qquad c = \frac{1}{\sigma_{ur}}, \qquad d = \frac{1}{\tau_{ux'\varphi'}}, \qquad e = \frac{1}{\tau_{u\varphi' r}}, \qquad f = \frac{1}{\tau_{urx'}},$$

$$g = \frac{4}{\sigma_{ux'\varphi'}^{(45)}} - a - b - d, \qquad h = \frac{4}{\sigma_{u\varphi' r}^{(45)}} - b - c - e, \qquad m = \frac{4}{\sigma_{urx'}^{(45)}} - c - a - f,$$

$\sigma_{ux'}$, $\sigma_{u\varphi'}$, and σ_{ur} are the ultimate tension or compression strength values along the respective principal directions of anisotropy, $\tau_{ux'\varphi'}$, $\tau_{u\varphi' r}$, and $\tau_{urx'}$ are ultimate pure-shear strength values in the respective principal planes of anisotropy, and $\sigma_{ux'\varphi'}^{(45)}$, $\sigma_{u\varphi'r}^{(45)}$, and $\sigma_{urx'}^{(45)}$ are the ultimate strength values along the diagonal direction at an angle of 45° to the symmetry axis in the principal plane of anisotropy corresponding to the subscripts.

Thus, the limiting surface under the Ashkenazi criterion (2) is plotted by the data from nine independent experiments: six uniaxial tension (or compression) experiments and three pure-shear ones. It identically satisfies these experimental (reference) points: the function Φ in them is identically equal to unity.

Since the Ashkenazi criterion implies uniform strength of a composite material (CM) in tension and compression, which is rarely the case for real materials, in recent years the Tsai–Wu criterion has been used more often for strength analyses. This criterion is free from the above-mentioned drawback, and in the principal axes of anisotropy of the material it is written as [6–9]

$$\Psi \le 1, \tag{3}$$

$$\Psi = \sigma_{x'}\left(\frac{1}{\sigma_{ux'}^{+}} - \frac{1}{\sigma_{ux'}^{-}}\right) + \sigma_{\varphi'}\left(\frac{1}{\sigma_{u\varphi'}^{+}} - \frac{1}{\sigma_{u\varphi'}^{-}}\right) + \sigma_{r}\left(\frac{1}{\sigma_{ur}^{+}} - \frac{1}{\sigma_{ur}^{-}}\right)$$

$$+\frac{\sigma_{x'}^2}{\sigma_{ux'}^{+}\sigma_{ux'}^{-}} + \frac{\sigma_{\varphi'}^2}{\sigma_{u\varphi'}^{+}\sigma_{u\varphi'}^{-}} + \frac{\sigma_{r}^2}{\sigma_{ur}^{+}\sigma_{ur}^{-}} + \frac{\tau_{x'\varphi'}^2}{\tau_{ux'\varphi'}^2} + \frac{\tau_{\varphi'r}^2}{\tau_{u\varphi'r}^2} + \frac{\tau_{rx'}^2}{\tau_{urx'}^2} - F_{x'\varphi'}\sigma_{x'}\sigma_{\varphi'} - F_{\varphi'r}\sigma_{\varphi'}\sigma_{r} - F_{rx'}\sigma_{r}\sigma_{x'},$$

where the superscripts "+" and "–" denote ultimate strength in tension and compression, respectively:

$$F_{ij} = \frac{1}{\sqrt{\sigma_{ui}^{+}\sigma_{ui}^{-}\sigma_{uj}^{+}\sigma_{uj}^{-}}}, \qquad i,\ j = x',\ \varphi',\ r;\ \ i \ne j. \tag{4}$$

The limiting surface (3) is also plotted by the data of nine experiments which, however, are slightly different from those outlined above for the Ashkenazi criterion; namely: three experiments in pure shear and three in uniaxial tension in the principal planes and directions of anisotropy, respectively, as in the case of criterion (2). Moreover, three experiments in uniaxial compression along the principal directions of anisotropy were performed. Surface (3), much like the one given by (2), identically satisfies the experimental points: the function Ψ therein becomes equal to unity, doing so for any three arbitrary values of constants F_{ij}. Depending on the method of redefining these constants, one can construct various strength criteria. In particular, if these constants are defined by formulas (4), we arrive at a strength criterion which is conventionally referred to as the Tsai–Wu criterion in the publications [8, 9].

Depending on the strength criterion used, function Φ from (2) or function Ψ from (3) determines a region at the boundary of which SSS is critical: if the condition $\Phi < 1$ (or $\Psi < 1$) is met no fracture happens; otherwise, the fracture does occur and strength is not fulfilled. Since criteria (2) and (3) are written in the coordinate system coinciding with the principal anisotropy axes and SSS is computed in a global cylindrical coordinate system, during the determination of stresses involved in (2) or (3) the stress tensor should be rotated through an appropriate reinforcement angle α using the known relations [5, 12]. The strength verification procedure should be carried out for every finite-difference cell of the design region, at every time step. Thus, if the condition $\Phi < 1$ (or $\Psi < 1$) is obeyed within the whole region and over the entire time range, the structure is strong. If the said condition is not met at least at a single point and at any instant of time, the strength does not hold.

The Tsai–Wu limiting surface (3), (4) is the classical quadric and convex surface in which is also convex in the six-dimensional space of stresses. The difference between the CM's tensile and compressive strength is allowed for by both squared and linear terms in (3) and leads, in particular, to a shift of the surface center with respect of the origin of coordinates. The Ashkenazi limiting surface (2) is always centrally symmetric, representing also a particular case of a quartic surface without any linear or cubic terms; it can have an alternating-sign curvature and consist of convex, concave, and saddle-shaped segments that interface with each other in a perfectly smooth manner [4].

The absence of stresses is an optimum (sometimes not the only one) of the Ashkenazi strength function Φ from (2): with the stress tensor being zero, the function Φ reaches its global minimum ($\Phi_{\min} = 0$). On the other hand, the minimum of the Tsai–Wu function Ψ from (3) is almost never reached with the stress tensor being zero, except in the degenerated case where a composite has uniform strength along all the principal directions of anisotropy and then $\Psi_{\min} = 0$ and this optimum is not the only one. As evident from (3), the presence of linear terms (even if a single one) results in a situation that with some nonzero stresses the function Ψ will be negative and its minimum will be negative too and attained at a location other than the origin of coordinate in the six-dimensional space. Thus, from the standpoint of strict mathematics the total absence of stresses will almost never represent a strength-optimal SSS as per the Tsai–Wu criterion (3) and will always be an optimum (not always a single one) as per the Ashkenazi criterion (2).

If in (3) we take

$$F_{ij} = \frac{1}{\sigma^+_{ui}\sigma^-_{ui}} + \frac{1}{\sigma^+_{uj}\sigma^-_{uj}} - \frac{\sigma^+_{ui}\sigma^-_{ui}\sigma^+_{uj}\sigma^-_{uj}}{\sigma^+_{ux'}\sigma^-_{ux'}\sigma^+_{u\varphi'}\sigma^-_{u\varphi'}\sigma^+_{ur}\sigma^-_{ur}}, \qquad i,\ j = x',\ \varphi',\ r;\ i \neq j, \tag{5}$$

we will arrive at the well-known Hoffman failure criterion [7, 8]. This criterion suffers a number of drawbacks and is used much more rarely in comparison to the Tsai–Wu criterion. In particular, for the whole, quite wide class of essentially anisotropic materials the Hoffman criterion (5) fails to meet the necessary stability conditions [7]:

$$|F_{ij}| < \frac{2}{\sqrt{\sigma^+_{ui}\sigma^-_{ui}\sigma^+_{uj}\sigma^-_{uj}}}, \qquad i,\ j = x',\ \varphi',\ r;\ i \neq j, \tag{6}$$

which are automatically satisfied by the Tsai–Wu criterion due to (4). Nevertheless, despite the fact that the necessary stability conditions are identically satisfied, the Tsai–Wu limiting surface (3), (4) in the six-dimensional space is an open one – it represents an elliptic paraboloid or, in degenerated cases, a cylinder [7].

For the Ashkenazi criterion (2) the appropriate necessary stability conditions are written as

$$\left| \frac{4}{\sigma^{(45)}_{uij}} - \frac{1}{\sigma_{ui}} - \frac{1}{\sigma_{uj}} - \frac{1}{\tau_{uij}} \right| < \frac{2}{\sqrt{\sigma_{ui}\sigma_{uj}}}, \qquad i,\ j = x',\ \varphi',\ r;\ i \neq j. \tag{7}$$

As in the case of the Tsai–Wu criterion, the fulfillment of conditions (7) ensures only the simple connectedness of the Ashkenazi surface (2) but not its being closed. Since surface (2) is centrally symmetric, it can be either closed or open on two sides. In case the necessary stability conditions (6) or (7) are not met, one cannot apply the respective strength criterion to the given anisotropic material.

Furthermore, for linear-quadratic criteria of type (3) in the case of a transtropic CM in the isotropy plane *ij* the condition of the criterion's invariance to the rotation of axes should be met for any two mutually orthogonal directions in this plane can be considered as principal axes of anisotropy [6]:

$$F_{ij} = \frac{1}{\tau^2_{uij}} - \frac{2}{\sigma^+_{ui}\sigma^-_{ui}}, \tag{8}$$

naturally, the fulfillment of the condition $\sigma^+_{ui} \equiv \sigma^+_{uj}$ and $\sigma^-_{ui} \equiv \sigma^-_{uj}$ is mandatory in this case.

Thus, despite the fact that the constants F_{ij} in (3) are formally used as free parameters, in fact they are not such parameters because (I) they must meet the stability conditions (6) and (ii) for transtropic materials the respective constant is quite a uniquely determined quantity (8) rather than a free parameter. Clearly, the correct mathematical (formulaic) determination of the constants F_{ij} during the limiting transition from orthotropy to transtropy or isotropy must identically satisfy condition (8). Unfortunately, the Tsai–Wu (4) and Hoffman (5) determinations have no such property (unlike the Ashkenazi criterion).

In view of (4), the condition (8) for the Tsai–Wu criterion is written as

$$\tau_{uij} = \sqrt{\frac{\sigma_{ui}^{+}\sigma_{ui}^{-}}{3}}, \tag{9}$$

and for the Hoffman criterion, in view of (5), it takes the form:

$$\tau_{uij} = \sqrt{\frac{\sigma_{ui}^{+}\sigma_{ui}^{-}}{4 - \dfrac{(\sigma_{ui}^{+}\sigma_{ui}^{-})^3}{\sigma_{ux'}^{+}\sigma_{ux'}^{-}\sigma_{u\varphi'}^{+}\sigma_{u\varphi'}^{-}\sigma_{ur}^{+}\sigma_{ur}^{-}}}}, \tag{10}$$

while the Ashkenazi criterion (2) for transtropic CMs is identically invariant with respect to choice of coordinates in the isotropy plane ij by definition without any restrictions like (9) or (10) for the respective ultimate strengths.

It should be mentioned that for the composites that have uniform strength in tension and compression the Tsai–Wu limiting surface (3), (4) degenerates into an elliptic cylinder:

$$\Psi = S_{x'}^2 + S_{\varphi'}^2 + S_r^2 - S_{x'}S_{\varphi'} - S_{\varphi'}S_r - S_r S_{x'} + 3(T_{x'\varphi'}^2 + T_{\varphi' r}^2 + T_{rx'}^2) \le 1, \tag{11}$$

where the respective dimensionless stresses are given by

$$S_i = \frac{\sigma_i}{\sigma_{ui}}, \qquad T_{ij} = \frac{\tau_{ij}}{\tau_{uij}\sqrt{3}}, \qquad i,\ j = x',\ \varphi',\ r;\ \ i \ne j. \tag{12}$$

Since the left-hand part of criterion (11) contains squared intensity of dimensionless stresses (12), for such materials it would be more logical to represent (11) as

$$\sqrt{\Psi} \le 1. \tag{13}$$

Thus, in the left-hand part of (13) we have written an analog of a ratio of some generalized equivalent stress to some generalized allowable one, which is given in the form (2) as in the case of the Ashkenazi criterion. For isotropic materials that have uniform strength in tension and compression, subject to the condition

$$\tau_u = \sigma_u / \sqrt{3} \tag{14}$$

the Tsai–Wu criterion in the form (13) automatically passes into the fourth theory of strength – the criterion for specific potential distortion energy [13]:

$$\sigma_{int} / \sigma_u \le 1, \tag{15}$$

where σ_{int} is the stress intensity, σ_u and τ_u are the ultimate uniaxial tension and pure-shear strengths, respectively. In this case, condition (14) is the invariance condition (8) for an isotropic material with uniform strength in tension and compression.

For materials whose ultimate tensile and compressive strengths differ, the Tsai–Wu strength condition in the form (13) becomes absurd because for such materials the global minimum of function Ψ will always be below zero, i.e., the function may take positive as well as negative values. In such case, the Tsai–Wu criterion should be used in the form (3), (4). In doing so, one can perform only an alternative strength assessment to answer the question: will the material fail under the given SSS or not? To carry out a quantitative assessment of strength margin or, inversely, of the degree of increasing the allowable stress limits would be rather problematic because mathematically even the complete absence of stresses would not be optimal for criterion (3).

The foregoing suggests that a comparative analysis of the Ashkenazi (2) and Tsai–Wu failure criteria makes sense only for CMs with uniform strength in tension and compression, and in that case the last-mentioned criterion should be used in the form (13). For such composites the Ashkenazi limiting surface will identically satisfy the results of nine independent experiments, while the Tsai–Wu surface will do so only six experiments. By definition of (11) and (12) the Tsai–Wu criterion may fail to satisfy the results of three experiments in uniaxial tension (compression) at an angle of 45° to the principal axes i, j of anisotropy $(\sigma_{uij}^{(45)})$ in the planes of these axes. This loading is equivalent to the following plane stress state (PSS) along the principal anisotropy axes: $\sigma_i = \sigma_j = \pm\tau_{ij} = \pm\sigma_{uij}^{(45)}/2$. In this case, the function Φ from (2) identically becomes unity, while according to the Tsai–Wu criterion (11)–(13) we have

$$\sqrt{\Psi} = \lambda_{ij}, \tag{16}$$

$$\lambda_{ij} = \frac{\sigma_{uij}^{(45)}}{2}\sqrt{\frac{1}{\sigma_{ui}^2} + \frac{1}{\sigma_{uj}^2} + \frac{1}{\tau_{uij}^2} - \frac{1}{\sigma_{ui}\sigma_{uj}}}, \qquad i,\ j = x',\ \varphi',\ r;\ i \neq j.$$

It is obvious that to have a satisfactory agreement between the Ashkenazi criterion (2) and Tsai–Wu criterion (11)–(13) it is desirable that the following six conditions are met simultaneously:

$$\frac{\sigma_{ui}^{+}}{\sigma_{ui}^{-}} = 1, \qquad \lambda_{ij} = 1, \qquad i,\ j = x',\ \varphi',\ r;\ i \neq j. \tag{17}$$

The greater is deviation of the quantities in the left-hand parts in (17) from unity, the worse is the agreement between the criteria. Even an accurate fulfillment of conditions (17) will not always guarantee their good matching.

The identical satisfaction of the last three requirements $\lambda_{ij} = 1$ from (17) can be attained if for the composites with uniform strength in tension and compression instead of using Tsai–Wu formulas (4) one re-defines the constants F_{ij} as follows:

$$F_{ij} = \frac{1}{\sigma_{ui}^2} + \frac{1}{\sigma_{uj}^2} + \frac{1}{\tau_{uij}^2} - \frac{4}{(\sigma_{uij}^{(45)})^2}, \qquad i,\ j = x',\ \varphi',\ r;\ i \neq j, \tag{18}$$

i.e., to concretize criterion (3) in this case one will need the same number (nine) of experiments as in the case of Ashkenazi criterion (2), and the coincidence of these two criteria at the experimental (reference) points would be complete. Thus, for the composites that have uniform strength in tension and compression we can recommend an updated criterion to replace the Tsai–Wu criterion (11)–(13). If represented in terms of dimensionless variables (12) the updated criterion is written as

$$\Omega \leq 1, \tag{19}$$

$$\Omega = [S_{x'}^2 + S_{\varphi'}^2 + S_r^2 + 3(T_{x'\varphi'}^2 + T_{\varphi' r}^2 + T_{rx'}^2) - \xi_{x'\varphi'}S_{x'}S_{\varphi'} - \xi_{\varphi' r}S_{\varphi'}S_r - \xi_{rx'}S_r S_{x'}]^{1/2},$$

where

$$\xi_{ij} = \frac{\sigma_{ui}}{\sigma_{uj}} + \frac{\sigma_{uj}}{\sigma_{ui}} + \sigma_{ui}\sigma_{uj}\left[\frac{1}{\tau_{uij}^2} - \frac{4}{(\sigma_{uij}^{(45)})^2}\right], \qquad |\xi_{ij}| < 2, \qquad i,\ j = x',\ \varphi',\ r;\ i \neq j. \tag{20}$$

The last-written inequality in (20) represents the necessary stability conditions (6) for criterion (19). For the composites for which the condition $|\xi_{ij}| \geq 2$ is met for at least one combination of i and j one cannot use the updated criterion (19). Like the Ashkenazi criterion (2), the criterion (19) identically satisfies the requirement for invariance with respect to the choice of coordinates in the isotropy plane, without any restrictions on strength characteristics.

The drawback of the criterion (19) is that it is acceptable only for the materials with uniform strength in tension and compression, for which the Ashkenazi criterion (2) has been proved effective. The Tsai–Wu criterion is used, first of all, to allow for just the difference (sometimes quite significant) between tensile and compressive strengths of a CM, which is the case with the majority of real composites. For such materials the definition of F_{ij} by formula (18) makes no sense because while satisfying the experimental data in one quadrant (for the sake of definiteness, let is be tension–tension) it will thereby introduce a considerable error into other quadrants, in particular, compression–compression. To simultaneously satisfy the test data in all the quadrants, the constants F_{ij} should undergo a stepwise change when passing from quadrant to quadrant; in that case, the number of additional independent experiments grows from three to twelve. The difficulties associated with an optimal definition of the constants F_{ij} were repeatedly mentioned in earlier publications [6, 7]. If we restrict ourselves to the minimum number of independent simplest experiments, namely, nine, then perhaps the Tsai–Wu expressions (4) would be optimal for solving such a problem. For the materials with uniform strength in tension and compression we can recommend an update of the Tsai–Wu criterion according to (19) and (20).

Thus, let us state the rules one should follow when choosing among the four CM strength criteria as outlined above.

1. An orthotropic material with nonuniform strength in tension and compression.

1.1. The Ashkenazi criterion (2) is inapplicable.

1.2. The Tsai–Wu criterion (3) and (4) is applicable.

1.3. The Hoffman criterion (3) and (5) can be used if the stability conditions (6) are met.

1.4. The updated criterion (19) is inapplicable.

2. An orthotropic material with uniform strength in tension and compression.

2.1. The Ashkenazi criterion can be used if the stability conditions (7) are satisfied.

2.2. The Tsai–Wu criterion (3) and (4) can be applied, preferably in the form (13).

2.3. The Hoffman criterion (3) and (5) can be used, preferably in the form (13), if the following stability conditions are met:

$$|F_{ij}| < \frac{2}{\sigma_{ui}\sigma_{uj}}, \qquad i,\ j = x',\ \varphi',\ r;\ \ i \neq j. \tag{21}$$

2.4. The updated criterion (19) is applicable if the stability conditions (20) are satisfied.

3. A transtropic material with nonuniform strength in tension and compression.

3.3. See clause 1.1.

3.2. The Tsai–Wu criterion (3) and (4) can be used if the invariance condition (9) is met.

3.3. The Hoffman criterion (3) and (5) is applicable if the stability conditions (6) and the invariance condition (10) are satisfied.

3.4. See clause 1.4.

4. A transtropic material with uniform strength in tension and compression.

4.1. See clause 2.1.

4.2. The Tsai–Wu criterion (3) and (4) can be used, preferably in the form (13), if the following invariance condition is met: in the isotropy plane ij we have

$$\tau_{uij} = \sigma_{ui} \big/ \sqrt{3}. \tag{22}$$

4.3. The Hoffman criterion (3), (5) is applicable, preferably in the form (13), if the stability conditions (21) and the following invariance condition are satisfied: in the isotropy plane ij we have

$$\tau_{uij} = \frac{\sigma_{ui}}{\sqrt{4 - \dfrac{\sigma_{ui}^6}{\sigma_{ux'}^2 \sigma_{u\varphi'}^2 \sigma_{ur}^2}}}. \tag{23}$$

4.4. See clause 2.4.

5. An isotropic material with nonuniform strength in tension and compression.

5.1. See clause 1.1.

5.2. The Tsai–Wu criterion (3) and (4) can be used if the following invariance condition is met:

$$\tau_u = \sqrt{\frac{\sigma_u^+ \sigma_u^-}{3}}. \tag{24}$$

5.3. The Hoffman criterion (3), (5) is identified with the Tsai–Wu criterion (3) and (4); see clause 5.2.

5.4. See clause 1.4.

6. An isotropic material with uniform strength in tension and compression.

6.1. The Ashkenazi criterion (2) is applicable if the following stability condition is satisfied:

$$\tau_u > \sigma_u / 4. \tag{25}$$

6.2. The Tsai–Wu criterion (3) and (4) can be used, preferably in the form (15), if the invariance condition (14) is met.

6.3. The Hoffman criterion (3) and (5) is identified with the Tsai–Wu criterion (3) and (4); see clause 6.2.

6.4. The updated criterion (19) is applicable if the following stability condition is satisfied:

$$\tau_u > \sigma_u / 2. \tag{26}$$

The above rules impose rather serious restrictions on the applicability of the strength criteria under consideration. Specifically:

(i) an isotropic material with nonuniform strength in tension and compression, for which the relation (24) is not obeyed, cannot have its strength verified by any of the four criteria;

(ii) the same can be said about an isotropic material with uniform strength in tension and compression, for which $\tau_u \le \sigma_u / 4$;

(iii) also, none of the four criteria is applicable to transtropic materials with nonuniform strength in tension and compression, if in the isotropy plane ij neither of the invariance conditions (9) and (10) is met.

The cases (i) and (ii) cover all the possible situations for isotropic materials, where none of the four criteria is applicable (see clauses 5.1–6.4 of the above rules). For transtropic materials, in addition to (iii) there are also other cases of inapplicability for all of the above-listed criteria; we will not dwell on them for they directly follow from clauses 3.1–4.4. As to orthotropic composites, the clauses 1.1–2.4 suggest that the Tsai–Wu criterion (3) and (4) for such materials can always be used for it is stable by definition (4), and there are no grounds for imposing any additional restrictions [such as invariance conditions (8)] on constants F_{ij}. Speaking about stability, if we compare (25) with (26) it can be inferred that the Ashkenazi criterion would most likely have a better stability in comparison to the updated criterion (19).

The above rules were derived through strict mathematical reasoning and represent the prerequisites for the possibility of using the isotropic material failure criteria – Ashkenazi, Tsai–Wu, Hoffman, and (19). If more than one criterion are applicable to a given CM, there is no guarantee that the results of the strength analysis will agree well with each other.

Let us consider an orthotropic CM (organic-plastic material) [14, 15] with the following physical-mechanical characteristics along the principal anisotropy axes X, Y, Z: density $\rho = 1350$ kg/m^3, E_X, E_Y, and E_Z are the Young moduli along the respective principal directions of anisotropy, $E_X = 48{,}600$ MPa, $E_Y = 21{,}300$ MPa, and $E_Z = 7140$ MPa, G_{XY}, G_{YZ}, and G_{ZX} are the shear moduli along the respective principal anisotropy planes, $G_{XY} = 930$ MPa, $G_{YZ} = 900$ MPa, and $G_{ZX} = 850$ MPa, $\sigma_{uX} = 2670$ MPa, $\sigma_{uY} = 1180$ MPa, $\sigma_{uZ} = 390$ MPa,

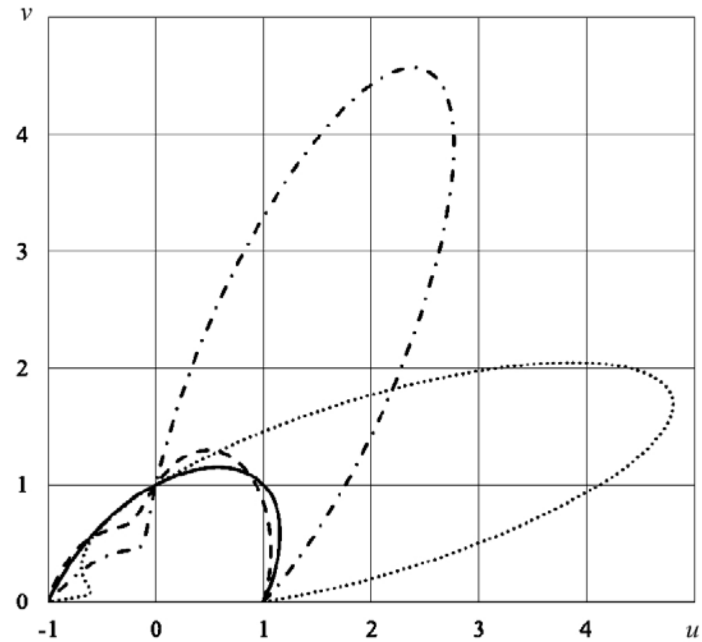

Fig. 1. A comparative analysis of the Ashkenazi and Tsai–Wu criteria for PSS in the planes $\tau_{ij} = 0$.

$\tau_{uXY} = 975$ MPa, $\tau_{uYZ} = 800$ MPa, $\tau_{uZX} = 607$ MPa, $\sigma_{uXY}^{(45)} = 1850$ MPa, $\sigma_{uYZ}^{(45)} = 1600$ MPa, $\sigma_{uZX}^{(45)} = 1215$ MPa, ν_{XY}, ν_{YZ}, and ν_{ZX} are the Poisson's ratios such that the following relation [12] should be obeyed:

$$E_i \nu_{ji} = E_j \nu_{ij}, \qquad i, j = X, Y, Z; \; i \neq j, \tag{27}$$

$\nu_{XY} = 0.28$, $\nu_{YZ} = 0.26$, and $\nu_{ZX} = 0.037$.

Note that this organic-plastic material has a very low shear stiffness in comparison to the tension (compression) stiffness. As evident from the experimental data, this material is subject to clause 2. It is easy to verify that the requirements under clauses 2.3 and 2.4 are not obeyed (the stability conditions are met for neither the Hoffman criterion nor the updated criterion), i.e., among the four strength criteria it is only the Ashkenazi criterion (2) or Tsai–Wu criterion (11)–(13) which can be used for checking strength of this composite. Figure 1 shows limit curves plotted in dimensionless stresses (12) for three plane stress states (PSS) with absence of any shear stresses along the principal anisotropy axes, where the solid curve (the ellipse) corresponds to the Tsai–Wu criterion and is invariant to the choice of axes; the other curves correspond to the Ashkenazi criterion and look different depending on the choice of axes: dashed line ($u = S_X$, $v = S_Y$), dash-and-dot line ($u = S_Y$, $v = S_Z$), and dotted line ($u = S_Z$, $v = S_X$). It is obvious that the difference in data between the Ashkenazi criterion and Tsai–Wu criterion can be rather large (up to five-fold). Considering that all the curves are centrally symmetric, they have been plotted only in the upper half-plane of each of three coordinate systems.

For other PSS types the situation will be equally critical. For example, let us consider PSS in the plane ZX under the condition that $\sigma_X / \sigma_Z = 2.5$ which in the terms of dimensionless stresses (12) is equivalent to $S_X / S_Z = 0.365$. In Fig. 2a the limit curves for this PSS have been plotted only first quadrant considering the simultaneous symmetry of all the curves with respect to abscissa and ordinate axes: solid line by the Tsai–Wu criterion and dotted line by the Ashkenazi criterion. The data obtained by these criteria are seen to be in a very poor agreement: the maximum discrepancy is about five-fold. In this case, the Tsai–Wu criterion has turned out to be more conservative. In Fig. 2b, similar curves have been plotted for PSS in the YZ plane under the condition $\sigma_Y / \sigma_Z = -1.1$ which is equivalent to $S_Y / S_Z = -0.365$. Based on these data from Fig. 2a it is the Ashkenazi criterion which has turned out to be more conservative, and the maximum discrepancy is approximately two-fold. It is also evident that the Ashkenazi limiting surface has an alternating-sign curvature and is made up of mating convex,

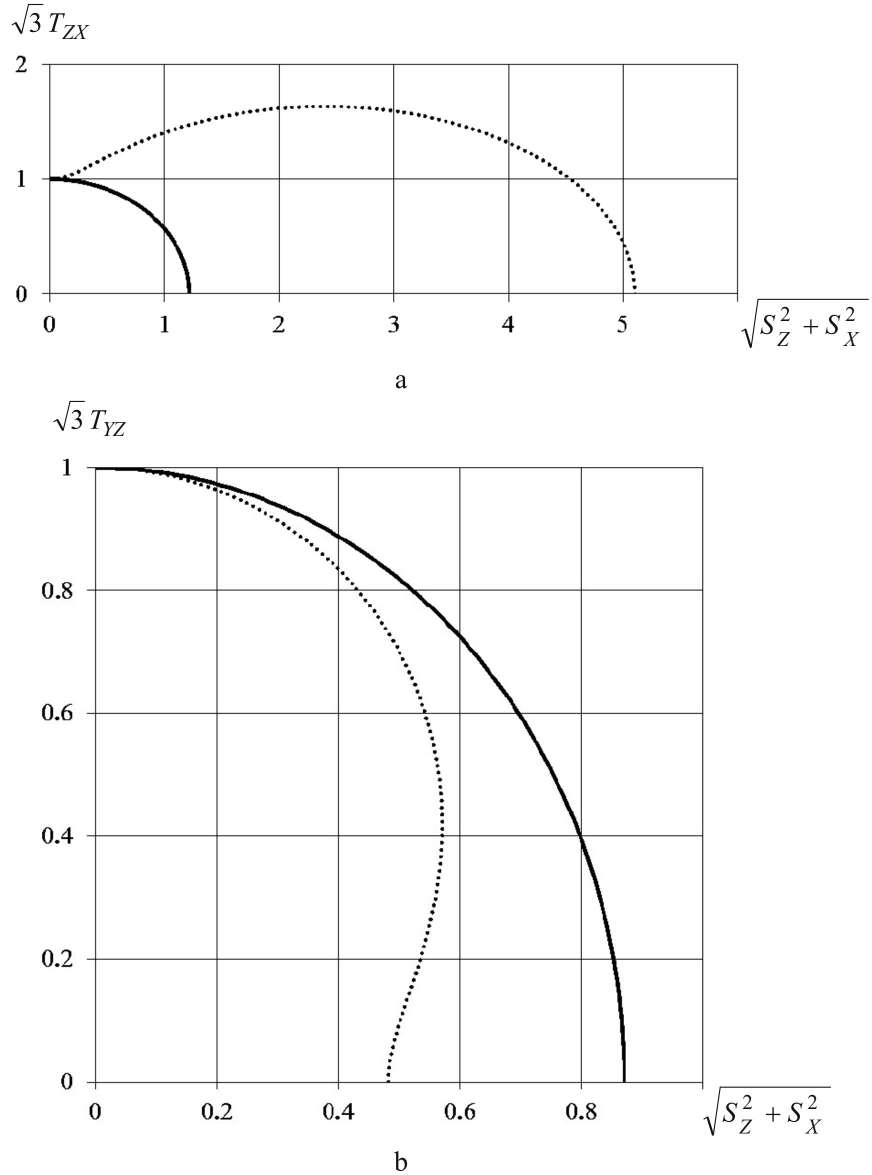

Fig. 2. Further to the comparative analysis of the Ashkenazi and Tsai–Wu criteria.

concave, and saddle-shaped segments, while the Tsai–Wu surface is all-convex. For triaxial SSS the agreement between the criteria can only become worse because PSSs represent partial cases of triaxial stress states.

Let us consider a cylinder made of this material, under loading (1) at $P_0 = 222$ MPa, $x_0 = 0$. The reinforcement is taken to be as follows: $x' = X$, $\varphi' = Y$, and $r = Z$, the reinforcement angle α will be varied, with $\alpha = 0$ corresponding to the axial reinforcement ($x' = x$) and $\alpha = 90°$ to the circumferential one ($x' = \varphi$). The calculations show that for any reinforcement angle the most dangerous point as per either of these two criteria is the point closest to the charge ($x = 0$, $r = R_1$). In this case, the most dangerous instant of time was only slightly dependent on α and the criterion chosen and varied between 10.2 and 16 μs. The values of the maximum strength functions Φ and $\sqrt{\Psi}$ also depended only weakly on reinforcement angle and varied within the ranges 0.86–1.07 and 0.6–0.62, respectively: while the Ashkenazi criterion pointed to the material failure, the Tsai–Wu criterion showed that the material had an 40% strength margin. Interestingly, the SSS at the dangerous point essentially depended on α: the maximum σ_φ, σ_x, and $\tau_{x\varphi}$ varied within the ranges 450–1030, (–380)–365, and (–260)–300 MPa, respectively. The period of radial vibrations of the cylinder was also significantly dependent on the

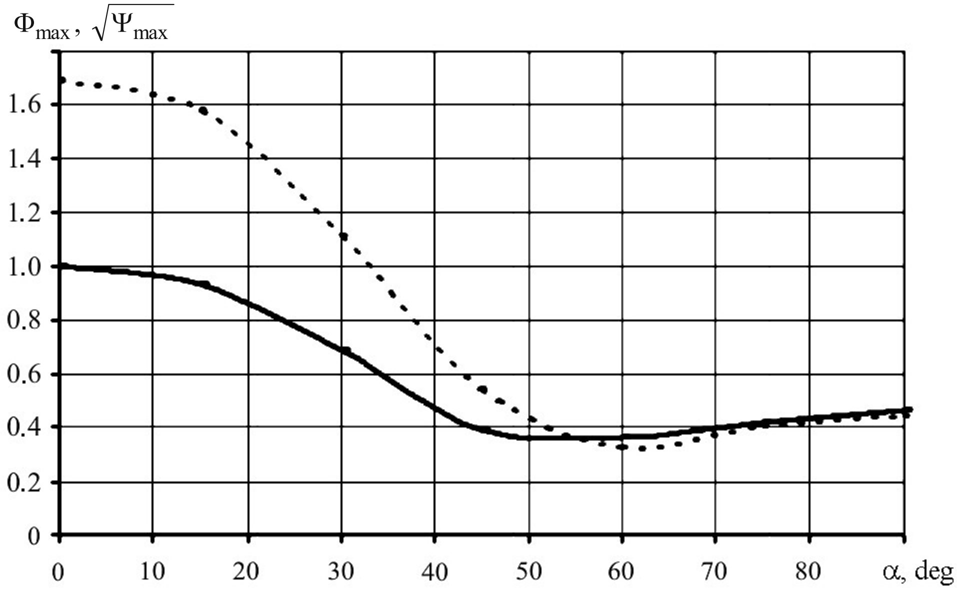

Fig. 3. A comparative analysis of the Ashkenazi and Tsai–Wu criteria for spirally orthotropic cylinders under dynamic loading.

reinforcement angle: at $\alpha = 0$ it was maximum, $2T = 380\,\mu s$, while at $\alpha = 90°$ it was minimum, $2T = 260\,\mu s$. In all the cases the calculations were performed till the instant of time 200 μm in order to cover the maximum of the possible half-periods of the shell radial vibrations.

Thus, for the reinforcement configuration chosen we observed only a qualitative satisfactory agreement between the Ashkenazi and Tsai–Wu criteria: they revealed the same dangerous point and almost the same dangerous instant of time, and the dependence of maximum values of the strength functions Φ and $\sqrt{\Psi}$ on the reinforcement angle was weak. Quantitatively, the discrepancy between these two criteria was almost two-fold.

If the reinforcement configuration is changed to $x' = X$, $\varphi' = Z$, and $r = Y$, the calculations show that even a qualitative agreement between the criteria cannot be achieved. These two criteria will diagnose different dangerous points and different instants of time and the dependence of maximum values of Φ and $\sqrt{\Psi}$ on the reinforcement angle will be as shown in Fig. 3, where the solid line corresponds to the Tsai–Wu criterion and the dotted one to the Ashkenazi criterion. Based on the aforesaid, the fact that the curves coincide along the segment $55° < \alpha < 90°$ is most likely accidental: at $\alpha < 55°$ the Ashkenazi criterion was far more conservative (up to 70%), while at $\alpha > 55°$ both criteria coincided closely, whereas the Tsai–Wu criterion is slightly more conservative.

CONCLUSIONS

1. When analyzing strength of a particular CM one can apply only those criteria which comply with the necessary rules. There are many materials (at least experimental data thereon) to which none of the four strength criteria is applicable.

2. If more than one criterion simultaneously comply with the said rules, the calculated data on the CM strength can differ significantly (up to several hundred percent). Therefore, to finally choose the most adequate strength criterion one should undertake an additional analysis based on specific physical (phenomenological) considerations.

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