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THE ELASTIC THEORY DYNAMIC PROBLEM FOR A TRANSTROPIC MULTILAYER SPHERE

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The analytical solution is obtained in quadratures of the elastic theory dynamic centrally-symmetric boundary problems of the same class for multilayer transtropic spheres, the sonic speed in the radial direction of which tends to infinity. Convergence and accuracy of the derived solutions are analyzed. It is shown that the analytical solutions obtained can be used as the first approximation for the cases where the sonic speed in materials is finite.

Keywords: multilayer transtropic sphere, dynamics, central symmetry, analytical solution, finite difference method, stress-strain state.

Introduction. Multilayer spherical thin- and thick-walled shells find wide application in various fields of state-of-the-art engineering. Elastic reinforced composite materials (CM) frequently serve as laminate layers. In spherical vessel applications, all or some shell layers should be treated as spherically transtropic bodies [1] with elastic characteristics E and ν in radial direction r (which is the rotational symmetry axis) and E_0 and ν_0 in the orthogonal isotropy plane (or more precisely sphere). Conditions $E \equiv E_0$ and $\nu \equiv \nu_0$ are valid for isotropic layers.

In many cases, such structural components as multilayer closed spheres are subjected to centrally symmetrical pulse inner or outer pressure, i.e., during blast loading by explosive material charge. The dynamic behavior of such shells has been experimentally studied in [2]. The analytical calculation of nonstationary centrally symmetric stress-strain state of the above structural components is of practical interest, since it can be used for the shell strength assessment under the particular blast loading conditions. Obtaining solutions based on the elastic theory equations is quite critical, insofar as results based on various shell theories can be inaccurate due to such factors as large thickness of shell walls, short-term loading and steep wavefronts. Precise analytical solutions of such problems are quite cumbersome even for isotropic layers and are available only for few particular cases, such as one-layered spheres and spherical cavities [3, 4]. This study is aimed at development of calculation techniques for solving dynamical centrally symmetrical boundary problems of the elastic theory for multilayered spherically transtropic spheres.

Problem Mathematical Statement. Let us use the spherical coordinate system r, ψ, where r is a radial coordinate and ψ is angular coordinate orthogonal to r. Due to the central symmetry of the problem, usage of the third coordinate is not required, since all directions along ψ are identically equivalent.

Within this coordinate system motion equations will take the following form [3, 4]

$$\frac{\partial \sigma_r}{\partial r} + \frac{2(\sigma_r - \sigma_\psi)}{r} = \rho_i \frac{\partial^2 u}{\partial t^2}, \tag{1}$$

where t is time, i is layer number (layer enumeration is performed from the internal to the external ones), ρ_i is material density of the layer under study, u is radial displacement, and σ_r and σ_ψ are radial and circumferential stresses, respectively.

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The Cauchi geometrical relations can be written as

$$\varepsilon_r = \frac{\partial u}{\partial r}, \qquad \varepsilon_\psi = \frac{u}{r}, \tag{2}$$

where ε_r and ε_ψ are radial and circumferential strains, respectively.

Hooke's law for spherically transtropic layer with account of Eq. (2) and a central symmetry can be written as [1]:

$$E\frac{\partial u}{\partial r} = \sigma_r - 2\nu\sigma_\psi, \qquad E\frac{u}{r} = \sigma_\psi(1-\tilde{\nu}) - \nu\sigma_r, \tag{3}$$

where $\tilde{\nu}$ is reduced Poisson's ratio,

$$\tilde{\nu} = \frac{E}{E_0}(\nu_0 - 1) + 1, \tag{4}$$

while $\tilde{\nu} \equiv \nu_0 \equiv \nu$ for isotropic material. In (3) and (4), index i characterizing the layer number is omitted for brevity.

In case of a multilayer shell, contact between layers is considered ideal:

$$\sigma_r^- = \sigma_r^+, \qquad u^- = u^+ \qquad \text{for} \qquad r = R_{i,i+1}, \tag{5}$$

where $R_{i,i+1}$ is the radius of contact surface of ith layer with $(i+1)$th one, whereas "–" and "+" indices correspond to contacting layers from below and from above, respectively.

Boundary conditions are

$$\sigma_{r}\big|_{r=R_j} = -P_j(t), \qquad j = 1,\, 2, \tag{6}$$

where R_1 and R_2 are the inner and outer shell radii, respectively, and $P_1(t)$ and $P_2(t)$ are internal and external pulse pressure on the shell.

Zero initial conditions are postulated:

$$u = \frac{\partial u}{\partial t} = 0 \qquad \text{for} \qquad t = 0. \tag{7}$$

Solution technique of boundary problem (1)–(7) significantly depends on the principal determinant value of the system (3):

$$D = 1 - \tilde{\nu} - 2\nu^2 \geq 0. \tag{8}$$

Although $D > 0$ for the majority of materials, there is a certain class of materials (e.g., incompressible ones) for which $D = 0$.

Provided stringent inequality $D_i > 0$ is valid for all layers under study, the problem solution should be sought in displacements. Solution of system (3) with respect to stresses and substitution of the solutions obtained into (1) yields the following wave equation:

$$\frac{1}{c_i^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - 2\beta_i \frac{u}{r^2}, \tag{9}$$

where

$$\beta_i = \frac{1-\nu_i}{1-\tilde{\nu}_i}, \qquad c_i = \sqrt{\frac{E_i(1-\tilde{\nu}_i)}{\rho_i(1-\tilde{\nu}_i - 2\nu_i^2)}}. \tag{10}$$

Boundary conditions (6) can be expressed as

$$\begin{aligned} &\frac{E_1}{D_1}\left[(1-\tilde{\nu}_1)\frac{\partial u}{\partial r}+2\nu_1\frac{u}{r}\right]=-P_1(t), \qquad r=R_1,\\ &\frac{E_N}{D_N}\left[(1-\tilde{\nu}_N)\frac{\partial u}{\partial r}+2\nu_N\frac{u}{r}\right]=-P_2(t), \quad r=R_2, \end{aligned} \tag{11}$$

where N is the last (upper) layer number.

The initial conditions (3) remain unchanged, while the contact ones (3) take the following form

$$E_i D_{i+1}\left[(1-\tilde{\nu}_i)\frac{\partial u}{\partial r}+2\nu_i\frac{u}{r}\right]=E_{i+1}D_i\left[(1-\tilde{\nu}_{i+1})\frac{\partial u}{\partial r}+2\nu_{i+1}\frac{u}{r}\right]. \tag{12}$$

Though analytical integration of system (7), (9)–(12) is quite problematic, it is possible in some cases. If the following condition is valid for all layers

$$\beta_i=1, \qquad \text{i.e.,} \qquad \nu_i=\tilde{\nu}_i, \tag{13}$$

then an analytical solution exists in a form of quite complex linear superpositions of spherical running and reflected waves (of the D'Alembert type) [3, 4]. Obtaining such solutions for quite large time values, especially in case of a multilayer shell, is very difficult, though realizable in principle. Noteworthy is that requirement (13) is valid not only for isotropic materials, but for a certain class of spherically transtropic ones, i.e., in the case of central symmetry, certain spherical orthotropic shells will have similar behavior to the isotropic ones.

Violation of condition (13) indicates that no solution in a form of superpositions of spherical running and reflected waves is likely to exist, while the problem analytical solution becomes more complicated by an order of magnitude.

In order to obtain relatively accurate solutions of problem (9)–(12) for quite large time values in case of $D_i>0$, it is expedient to apply the numerical finite difference methods (FDM). The explicit FDM cross-type scheme of the second order of accuracy has demonstrated its applicability to such calculations in [2].

If conditions $D_i=0$ are valid for each shell layer, the boundary problem (1)–(7) is precisely integrable in quadratures and for some loading conditions – in elementary functions.

We'll show that these solutions can also be applied as the first approximation to assessment of dynamical stress-strain state of multilayer spheres for which $D_i>0$.

Problem Solution for $D_i=0$. Since the principal determinant of system (3) is equal to zero, stresses cannot be unambiguously expressed via strains. Let us discuss in detail the methods for obtaining analytical solutions for one- and two-layered spheres. Solutions for the case where the number of layers is more than two can be obtained by analogy following the technique described for the two-layered shell.

One-Layer Sphere. Taking into account that $D=0$, we have

$$1-\tilde{\nu}=2\nu^2. \tag{14}$$

By substituting (14) into (3), we get an ordinary differential equation in relation to deflection:

$$\nu\frac{\partial u}{\partial r}+\frac{u}{r}=0, \tag{15}$$

solution of which yields

$$u=y(t)\big/r^M, \qquad M=1/\nu, \tag{16}$$

where $y(t)$ is a one-variable function, which requires further specification.

Upon substituting (16) into any of the two equations of system (3), we get

$$2(\sigma_r - \sigma_\psi) = (2 - M)\sigma_r - \frac{EM^2}{r^{1+M}} y(t). \tag{17}$$

Taking into account

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\rho}{r^M} y''(t), \tag{18}$$

and substituting (17) and (18) in motion equation (1), we get

$$\frac{\partial}{\partial r}(r^{2-M}\sigma_r) = \rho r^{2-2M} y'' + EM^2 r^{-2M} y. \tag{19}$$

Introduce the following designation

$$\Phi(a,\ b,\ c) = \int_a^b \frac{dx}{x^c} = \begin{cases} \dfrac{1}{1-c}(b^{1-c} - a^{1-c}), & c \neq 1, \\ \ln \dfrac{b}{a}, & c = 1. \end{cases} \tag{20}$$

Integration of (19) over r from the inner shell radius R_1 to the outer one R_2 with account taken of boundary conditions (6) yields an ordinary differential equation for specification of function $y(t)$:

$$y'' + \omega^2 y = F(t), \tag{21}$$

where

$$F(t) = \frac{R_1^{2-M} P_1(t) - R_2^{2-M} P_2(t)}{\rho\Phi(R_1;\ R_2;\ 2M-2)}, \qquad \omega = \sqrt{\frac{EM^2\Phi(R_1;\ R_2;\ 2M)}{\rho\Phi(R_1;\ R_2;\ 2M-2)}}. \tag{22}$$

By integrating (21) with account of zero initial conditions $[y(0) = y'(0) = 0]$ which follow from (7) we get the sought function $y(t)$ [6]:

$$y(t) = \omega^{-1} \int_0^t F(\xi) \sin \omega(t - \xi) d\xi. \tag{23}$$

Upon determination of function $y(t)$ and integration of the motion equation in form (19) from R_1 to the current radius r, we get

$$\sigma_r = r^{M-2}[EM^2 y(t)\Phi(R_1;\ r;\ 2M) + \rho F(t)\Phi(R_1;\ r;\ 2M-2) - R_1^{2-M} P_1(t) - \rho\omega^2 y(t)\Phi(R_1;\ r;\ 2M-2)]. \tag{24}$$

Then from Hooke's law we determine

$$\sigma_\psi = \frac{M}{2}\sigma_r + \frac{EM^2}{2}\frac{y(t)}{r^{1+M}}. \tag{25}$$

In particular, at the internal surface $(r = R_1)$

$$\sigma_\psi = -\frac{M}{2} P_1(t) + \frac{EM^2}{2}\frac{y(t)}{R_1^{1+M}}. \tag{26}$$

Similarly, at the external surface ($r = R_2$)

$$\sigma_\psi = -\frac{M}{2} P_2(t) + \frac{EM^2}{2} \frac{y(t)}{R_2^{1+M}}. \tag{27}$$

Two-Layer Sphere. $D_1 = D_2 = 0$.

We write for the inner sphere

$$u = \frac{y_1(t)}{r^{M_1}}, \tag{28}$$

and for the outer sphere –

$$u = \frac{y_2(t)}{r^{M_2}}. \tag{29}$$

Using designation $y_1(t) = y(t)$, we get from the contact condition (5)

$$y_1(t) = y(t), \qquad y_2(t) = R_k^{M_2 - M_1} y(t), \tag{30}$$

where R_k is the contact surface radius.

Thus, in view of (30), the problem for a two-layer shell is also reduced to specification of a single governing function.

Similarly to the case of one-layer sphere, from (3), (28), and (29) we get two equations of type (17) for the inner and outer layers. By substituting the latter relations into the respective motion equations we obtain two equations of type (19). Integration of the first equation from R_1 to R_k and the second one from R_k to R_2 yields

$$\begin{aligned} &R_1^{2-M_1} P_1(t) + R_k^{2-M_1} \sigma_k(t) = \rho_1 y'' \Phi(R_1;\, R_k;\, 2M_1 - 2) + E_1 M_1^2 y \Phi(R_1;\, R_k;\, 2M_1), \\ &-R_k^{2-M_2} \sigma_k(t) - R_2^{2-M_2} P_2(t) = \rho_2 y'' R_k^{M_2-M_1} \Phi(R_k;\, R_2;\, 2M_2 - 2) + E_2 M_2^2 R_k^{M_2-M_1} y \Phi(R_k;\, R_2;\, 2M_2), \end{aligned} \tag{31}$$

where $\sigma_k(t)$ is the radial stress at the contact boundary $r = R_k$.

Upon excluding the unknown $\sigma_k(t)$ from system (31) we reduce the system to Eq. (21), where ω and $F(t)$ are replaced by other variables, i.e.,

$$\begin{aligned} &F(t) = [R_1^{2-M_1} R_k^{M_1-M_2} P_1(t) - R_2^{2-M_2} P_2(t)]/\Omega, \\ &\omega = [E_1 M_1^2 R_k^{M_1-M_2} \Phi(R_1;\, R_k;\, 2M_1) + E_2 M_2^2 R_k^{M_2-M_1} \Phi(R_k;\, R_2;\, 2M_2)]^{1/2} \big/ \sqrt{\Omega}, \\ &\Omega = \rho_1 R_k^{M_1-M_2} \Phi(R_1;\, R_k;\, 2M_1 - 2) + \rho_2 R_k^{M_2-M_1} \Phi(R_k;\, R_2;\, 2M_2 - 2). \end{aligned} \tag{32}$$

Given $y(t)$, find σ_r. At the inner layer ($R_1 \le r \le R_k$) this procedure is the same as in the case of one-layer shell. At the outer layer ($R_k \le r \le R_2$) integration over r can be performed initially from R_1 to R_k, which yields $\sigma_k(t)$, and then, given $\sigma_k(t)$, from R_k to r. However, in case of a two-layer shell, it is more expedient to perform integration over r directly from R_2 to r, omitting calculation of $\sigma_k(t)$, in view of (30). Then at the outer layer we get

$$\begin{aligned} \sigma_r = r^{M_2-2} [&E_2 M_2^2 R_k^{M_2-M_1} y(t) \Phi(R_2;\, r;\, 2M_2) + \rho_2 F(t) \Phi(R_2;\, r;\, 2M_2 - 2) \\ &- R_2^{2-M_2} P_2(t) - \rho_2 \omega^2 R_k^{M_2-M_1} y(t) \Phi(R_2;\, r;\, 2M_2 - 2)]. \end{aligned} \tag{33}$$

Given σ_r, value of σ_ψ is calculated from Hooke's law similar to the case of a one-layer sphere

$$\begin{aligned} \sigma_\psi &= \frac{M_1}{2}\sigma_r + \frac{E_1 M_1^2}{2}\frac{y(t)}{r^{1+M_1}}, \qquad R_1 \le r \le R_k, \\ \sigma_\psi &= \frac{M_2}{2}\sigma_r + \frac{E_2 M_2^2}{2} R_k^{M_2-M_1}\frac{y(t)}{r^{1+M_2}}, \qquad R_k \le r \le R_2. \end{aligned} \tag{34}$$

If the number of shell layers N exceeds two, we'll have N relations instead of (30)

$$\begin{cases} y_1(t) = y(t), \\ y_2(t) = R_{k_1}^{M_2-M_1} y(t), \\ y_3(t) = R_{k_2}^{M_3-M_2} R_{k_1}^{M_2-M_1} y(t), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_N(t) = R_{k_{N-1}}^{M_N-M_{N-1}} \cdot \ldots \cdot R_{k_1}^{M_2-M_1} y(t). \end{cases} \tag{35}$$

System (31) is replaced by the system of N equations of the same type. By excluding unknown terms $\sigma_{k_1}(t)$, $\sigma_{k_2}(t)$, ..., $\sigma_{k_{N-1}}(t)$, the system is reduced to the equation of type (21) with other representations of ω and $F(t)$: these become more cumbersome with increasing N. After determination of $y(t)$, in order to calculate $\sigma_r(t)$ in all layers, intermediate procedures for calculation of contact radial stresses should be executed at least $N-2$ times. Value of σ_ψ is easily determined from Hooke's law via the known values of σ_r similarly to the above two cases.

Insofar as we discuss model problems only for one- and two-layer spheres, the details of calculation procedures can be omitted.

Study of the Analytical Method Accuracy and Convergence. An analytical method could be applied for convergence study of wave solutions for $D_i \to +0$ in relation to the precise ones earlier obtained for $D_i = 0$. Similar approach has been used in [7] for incompressible cylinders under plane strain conditions.

However such approach is quite cumbersome and not enough demonstrative. Since this study covers both analytical solutions (for $D_i = 0$) and FDM numerical ones ($D_i > 0$) of the problems, it would be more expedient to compare numerical results for shells having low but positive values of D_i with analytical results for the same shells having identically zero values of D_i.

We restrict our attention to the case of internal pulse loading:

$$P_2(t) \equiv 0, \qquad P_1(t) = P_0 e^{-t/t_0} H(t), \tag{36}$$

where $H(t)$ is the Heaviside function.

Solution of Eq. (21) for loading (36) has the following form

$$y = \frac{A}{\omega^2 + t_0^{-2}}\left(e^{-t/t_0} - \cos\omega t + \frac{\sin\omega t}{\omega t_0}\right), \tag{37}$$

in particular, for a one-layer shell:

$$A = \frac{P_0 R_1^{2-M}}{\rho\Phi(R_1;\, R_2;\, 2M-2)}, \tag{38}$$

and for a two-layer sphere:

$$A = \frac{P_0 R_1^{2-M_1} R_k^{M_1-M_2}}{\Omega}. \tag{39}$$

It would appear reasonable that ω values will be different for one- and two-layer shells (22) and (32). For $t_0 \to \infty$, we use formulas (37)–(39) to get solutions for the case of internal surface loading by the Heaviside step function, which case is quite peculiar, since the slope of exponential pressure pulse in (36) can be quite flat due to the closed space within a closed spherical shell.

In test and further calculations we assumed to $t_0 = 10^{-5}$ s and to $t_0 = \infty$ (loading by the Heaviside function). Two-layer shell with inner radius $R_1 = 0.5$ m and outer radius $R_2 = 0.6$ m was studied. Both layers had the same thickness, therefore the contact surface radius was equal to $R_k = (R_1 + R_2)/2 = 0.55$ m.

As materials of the layers, fiberglass reinforced plastics with the following characteristics were considered [8]:

Material No. 1: $\rho = 2000$ kg/m^3, $E = 5.7 \cdot 10^4$ MPa, $E_0 = 1.4 \cdot 10^4$ MPa, $\nu_0 = 0.4$, and $\nu = 0.28$. Such material is characterized by relatively high D value (about 2.3). In order to make this value negligibly small (for the purpose of the analytical method convergence analysis), the above characteristics except ν remained unchanged, whereas the value $\nu = 1.1$ used in the FDM numerical wave simulations provided hundred-fold reduction of D. In analytical calculations we assumed $\nu = 1.1052$, which provided identically zero D.

Material No. 2: $\rho = 1500$ kg/m^3, $E = 7 \cdot 10^3$ MPa, $E_0 = 2 \cdot 10^4$ MPa, $\nu_0 = 0.15$, and $\nu = 0.14$. This material also have a relatively high D value (about 0.26). Incorporation of $\nu = 0.38$ into numerical calculations resulted in 30-fold reduction of D. In analytical calculations for $D = 0$ we assumed $\nu = 0.3857$.

It was postulated that the internal layer consisted of material No. 1, while the external one – from material No. 2. Noteworthy is that material No. 1 is significantly more rigid (reinforced) in the radial direction, while material No. 2 is more rigid in the circumferential direction (isotropic sphere).

The results of numerical (wave and analytical calculations presented in Fig. 1) illustrate variation of dimensionless circumferential stresses σ_ψ / P_0 in time at the inner (curves *1* and *2*) and outer (curves *3* and *4*) surfaces of the two-layer sphere. It is seen that the exponential and the Heaviside loading curves are close to each other, which confirms convergence of wave solutions for $D_i \to 0$ and the analytical ones for $D_i = 0$. The discrepancy between these curves is practically zero for the exponential loading and does not exceed 2.5% for the Heaviside loading ($t_0 = \infty$). This difference can be attributed to the much higher power (energy) of the Heaviside loading, as compared to the exponential one, within a closed space. Therefore the respective loading energy transferred to the shell is also much higher, which results in more pronounced discrepancy between curves. The results in Fig. 1 confirm the accuracy of the above analytical solutions for materials with $D = 0$.

Thus the above analytical method is equally applicable for materials with $D = 0$ and $D \approx 0$. The question arises of whether solutions can be found for much higher than zero values of D. The numerical FDM procedure seems to be quite applicable, but the analytical solution with a certain degree of accuracy can be obtained as well. Discuss this issue in more detail for particular examples of one- and two-layer spheres exponentially loaded by internal pulse loading according to (36).

One-Layer Sphere. Consider a composite material with $D > 0$. There are the following three options, which allow one to replace it by a hypothetical material with $D = 0$:

1) the true value of $\tilde{\nu}$ is used, while ν is calculated from condition (14):

$$\nu = \sqrt{(1 - \tilde{\nu})/2}; \tag{40}$$

2) the material is treated as incompressible: $\nu = \tilde{\nu} = 1/2$;

3) the true value of ν is used, while $\tilde{\nu}$ is calculated from condition (14):

$$\tilde{\nu} = 1 - 2\nu^2. \tag{41}$$

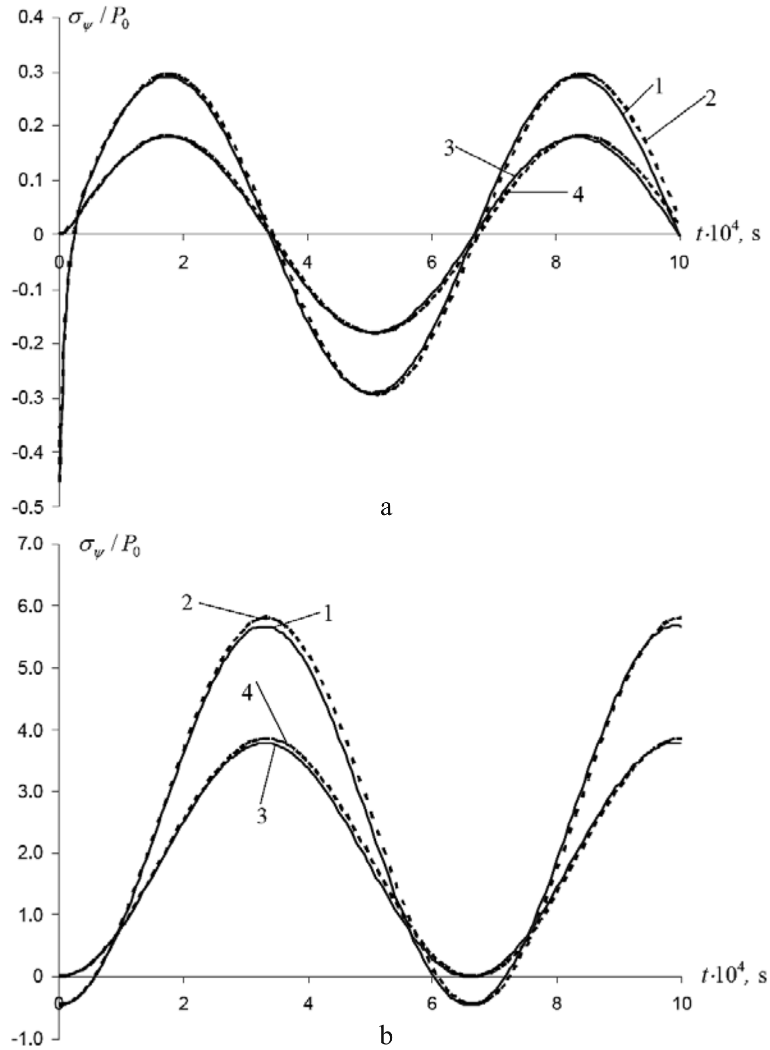


Fig. 1. Test calculations for exponential (a) and Heaviside function (b) loading. Here and in Fig. 5: (*1*, *3*) numerical results; (*2*, *4*) analytical results.

The following three types of composite materials are considered:

Material No. 1 (earlier described), $E/E_0 = 4.07$ and $D = 2.286$.

Material No. 2 (earlier described), $E/E_0 = 0.35$ and $D = 0.2583$.

Material No. 3: of steel type, $\rho = 8000$ kg/m^3, $E = E_0 = 2 \cdot 10^5$ MPa, $\nu = \nu_0 = \tilde{\nu} = 0.25$, $E/E_0 = 1$, and $D = 0.625$.

The numerical and analytical calculation results are shown in Figs. 2–4. Their analysis shows that the second and third options for determination of a hypothetical material are not adequate, in contrast to the first one. Curves *1* and *2* in Figs. 1–5 are located close to each other, while the best correlation is obtained when $E/E_0 \geq 1$. Although for material No. 2 this correlation is weaker, the analytical calculation can be used as the first approximation. The basic oscillation frequency remains unchanged, while curves *2* describe precise curves *1* "on average": curves *1* oscillate around curves *2* with a small amplitude. Then, replacing the real composite material with a hypothetical one, we'll apply the first option – keep the true $\tilde{\nu}$ constant and calculate the approximate (hypothetical) ν by formula (40).

The results obtained confirm that, in case of spherical symmetry, it is more convenient to replace even isotropic materials by hypothetical transtropic ones. In order to satisfy the condition $D = 0$ for an isotropic material, it should be considered incompressible ($\nu = 0.5$). As shown in Fig. 4, this yields worse results than replacement of isotropic material by a hypothetical transtropic one by formula (40).

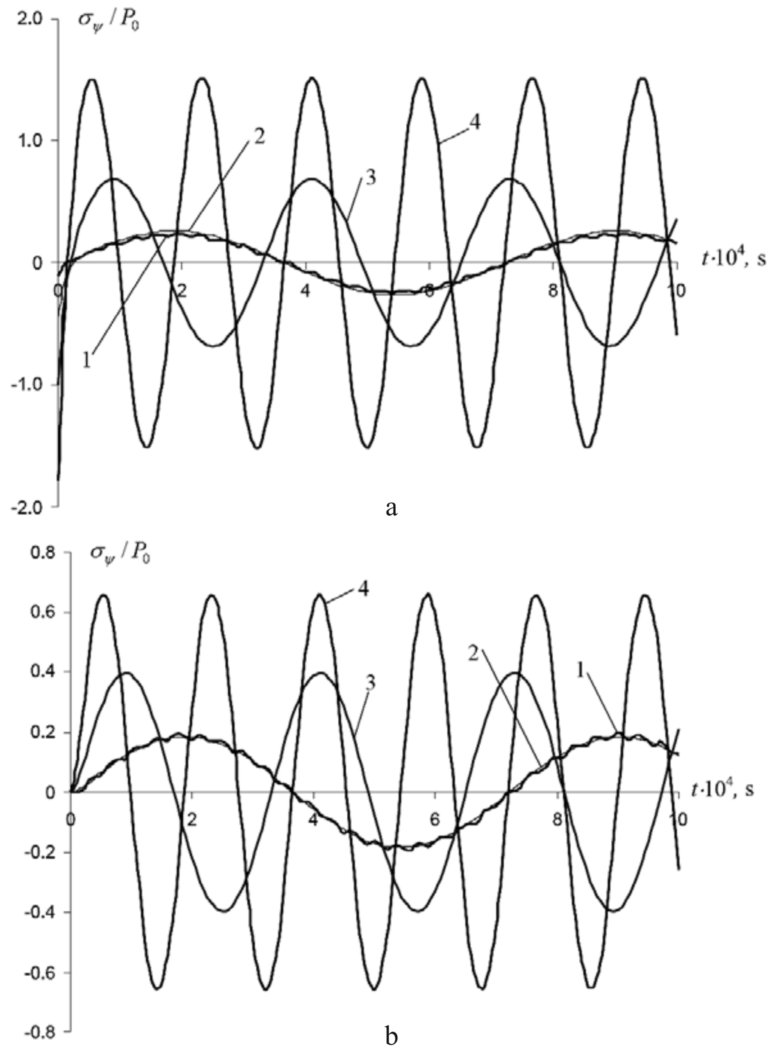
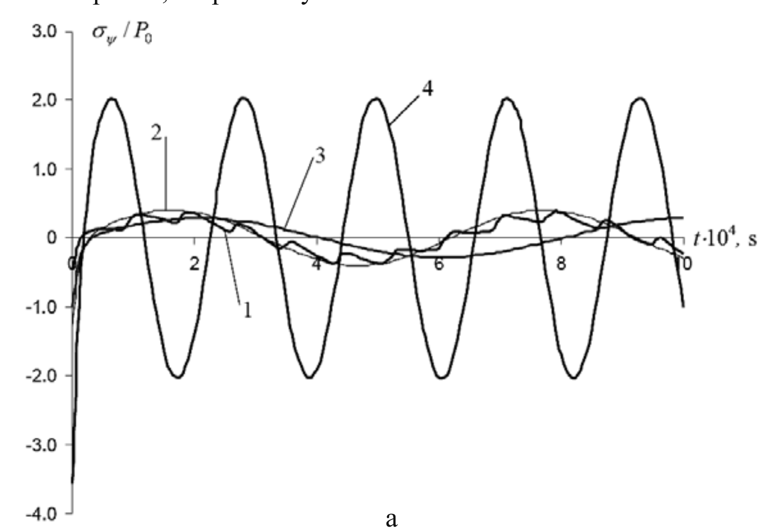
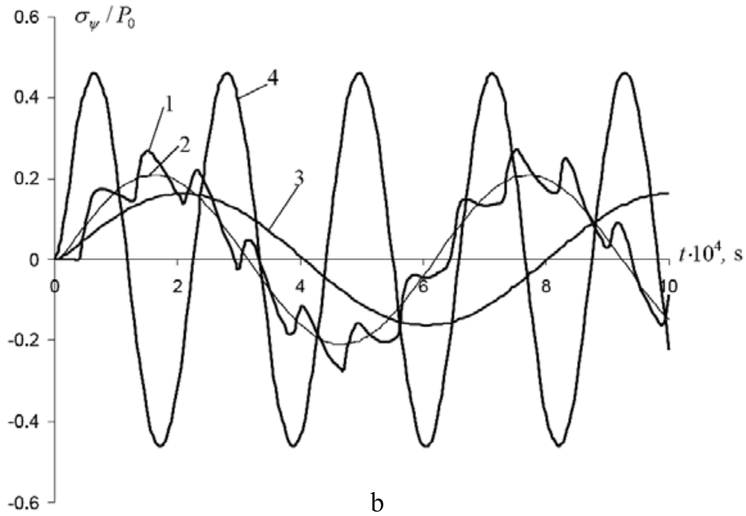


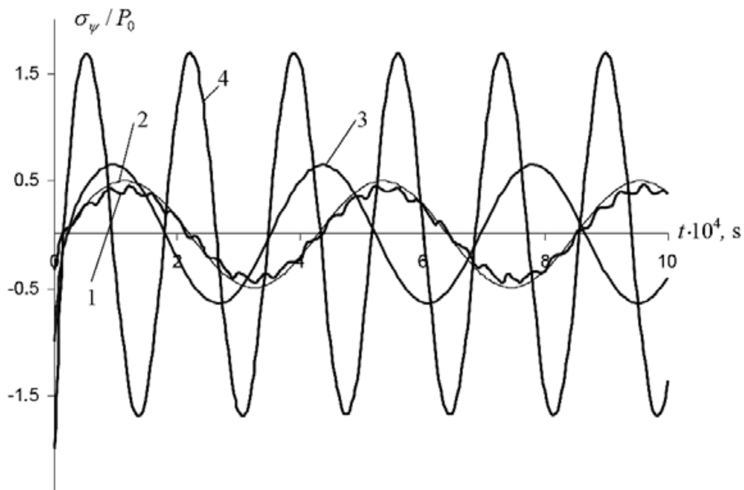
Fig. 2. Variation of σ_ψ / P_0 vs t for a sphere from material No. 1 at the internal (a) and external (b) shell surfaces. Here and in Figs. 3 and 4: (*1*) a precise wave solution for a real composite material obtained via FDM; (*2*, *3*, *4*) analytical calculations for hypothetical composite materials obtained by the first, second, and third options, respectively.





b

Fig. 3. Variation of σ_ψ / P_0 vs t for a sphere from material No. 2 at the internal (a) and external (b) shell surfaces.



a

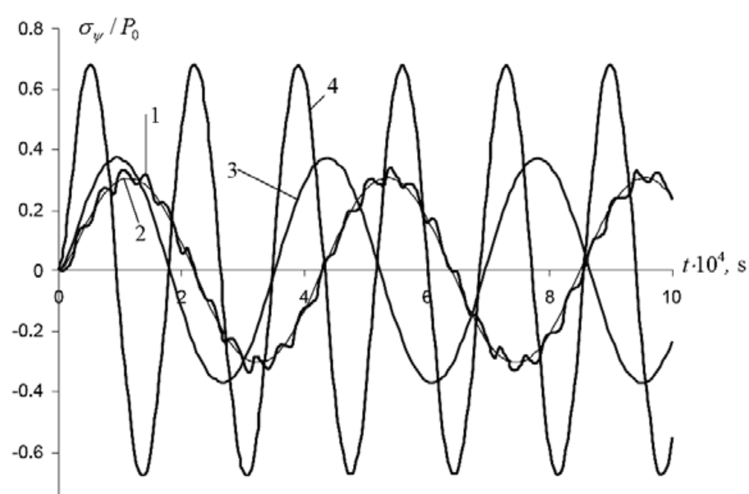


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Fig. 4. Variation of σ_ψ / P_0 vs t for a sphere from material No. 3 at the internal (a) and external (b) shell surfaces.

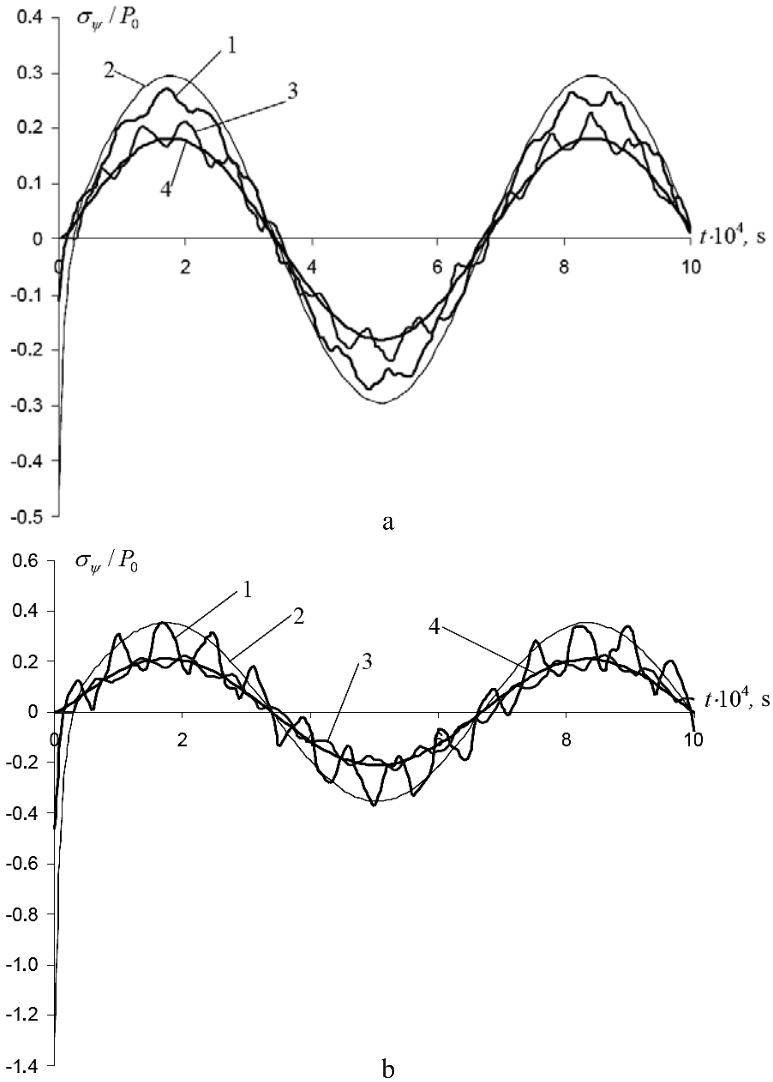

Fig. 5. Variation of σ_ψ / P_0 vs t for a two-layer sphere with: the internal layer from material No. 1 and external layer from material No. 2 (a); the internal layer from material No. 2 and external layer from material No. 1 (b).

Two-Layer Sphere. We discuss two combinations of layers of the same thickness:

1) the internal layer is from material No. 1 and external layer – from material No. 2;

2) the internal layer is from material No. 2 and external layer – from material No. 1.

The calculation results are presented in Fig. 5, which demonstrates a good fit of the analytical (approximate with $D_i = 0$) results with numerical ones obtained by FDM for exact problem statement ($D_i > 0$). In case of analytical approach, the basic frequency of the shell radial oscillations remains unchanged. Meanwhile curves constructed by the numerical method ($D_i > 0$) oscillate with a small amplitude around relatively smooth analytically constructed curves ($D_i = 0$). In some cases, analytical curves envelope the respective FDM ones, and the approximation error can be treated as a safety factor.

CONCLUSIONS

1. We have proposed an effective analytical method for integration in quadratures of boundary nonstationary centrally symmetric problems of the elastic theory for multilayer transtropic spheres in case of $D_i = 0$.

2. The proposed method accuracy and convergence of solutions for $D_i \to 0$ and the analytical solutions for $D_i = 0$ has been analyzed.

3. It is shown that the analytical solutions obtained for composite materials with $D > 0$ can be used as the effective first approximation.

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