Strength of Materials, Vol. 42, No. 4, 2010

SCIENTIFIC AND TECHNICAL SECTION

APPROXIMATE ANALYTICAL DETERMINATION OF VIBRODIAGNOSTIC PARAMETERS OF THE PRESENCE OF A CRACK IN AN ELASTIC BODY UNDER SUPERHARMONIC RESONANCE

V. V. Matveev and O. E. Boginich UDC 620.178.5:620.179

We discuss an approximate analytical method for calculating the main parameter of nonlinearity of vibration of an elastic body with a closing crack, which is simulated by a single-degree-of-freedom system with an asymmetric bilinear characteristic of the restoring force, under a strong 2nd-order superharmonic resonance.

Keywords: nonlinear vibration, superharmonic resonance, vibrodiagnostics of fatigue damage.

Introduction. As mentioned in [1, 2], one of the least studied lines in the field of assessment of possible changes of the vibration state of a structural member during its long-term operation and in the development of vibrodiagnostic methods for detecting damages such as fatigue cracks is to establish relations between a closing crack parameters and the vibration parameters in nonlinear resonances.

Some approximate analytical solutions were derived earlier for the determination of vibrodiagnostic parameters of the above-mentioned type of damage in an elastic body under forced vibration in the region of a strong and weak 1/2-order resonances [2, 3] as well as a weak 2nd-order superharmonic resonance [1].

To further elaborate on the publication [1], we will discuss here an approximate analytical solution for the case of a strong superharmonic resonance.

Approximate Solution Procedure. The procedure is based on the fundamentals as stated in [1, 2]:

1) in the case of a fairly small mode I crack we can disregard some difference in the modes of vibration of an elastic body between its deformation half-cycles, where the crack is open and closed, respectively, and for the given natural mode of vibration we can represent the body by a model of a single-degree-of-freedom system with an asymmetric bilinear characteristic of the restoring force. The forced vibration of this system is described by a differential equation of the form

$$\frac{d^2u}{dt^2} + 2h\frac{du}{dt} + \omega^2[1 - 0.5\alpha(1 + \operatorname{sign} u)]u = q_0 \sin \nu t. \tag{1}$$

Here ω is the natural frequency of an intact body of a body with a closed crack, α is the parameter that represents a relative change of the natural frequency squared,

$$\alpha = 1 - \left(\frac{\omega_o}{\omega}\right)^2, \tag{2}$$

where ω_o is the natural frequency of the elastic body when the crack is open.

Pisarenko Institute of Problems of Strength, National Academy of Sciences of Ukraine, Kiev, Ukraine.
Translated from Problemy Prochnosti, No. 4, pp. 5 – 19, July – August, 2010. Original article submitted April 6, 2009.

The parameter α depends on the crack type, relative size, and location as well as on the relative dimensions of the structural member and its mode of vibration and can be defined in terms of the energy characteristic of the member damage, $\kappa = \Delta\Pi_c/\Pi_0$,

$$\alpha = \frac{\kappa}{1+\kappa}, \tag{3}$$

where Π_0 is the potential energy of deformation in an intact body and $\Delta\Pi_c$ is the increment of the potential energy due to an increase in the body's compliance because of the presence of a crack and is defined in terms of the stress intensity factor. The examples of how to determine the parameters κ and α for rods of rectangular cross section under tension and bending as well as for rectangular plates in bending under the conditions of deformation by natural modes of vibrations and in the presence of various types of mode I cracks were discussed in the publications [3, 4] and [5, 6], respectively;

2) in a superharmonic resonance $\nu = \omega_0/2$, in addition to the fundamental – first – harmonic $A_1 \sin(\nu t - \gamma_1)$ corresponding to the exciting force frequency ν, there arises vibration with a spectrum of harmonic components of the fundamental resonance ($\nu = \omega_0$), which is determined using an asymptotic method of the nonlinear mechanics [1, 7]. Here, ω_0 is the natural frequency of an elastic body when the crack in it is closing [2],

$$\omega_0 = \frac{2\omega\omega_o}{\omega + \omega_o} = \frac{2\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}\omega; \tag{4}$$

3) the first harmonic amplitude A_1 roughly corresponds to solving problem of forced vibrations of a linearized system with a natural frequency ω_0,

$$A_1 = q_0[(\omega_0^2 - \nu^2)^2 + 4h^2\nu^2]^{-1/2} \tag{5}$$

or an approximately initial linear system, i.e., with $\omega_0 \equiv \omega$.

In view of the fundamentals outlined above, a solution to the essentially nonlinear equation (1) for the 2nd-order superharmonic resonance ($\nu = 0.5\omega_0$) is given as [1]

$$u = A_0 + A_1 \sin(\nu t - \gamma_1) + A_2 \sin(2\nu t - \gamma_2) + \sum_{n=2,4,\ldots} A_{2n} \cos n(2\nu t - \gamma_2), \tag{6}$$

where

$$A_0 = \frac{2\alpha}{\pi(2-\alpha)} A_1 \approx \frac{\alpha}{\pi} A_1, \qquad A_{2n} = (-1)^{(n/2)+1} \frac{2\alpha}{\pi(n^2-1)^2} A_2. \tag{7}$$

To find the unknown parameters A_2, γ_1, and γ_2 we will use a simple straightforward method by satisfying Eq. (1) for each half-cycle of vibration with a higher frequency (2ν) over the entire period of forced vibrations ($T_\nu = 2\pi/\nu$) at the time instants when the values of the bilinear characteristic of the restoring force are known:

$$\begin{gathered} t_1' = \frac{\beta + \gamma_2}{2\nu}, \quad t_1'' = \frac{\pi - \beta + \gamma_2}{2\nu}, \quad t_2' = \frac{\pi + \beta + \gamma_2}{2\nu}, \quad t_2'' = \frac{2\pi - \beta + \gamma_2}{2\nu}, \\ t_3' = \frac{2\pi + \beta + \gamma_2}{2\nu}, \quad t_3'' = \frac{3\pi - \beta + \gamma_2}{2\nu}, \quad t_4' = \frac{3\pi + \beta + \gamma_2}{2\nu}, \quad t_4'' = \frac{4\pi - \beta + \gamma_2}{2\nu}, \end{gathered} \tag{8}$$

where the parameter β is considered within the range $\beta_0 \le \beta \le \pi/2$ on the condition that $A_2 \sin\beta_0 \approx A_1$.

We substitute the above time instants (8) into Eq. (1), denote $\Delta\gamma = \gamma_2/2 - \gamma_1$, and thus arrive at the four pairs of input equations:

$$(1-\alpha)A_0 \pm \left\{\left[(1-\alpha)-\left(\frac{\nu}{\omega}\right)^2\right]\left(\sin\frac{\beta}{2}\cos\Delta\gamma+\cos\frac{\beta}{2}\sin\Delta\gamma\right)+2h\frac{\nu}{\omega^2}\left(\cos\frac{\beta}{2}\cos\Delta\gamma-\sin\frac{\beta}{2}\sin\Delta\gamma\right)\right\}A_1$$

$$+\left\{\left[(1-\alpha)-4\left(\frac{\nu}{\omega}\right)^2\right]\sin\beta+4h\frac{\nu}{\omega^2}\cos\beta\right\}A_2-4h\frac{\nu}{\omega^2}\sum_{n=2,4,\ldots}n\sin n\beta A_{2n}$$

$$+\sum_{n=2,4,\ldots}\left[(1-\alpha)-4n^2\left(\frac{\nu}{\omega}\right)^2\right]\cos n\beta A_{2n}=\pm\frac{q_0}{\omega^2}\left(\sin\frac{\beta}{2}\cos\frac{\gamma_2}{2}+\cos\frac{\beta}{2}\sin\frac{\gamma_2}{2}\right), \qquad (1'),\ (3')$$

$$(1-\alpha)A_0 \pm \left\{\left[(1-\alpha)-\left(\frac{\nu}{\omega}\right)^2\right]\left(\cos\frac{\beta}{2}\cos\Delta\gamma+\sin\frac{\beta}{2}\sin\Delta\gamma\right)+2h\frac{\nu}{\omega^2}\left(\sin\frac{\beta}{2}\cos\Delta\gamma-\cos\frac{\beta}{2}\sin\Delta\gamma\right)\right\}A_1$$

$$+\left\{\left[(1-\alpha)-4\left(\frac{\nu}{\omega}\right)^2\right]\sin\beta-4h\frac{\nu}{\omega^2}\cos\beta\right\}A_2+4h\frac{\nu}{\omega^2}\sum_{n=2,4,\ldots}n\sin n\beta A_{2n}$$

$$+\sum_{n=2,4,\ldots}\left[(1-\alpha)-4n^2\left(\frac{\nu}{\omega}\right)^2\right]\cos n\beta A_{2n}=\pm\frac{q_0}{\omega^2}\left(\cos\frac{\beta}{2}\cos\frac{\gamma_2}{2}+\sin\frac{\beta}{2}\sin\frac{\gamma_2}{2}\right), \qquad (1''),\ (3'')$$

$$A_0 \pm \left\{\left[(1-\alpha)-\left(\frac{\nu}{\omega}\right)^2\right]\left(\cos\frac{\beta}{2}\cos\Delta\gamma-\sin\frac{\beta}{2}\sin\Delta\gamma\right)-2h\frac{\nu}{\omega^2}\left(\sin\frac{\beta}{2}\cos\Delta\gamma+\cos\frac{\beta}{2}\sin\Delta\gamma\right)\right\}A_1$$

$$-\left\{\left[(1-\alpha)-4\left(\frac{\nu}{\omega}\right)^2\right]\sin\beta+4h\frac{\nu}{\omega^2}\cos\beta\right\}A_2-4h\frac{\nu}{\omega^2}\sum_{n=2,4,\ldots}n\sin n\beta A_{2n}$$

$$+\sum_{n=2,4,\ldots}\left[(1-\alpha)-4n^2\left(\frac{\nu}{\omega}\right)^2\right]\cos n\beta A_{2n}=\pm\frac{q_0}{\omega^2}\left(\cos\frac{\beta}{2}\cos\frac{\gamma_2}{2}-\sin\frac{\beta}{2}\sin\frac{\gamma_2}{2}\right), \qquad (2'),\ (4')$$

$$A_0 \pm \left\{\left[(1-\alpha)-\left(\frac{\nu}{\omega}\right)^2\right]\left(\sin\frac{\beta}{2}\cos\Delta\gamma-\cos\frac{\beta}{2}\sin\Delta\gamma\right)-2h\frac{\nu}{\omega^2}\left(\cos\frac{\beta}{2}\cos\Delta\gamma+\sin\frac{\beta}{2}\sin\Delta\gamma\right)\right\}A_1$$

$$-\left\{\left[(1-\alpha)-4\left(\frac{\nu}{\omega}\right)^2\right]\sin\beta-4h\frac{\nu}{\omega^2}\cos\beta\right\}A_2+4h\frac{\nu}{\omega^2}\sum_{n=2,4,\ldots}n\sin n\beta A_{2n}$$

$$+\sum_{n=2,4,\ldots}\left[(1-\alpha)-4n^2\left(\frac{\nu}{\omega}\right)^2\right]\cos n\beta A_{2n}=\pm\frac{q_0}{\omega^2}\left(\sin\frac{\beta}{2}\cos\frac{\gamma_2}{2}-\cos\frac{\beta}{2}\sin\frac{\gamma_2}{2}\right). \qquad (2''),\ (4'')$$

(9)

For the sake of convenience, Eqs. (9) are additionally numbered depending of the notation of the indices of the time instants (8): 1′, 1″, 2′, 2″, 3′, 3″, 4′, and 4″. In this case, the upper signs "+" pertain to equations (1′), (1″), (2′), and (2″), and the lower signs "–" to (3′), (3″), (4′), and (4″).

The results of the earlier calculations [3, 8] demonstrate that under the resonances at hand the value of the main vibrodiagnostic parameter (the relative amplitude of the resonant harmonic) is independent of the level of the exciting force relative amplitude q_0 but is governed by the value of the parameters of nonlinearity α and damping capacity h of the vibrating system. During the calculations, the constitutive equations for the vibrodiagnostic parameter were found by considering algebraic sums of input equations like (9), which corresponded to the sum and difference of the characteristic pairs of equations ($N' \pm M''$) and ($N'' \pm M'$) covering the entire period of the lowest harmonic. Also, the values of trigonometric functions of the β angle were averaged over the interval $\beta_0 \le \beta \le \pi/2$, where β_0 was given by the appropriate condition which dictated the known value of the bilinear stiffness for a given displacement $u(\beta)$.

Specifically, for a region of a weak 2nd-order superharmonic resonance, where the relative amplitude of the resonant (second) harmonic $\overline{A}_2 = A_2 / A_1 < 1$, we derived fairly simple approximate expressions for the vibro-diagnostic parameter [8],

$$\overline{A}_2 = \frac{\alpha\left[(2-\alpha)\frac{\pi}{4} - 2\right]}{8h\frac{\nu}{\omega^2}}\cos(\gamma_2 - 2\gamma_1) = \frac{\alpha\left[(2-\alpha)\frac{\pi}{4} - 2\right]}{(2-\alpha) - 8\left(\frac{\nu}{\omega}\right)^2}\sin(\gamma_2 - 2\gamma_1), \tag{10}$$

where $(\gamma_2 - 2\gamma_1)$ is the phase shift given by

$$(\gamma_2 - 2\gamma_1) = \arctan\frac{(2-\alpha) - 8\left(\frac{\nu}{\omega}\right)^2}{8h\frac{\nu}{\omega^2}}. \tag{11}$$

A comparison between the results of calculation of the functions $\overline{A}_2(\alpha, h)$ and the data of numerical solution of the differential equation (1) by the acceleration averaging technique reveals a good agreement when $\alpha\omega/h \le 5$.

The use of a similar approach for finding the constitutive equations from the given set of equations (9) for a strong superharmonic resonance ($\overline{A}_2 > 1$) implies, as distinct from the earlier solution [1–3, 8], the presence of a right-hand part and the absence of the explicit amplitude of the resonant harmonic. In particular, for the case of a tuned resonance ($\nu = \omega_0/2$), from the consideration of the algebraic sum of equations {[(1′)+(4″)]–[(1″)+(4′)]}+{[(2′)+(3″)]–[(2″)+(3′)]} we have

$$\left[(2-\alpha) - \frac{1}{2}\left(\frac{\omega_0}{\omega}\right)^2\right]\sin\Delta\gamma + \left(\alpha + 2h\frac{\omega_0}{\omega^2}\right)\cos\Delta\gamma = \frac{2q_0}{\omega^2 A_1}\sin\frac{\gamma_2}{2}, \tag{12}$$

while from the algebraic sum {[(1′)–(4″)]–[(1″)–(4′)]}–{[(2′)–(3″)]–[(2″)–(3′)]} we arrive at

$$\left[(2-\alpha) - \frac{1}{2}\left(\frac{\omega_0}{\omega}\right)^2\right]\cos\Delta\gamma + \left(\alpha - 2h\frac{\omega_0}{\omega^2}\right)\sin\Delta\gamma = \frac{2q_0}{\omega^2 A_1}\cos\frac{\gamma_2}{2}, \tag{13}$$

where

$$\Delta\gamma = \frac{\gamma_2}{2} - \gamma_1. \tag{14}$$

However, this seemingly complicated solution is simplified owing to the fact the equations contain absolutely no trigonometric functions of the β angle.

To find the sought-for diagnostic parameter $\overline{A}_2$, we rearrange the right-hand parts of Eqs. (12) and (13). Using the relation $\gamma_2/2 = \Delta\gamma + \gamma_1$ for the trigonometric functions and the expression for $\sin\gamma_1$ as defined from the balance of input and dissipated energy

$$\sin\gamma_1 = h\omega_0 \frac{A_1}{q_0}\left\{1 + 4\left[1 + \left(\frac{2\alpha}{\pi}\right)^2 \sum_{n=2,4,\dots} \frac{n^2}{(n^2-1)^4}\right]\overline{A}_2^2\right\}, \tag{15}$$

we have

$$\frac{2q_0}{\omega^2 A_1}\sin\frac{\gamma_2}{2} = \frac{2q_0}{\omega^2 A_1}\cos\gamma_1 \sin\Delta\gamma + 2h\frac{\omega_0}{\omega^2}\left\{1 + 4\left[1 + \left(\frac{2\alpha}{\pi}\right)^2 \sum_{n=2,4,\dots} \frac{n^2}{(n^2-1)^4}\right]\overline{A}_2^2\right\}\cos\Delta\gamma, \tag{16a}$$

$$\frac{2q_0}{\omega^2 A_1}\cos\frac{\gamma_2}{2} = \frac{2q_0}{\omega^2 A_1}\cos\gamma_1 \cos\Delta\gamma - 2h\frac{\omega_0}{\omega^2}\left\{1 + 4\left[1 + \left(\frac{2\alpha}{\pi}\right)^2 \sum_{n=2,4,\dots} \frac{n^2}{(n^2-1)^4}\right]\overline{A}_2^2\right\}\sin\Delta\gamma. \tag{16b}$$

We disregard the term $\left(\frac{2\alpha}{\pi}\right)^2 \sum_{n=2,4,\dots} \frac{n^2}{(n^2-1)^4}$ due to its smallness and with a sufficient approximation we take that $\cos\gamma_1 = 1$ and that $\frac{q_0}{\omega^2 A_1} \approx \frac{3}{4}\left(\frac{\omega_0}{\omega}\right)^2$ according to (5); then, by substituting $\frac{h}{\omega} = \frac{\delta}{2\pi}$, where δ is the logarithmic decrement of vibration, we obtain the following constitutive equations for a tuned 2nd-order super-harmonic resonance:

$$\begin{gathered}\left\{2\left[1 - \left(\frac{\omega_0}{\omega}\right)^2\right] - \alpha\right\}\sin\Delta\gamma + \alpha\cos\Delta\gamma = 4\frac{\delta}{\pi}\frac{\omega_0}{\omega}\overline{A}_2^2\cos\Delta\gamma, \\ \left\{2\left[1 - \left(\frac{\omega_0}{\omega}\right)^2\right] - \alpha\right\}\cos\Delta\gamma + \alpha\sin\Delta\gamma = -4\frac{\delta}{\pi}\frac{\omega_0}{\omega}\overline{A}_2^2\sin\Delta\gamma.\end{gathered} \tag{17}$$

From the set of equations (17) we derive the expressions for the phase shift $\Delta\gamma$

$$\Delta\gamma = \frac{1}{2}\arcsin\left\{1 - \frac{2}{\alpha}\left[1 - \left(\frac{\omega_0}{\omega}\right)^2\right]\right\} \tag{18}$$

and for the vibrodiagnostic parameter

$$\overline{A}_2 = \sqrt{\frac{\left\{2\left[1 - \left(\frac{\omega_0}{\omega}\right)^2\right] - \alpha\right\}\tan\Delta\gamma + \alpha}{4\frac{\delta}{\pi}\frac{\omega_0}{\omega}}} = \sqrt{\frac{\left\{\alpha - 2\left[1 - \left(\frac{\omega_0}{\omega}\right)^2\right]\right\}\cot\Delta\gamma - \alpha}{4\frac{\delta}{\pi}\frac{\omega_0}{\omega}}}. \tag{19}$$

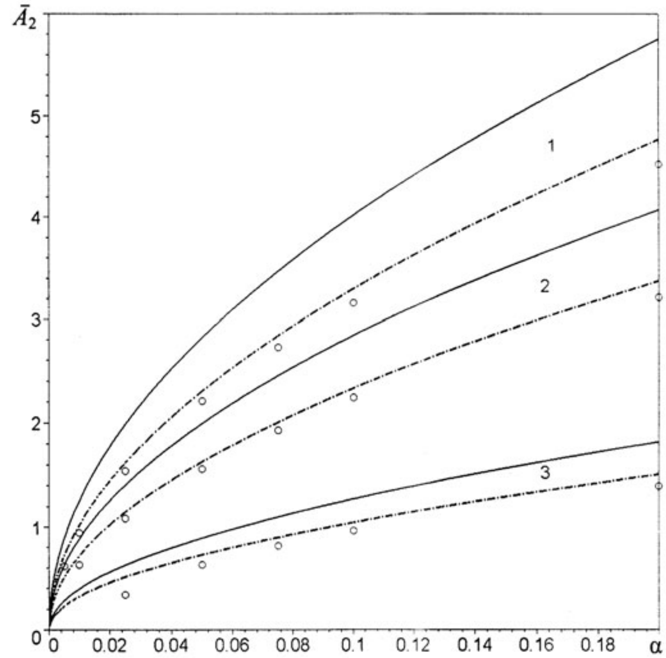

Fig. 1. The curves of the second-harmonic relative amplitude $\overline{A}_2$ vs. the parameter α constructed using formulas (19) (solid lines) and (20) (dashed lines) for various values of the logarithmic decrement δ: (*1*) $\delta = 0.005$; (*2*) $\delta = 0.010$; (*3*) $\delta = 0.050$. [Here and in Figs. 2 and 3: dots represent results of the numerical solution of Eq. (1).]

The value of the first harmonic amplitude A_1 can be also taken to correspond to the solution for forced vibration of the initial linear system with an excitation frequency $\nu = 0.5\omega$, i.e., $A_1 \approx \frac{4q_0}{3\omega^2}$ and $\frac{h\nu}{\omega^2} = \frac{\delta}{4\pi}$. Then, formulas (19) become

$$\overline{A}_2 = \sqrt{\frac{\left\{0.5\left[1-\left(\frac{\omega_0}{\omega}\right)^2\right]-\alpha\right\}\tan\Delta\gamma+\alpha}{4\frac{\delta}{\pi}}} = \sqrt{\frac{\left\{\alpha-0.5\left[1-\left(\frac{\omega_0}{\omega}\right)^2\right]\right\}\cot\Delta\gamma-\alpha}{4\frac{\delta}{\pi}}}, \tag{20}$$

where

$$\Delta\gamma = \frac{1}{2}\arcsin\left\{1-\frac{0.5}{\alpha}\left[1-\left(\frac{\omega_0}{\omega}\right)^2\right]\right\}. \tag{21}$$

Calculated Results. The curves of the relative amplitude $\overline{A}_2$ vs. the system nonlinearity parameter α, which were calculated by formulas (19) using (18) and by formula (20) using (21) are given in Fig. 1 for various values of the logarithmic decrement δ. Strictly speaking, one should take into account the values $\overline{A}_2 \geq 1$.

To verify the reliability of the calculated results, the dots in Fig. 1 represent the data of numerical solution of Eq. (1).* It is evident that the functions $\overline{A}_2(\alpha)$ are of similar trend, while the values of the vibrodiagnostic parameter obtained through the numerical solution are approximately 21–24% lower than those calculated by formulas (19) and 5% lower in comparison to the calculations by (20), i.e., in this case, the use of (20) is preferable.

* The data were obtained by Dr. O. A. Bovsunovskii using the Runge-Kutta method.

Analysis of formulas (19) and (20) reveals the following special feature of the first terms of in the numerator of the radicand. Regardless of the value of the parameter $0 \le \alpha \le 0.2$, these terms are

for formulas (19)

$$\left\{2\left[1-\left(\frac{\omega_0}{\omega}\right)^2\right]-\alpha\right\}\tan\Delta\gamma \approx 0, \qquad \left\{\alpha-2\left[1-\left(\frac{\omega_0}{\omega}\right)^2\right]\right\}\cot\Delta\gamma \approx 2\alpha,$$

and for formulas (20)

$$\left\{0.5\left[1-\left(\frac{\omega_0}{\omega}\right)^2\right]-\alpha\right\}\tan\Delta\gamma \cong -0.326\alpha, \qquad \left\{0.5\left[1-\left(\frac{\omega_0}{\omega}\right)^2\right]-\alpha\right\}\cot\Delta\gamma = 1.674\alpha.$$

Thus, the approximate analytical solution for the case of a strong resonance defines a directly proportional dependence of the diagnostic parameter $\bar{A}_2$ on the square root of the ratio of the system key parameters α/δ; for the α range at hand this dependence can be respectively represented, with an approximation acceptable for practical applications, as follows:

$$\bar{A}_2 = \sqrt{\frac{\pi}{4}\frac{\omega}{\omega_0}}\sqrt{\frac{\alpha}{\delta}} \approx 0.886\sqrt{\frac{\alpha}{\delta}} \quad \text{or} \quad \bar{A}_2 \approx 0.725\sqrt{\frac{\alpha}{\delta}}. \tag{22}$$

Relations (22) essentially differ from the function $A_2(\alpha/\delta)$ in the case of a weak resonance for which the approximate analytical solution (10) defines a roughly direct proportional relation between the parameter $\bar{A}_2$ and the ratio α/δ: $\bar{A}_2 \approx 0.69(\alpha/\delta)$.

Of some interest is to analyze the agreement between the results of approximate calculation of the vibrodiagnostic parameter for the regions of a weak ($\bar{A}_2 < 1$) and strong ($\bar{A}_2 > 1$) superharmonic resonance. The functions $\bar{A}_2(\alpha)$ calculated by formulas (10) for a weak resonance and by the second formula (22) for a strong one, including the range $\bar{A}_2 \approx 1$, are given in Fig. 2 for three values of the logarithmic decrement δ. It is obvious that the best results for the values $\bar{A}_2 \le 0.75$ are obtained by formulas (10) and for the values $\bar{A}_2 \ge 0.75$ by formula (22).

The proposed approximate analytical method for the determination of the vibrodiagnostic parameter of the presence of a closing crack in an elastic body clearly reveals a significant difference between the values of the vibration nonlinearity parameter in the super- and subharmonic resonances, with the values of the system nonlinearity parameter α and damping capacity δ being equal.

Figure 3 shows the calculated dependence of the relative amplitude $\bar{A}_{1/2}$ of the lowest resonant harmonic in the 1/2-order subharmonic resonance ($\nu = 2\omega_0$) and the relative amplitude $\bar{A}_2$ of the second resonant harmonic in the superharmonic resonance ($\nu = 0.5\omega_0$), which were found by formula (20), on the system nonlinearity parameter α with $\delta = 0.005$. The function $\bar{A}_{1/2}(\alpha)$ was calculated, according to the data provided in [3], by the approximate formula derived with the value $\beta = 0$,

$$\bar{A}_{1/2} = \frac{\alpha\sin\Delta\gamma}{\left[(2-\alpha)-2\left(\frac{\omega_0}{\omega}\right)^2\right]-D\alpha^2}, \tag{23}$$

where

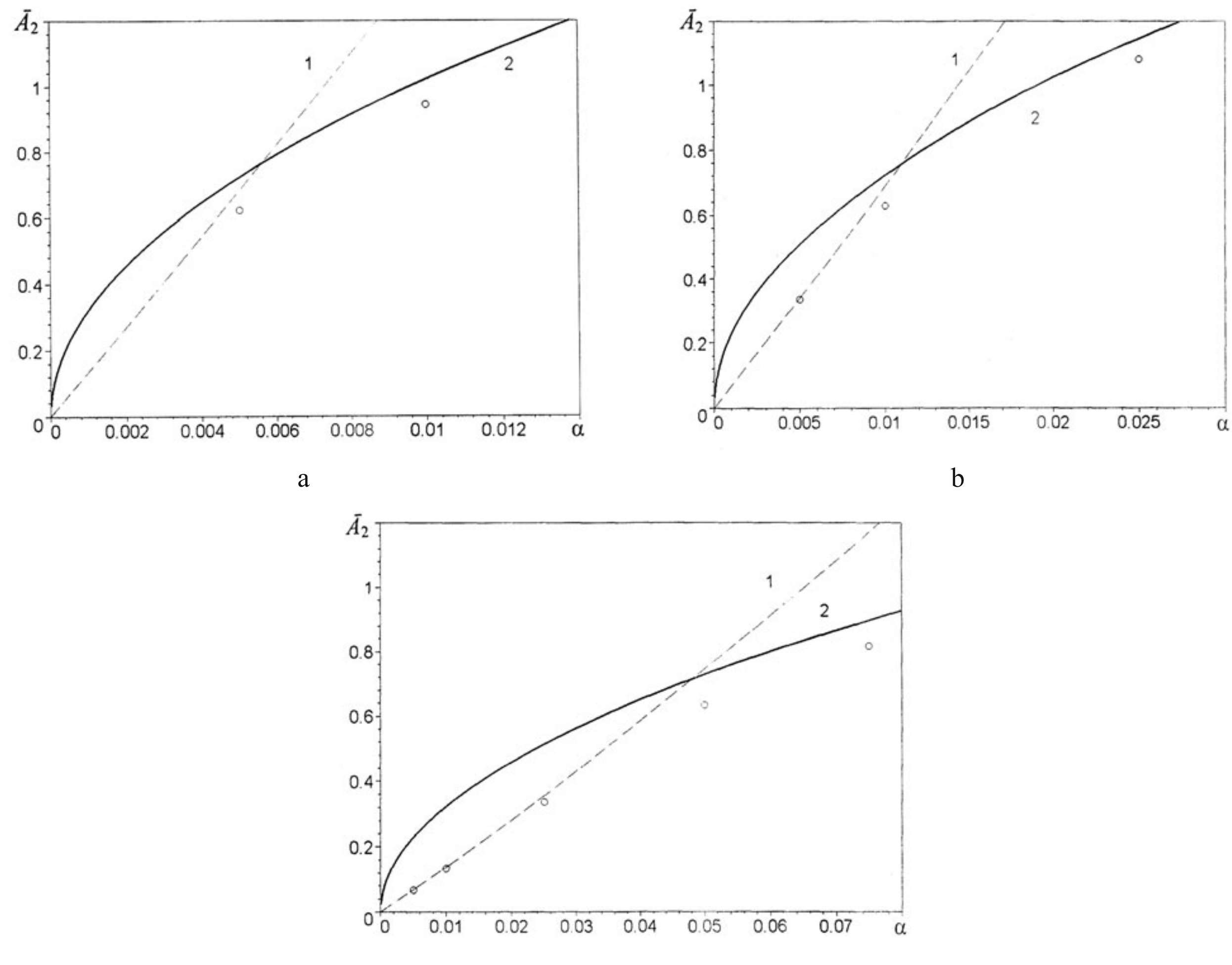

Fig. 2. The curves of the second-harmonic relative amplitude $\overline{A}_2$ vs. the parameter α constructed using formulas (10) (dashed lines) and (22) (solid lines) for various values of the logarithmic decrement δ: (a) $\delta = 0.005$; (b) $\delta = 0.010$; (c) $\delta = 0.050$.

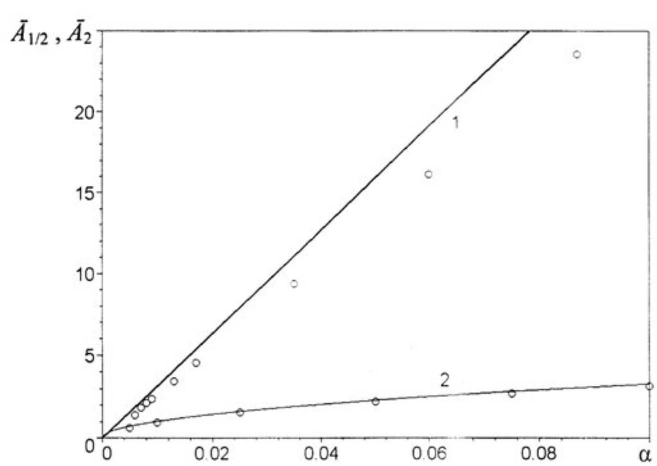

Fig. 3. The calculated dependence of the relative amplitudes $\overline{A}_{1/2}$ (curve *1*) and $\overline{A}_2$ (curve *2*) on the parameter α for $\delta = 0.005$.

$$\Delta\gamma = \arctan\left\{\left[(2-\alpha)-2\left(\frac{\omega_0}{\omega}\right)^2\right] - D\alpha^2\right\}\left(2\frac{\delta}{\pi}\frac{\omega_0}{\omega}\right)^{-1}$$

[D can take on the values 0.5, 0.570736, and $0.25(2-\alpha)+0.070736$].

For comparison, dots in Fig. 3 represent results obtained by the numerical solutions.

It should be mentioned that for a strong 1/2-order subharmonic resonance ($\nu = 2\omega_0$) the function of the relative amplitude of the lowest resonant harmonic $\bar{A}_{1/2}(\alpha, \delta)$ for the ratio $\alpha/\delta \le 10$ is adequately described by the formula $\bar{A}_{1/2} = \frac{\pi}{2}\frac{\alpha}{\delta}$ [13], i.e., $\bar{A}_{1/2}$ is directly proportional to the ratio α/δ, while for the a strong superharmonic resonance the value of $\bar{A}_2$ is proportional to square root of this ratio.

In conclusion, let us analyze the function $\bar{A}_{1/2}(\alpha, \delta)$ which is determined by formulas (23) using the above method of solving the set of equations with the right-hand term available. In publication [3], the above-mentioned formulas were derived from the solution to the set of two constitutive equations that had not right-hand term but contained the trigonometric function of the parameter β. There, the authors assumed that the mean value of the function $\cos 2\beta$ involved in one of the equations would be equal to the mean value of the function's absolute magnitude on the interval $0 < \beta < \pi/2$. The equations were derived from the respective algebraic sums of the input equations written similar to (9) for the characteristic time instants t_1', t_1'', t_2', and t_2'' of the vibration process at hand, denoted with the appropriate indices.

To exclude the presence of the function $\cos 2\beta$ let us look at the algebraic sums of equations [(1′)–(2′)]–[(1″)–(2″)] and [(1′)–(2″)]–[(1″)–(2′)] [3] which, with the higher harmonics disregarded, enable us to arrive at the following pair of constitutive equations:

$$2h\frac{\nu}{\omega^2}\cos\beta \cdot A_{1/2} - \alpha \sin 2\beta \cos\Delta\gamma \cdot A_1 = 0, \tag{24}$$

$$\left[(2-\alpha)-2\left(\frac{\nu}{\omega}\right)^2\right]\cos\Delta\gamma - 4h\frac{\nu}{\omega^2}\sin\Delta\gamma = 2\frac{q_0}{\omega^2 A_1}\cos 2\gamma_{1/2}. \tag{25}$$

It is seen that the first equation does no contain the problematic function $\cos 2\beta$ and the second one has absolutely no trigonometric functions of the parameter β.

Let us consider now the case of a tuned resonance ($\nu = 2\omega_0$). From Eq. (5) we find $\frac{q_0}{\omega^2 A_1} = -3\left(\frac{\omega_0}{\omega}\right)^2$, while from the balance of the input and dissipated energy we have $\sin\gamma_1 = 2h\nu\frac{A_1}{q_0}(1+0.25\bar{A}_{1/2}^2)$. Here, $\bar{A}_{1/2} = A_{1/2}/A_1$, where A_1 is the amplitude of the harmonic of forced vibration with an excitation frequency ν and $A_{1/2}$ is the amplitude of the resonant harmonic with a frequency $\frac{1}{2}\nu = \omega_0$.

In view of the fact that $\cos 2\gamma_{1/2} = \cos(\Delta\gamma + \gamma_1)$ and under the assumption that $\frac{h}{\omega} = \frac{\delta}{2\pi}$, Eq. (25) is written as

$$\left\{2\left[1-\left(\frac{\omega_0}{\omega}\right)^2\right] - \alpha\right\}\cos\Delta\gamma \approx -\frac{\delta}{\pi}\bar{A}_{1/2}^2 \sin\Delta\gamma. \tag{26}$$

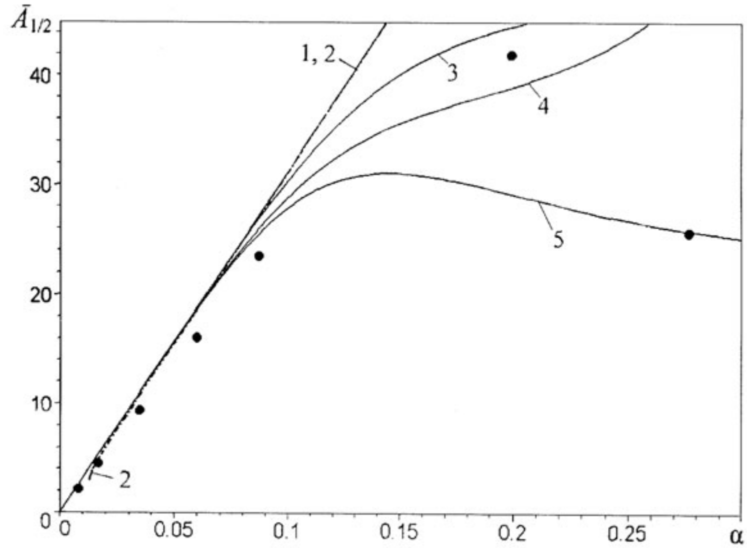

Fig. 4. The functions $\overline{A}_{1/2}$ calculated for $\delta = 0.005$ by formulas (27), (29) [curves *1* and *2*, respectively (curve *2* corresponds to approximate values of $\Delta\gamma$ and $\overline{A}_{1/2}$)] and by (23) (curves *3*, *4*, and *5* with a coefficient $D = 0.5$, $[0.25\,(2-\alpha) + 0.070736]$, and 0.570736, respectively).

With the functions $\cos\beta$ and $\sin 2\beta$ expressed in terms of their mean values on the interval $0 < \beta < \pi/2$, Eq. (24) is rearranged to the form:

$$\overline{A}_{1/2} = \frac{\alpha \cos\Delta\gamma}{2(\delta/\pi)}. \tag{27}$$

Substitution of (27) into (26) gives the required formula for the phase shift $\Delta\gamma$,

$$\Delta\gamma = \frac{1}{2}\arcsin 8\left\{\alpha - 2\left[1-\left(\frac{\omega_0}{\omega}\right)^2\right]\right\}\frac{\delta}{\pi\alpha^2} \approx \frac{1}{2}\arcsin 8\frac{\delta}{\pi\alpha}. \tag{28}$$

From (26) we derive the formula for the calculation of $\overline{A}_{1/2}$ in terms of $\tan\Delta\gamma$:

$$\overline{A}_{1/2} = \sqrt{\frac{\alpha - 2\left[1-\left(\frac{\omega_0}{\omega}\right)^2\right]}{\frac{\delta}{\pi}\tan\Delta\gamma}} \approx \sqrt{\frac{\pi\alpha}{\delta\tan\Delta\gamma}}. \tag{29}$$

To compare the earlier results [3] with the present ones, Fig. 4 shows the functions $\overline{A}_{1/2}(\alpha)$ calculated by formulas (27), (29), and (23) with $\delta = 0.005$ as well as the results of numerical solution of differential equation (1). It is evident that according to formulas (27) and (29) the function $\overline{A}_{1/2}$ is almost directly proportional and described by the formula $\overline{A}_{1/2} \approx \frac{\pi}{2}\frac{\alpha}{\delta}$, while according to (23) it exhibits the same trend but only to a certain α value which in this case is about 0.08. However, the functions determined by formula (23) go in line with the results of the numerical solution to a higher value of α. Thus, we can infer the adequacy of the assumption [3] on the mean value of the $\cos 2\beta$ function.

CONCLUSIONS

1. An approximate analytical method has been put forward for the determination of the vibration nonlinearity parameter of an elastic body with a closing crack, which is simulated by a single-degree-of-freedom system with an asymmetric bilinear characteristic of the restoring force, under a strong 2nd-order superharmonic resonance.

2. Approximate expressions have been derived for the calculation of the main vibrodiagnostic parameter of the presence of a crack – the second-harmonic relative amplitude $\overline{A}_2$ under a tuned superharmonic resonance.

3. It is demonstrated that with an accuracy good enough for practical applications the dependence of the relative amplitude $\overline{A}_2$ on the ratio between the nonlinearity parameter α and the damping capacity δ of the vibrating system is adequately described by the unified formula $\overline{A}_2 = 0.725\sqrt{\alpha/\delta}$, whereby the calculated results agree well with the data of the numerical solution.

4. The dependence of the vibration nonlinearity parameter on the ratio α/δ in a subharmonic resonance $(\overline{A}_{1/2})$ is shown to essentially differ from that in a superharmonic resonance $(\overline{A}_2)$. In the first case, $\overline{A}_{1/2}$ is directly proportional to α/δ, while in the second case it is proportional to the square root of this ratio, while the amplitude ratio is $\overline{A}_2/\overline{A}_{1/2} \approx 0.46\sqrt{\delta/\alpha}$.

REFERENCES

1. V. V. Matveev, "Approximate analytical determination of vibrodiagnostic parameters of non-linearity of elastic bodies due to the presence of a closing crack. Part 1. Available and proposed methods of solution," *Strength Mater.*, **36**, No. 4, 333–344 (2004).
2. V. V. Matveev and O. A. Bovsunovskii, "Approximate analytical determination of vibrodiagnostic parameters of a cracked elastic body under subharmonic resonance. Part 1. Weak resonance," *Strength Mater.*, **40**, No. 2, 180–191 (2008).
3. V. V. Matveev and O. A. Bovsunovskii, "Approximate analytical determination of vibrodiagnostic parameters of a cracked elastic body under subharmonic resonance. Part 2. Strong resonance," *Strength Mater.*, **40**, No. 3, 283–291 (2008).
4. V. V. Matveev and A. P. Bovsunovskii, "Efficiency of the method of spectral vibrodiagnostics for fatigue damage of structural elements. Part 3. Analytical and numerical determination of natural frequencies of longitudinal and bending vibrations of beams with transverse cracks. Solution," *Strength Mater.*, **31**, No. 4, 341–350 (1999).
5. V. V. Matveev and O. E. Boginich, "Vibrodiagnostic parameters of fatigue damage in rectangular plates. Part 1. A procedure of determination of damage parameters," *Strength Mater.*, **36**, No. 6, 549–557 (2004).
6. V. V. Matveev and O. E. Boginich, "Vibrodiagnostic parameters of fatigue damage in rectangular plates. Part 3. Through-the-thickness and surface semi-elliptical cracks," *Strength Mater.*, **38**, No. 5, 466–480 (2006).
7. V. V. Matveev, "Efficiency of the method of spectral vibrodiagnostics for fatigue damage of structural elements. Part 1. Longitudinal vibrations, analytic solution," *Strength Mater.*, **29**, No. 6, 561–572 (1997).
8. V. V. Matveev, "Approximate analytical determination of vibrodiagnostic parameters of non-linearity of elastic bodies due to the presence of a closing crack. Part 2. Determination of diagnostic parameters in principal and 2-nd order superharmonic resonances," *Strength Mater.*, **36**, No. 5, 449–462 (2004).