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APPROXIMATE ANALYTICAL DETERMINATION OF VIBRODIAGNOSTIC PARAMETERS OF A CRACKED ELASTIC BODY UNDER SUBHARMONIC RESONANCE. PART 2. STRONG RESONANCE

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Using the approach proposed in Part 1, an approximate calculation of vibration parameters is made for an elastic body with a closing crack, in the region of a strong 1/2-order subharmonic resonance with the lower-harmonic amplitude of free vibration spectrum larger than the main amplitude of forced vibrations.

Keywords: nonlinear vibrations, subharmonic resonance, fatigue damage vibrodiagnostics.

Introduction. Using the approach detailed in Part 1, we will discuss here an approximate calculation of vibration parameters of an elastic body with a closing crack, in the region of a strong 1/2-order subharmonic resonance where the lower-harmonic amplitude of the free vibration spectrum is larger than the main amplitude of forced vibration.

Calculation Procedure. Neglecting some difference between the vibration modes of a cracked elastic body in alternating half-cycles [1], the forced vibration of an equivalent single-mass nonlinear system is described by the differential equation

$$\frac{d^2u}{dt^2}+2h\frac{du}{dt}+\omega^2[1-0.5\alpha(1+\operatorname{sign}u)]u=q_0\sin\nu t, \tag{1}$$

where ω is the intact body natural frequency in the vibration mode at hand, α is the parameter that integrally represents a relative change in the body rigidity in the case of an open crack,

$$\alpha=\frac{K-K_{cr}}{K}, \qquad K_{cr}<K, \tag{2}$$

where K is the intact body rigidity corresponding to that of a body with a closing crack (in this case, with $u<0$), and K_{cr} is the rigidity of a body with an open crack ($u>0$).

Natural frequency of a body with a closing crack is taken as [2]

$$\omega_0=\frac{2\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}\omega. \tag{3}$$

Assuming that with $\nu\approx 2\omega_0$ there arise – in addition to the forced-mode fundamental harmonic – the vibrations with a spectrum of natural-mode harmonic components, which were defined earlier by the asymptotic

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method of nonlinear mechanics [3], a solution to equation (1) in the superharmonic resonance region is sought for in the form [1]

$$u(t) = A_0 + A_{1/2}\sin\left(\frac{\nu t}{2} - \gamma_{1/2}\right) + A_1\sin(\nu t - \gamma_1) + \sum_{n=2,4,\dots} A_n \cos n\left(\frac{\nu t}{2} - \gamma_{1/2}\right), \tag{4}$$

where

$$A_0 = \frac{\alpha}{\pi}A_{1/2}, \qquad A_n = (-1)^{\frac{n}{2}+1}\frac{2\alpha}{\pi(n^2-1)^2}A_{1/2}. \tag{5}$$

In order to find the unknown parameters $A_{1/2}$ and $\gamma_{1/2}$ by the method proposed in [1], we substitute solution (4) into equation (1), requiring its fulfillment for the instants t_i when the $u(t_i)$ displacement sign that dictates the elastic response value is known. In the case of the strong subharmonic resonance at hand ($A_{1/2} > A_1$), the above instants are taken as

$$\begin{aligned} &\nu t_1' = 2\beta + 2\gamma_{1/2}, && \nu t_2' = 2\pi + (2\beta + 2\gamma_{1/2}), \\ &\nu t_1'' = 2\pi - (2\beta - 2\gamma_{1/2}), && \nu t_2'' = 4\pi - (2\beta - 2\gamma_{1/2}). \end{aligned} \tag{6}$$

At the harmonic with a forced-mode half-frequency, to these instants (6) there correspond the points *1′*, *1″*, *2′*, and *2″* in Fig. 1. The value of β angle in (6) meets the condition $A_{1/2}\sin\beta > A_1$ and can be chosen in the range $\beta_0 \le \beta \le \pi/2$, where β_0 satisfies the condition $A_{1/2}\sin\beta = A_1$.

It is evident that according to the adopted law of rigidity variation points *1′* and *1″*, for which $u > 0$, determine the free frequency value $(1-\alpha)\omega^2$, while points *2′* and *2″* yield ω^2 (Fig. 1).

Substitution of solution (4) for the chosen instants (6) into equation (1) yields two pairs of input equations,

$$\left\{\begin{aligned} &(1-\alpha)A_0 + \left\{\left[(1-\alpha) - \frac{1}{4}\left(\frac{\nu}{\omega}\right)^2\right]\sin\beta + \sum_{n=2,4,\dots}(-1)^{\frac{n}{2}+1}\frac{2\alpha}{\pi(n^2-1)^2}\left[(1-\alpha) - \frac{1}{4}\left(\frac{n\nu}{\omega}\right)^2\right]\cos n\beta \right.\\ &\left.\pm\frac{\nu h}{\omega^2}\left[\cos\beta - \sum_{n=2,4,\dots}(-1)^{\frac{n}{2}+1}\frac{2\alpha n}{\pi(n^2-1)^2}\sin n\beta\right]\right\}A_{1/2} \pm \left\{\left[(1-\alpha) - \left(\frac{\nu}{\omega}\right)^2\right][\sin 2\beta\cos\Delta\gamma \pm \cos 2\beta \sin\Delta\gamma] \right.\\ &\left. + 2\frac{\nu h}{\omega^2}[\cos 2\beta \cos\Delta\gamma \mp \sin 2\beta\sin\Delta\gamma]\right\}A_1 = \pm\frac{q_0}{\omega^2}(\sin 2\beta\cos 2\gamma_{1/2} \pm \cos 2\beta \sin 2\gamma_{1/2}), \qquad (1'),\ (1'') \\ &A_0 - \left\{\left[1 - \frac{1}{4}\left(\frac{\nu}{\omega}\right)^2\right]\sin\beta - \sum_{n=2,4,\dots}(-1)^{\frac{n}{2}+1}\frac{2\alpha}{\pi(n^2-1)^2}\left[1 - \frac{1}{4}\left(\frac{n\nu}{\omega}\right)^2\right]\cos n\beta \right.\\ &\left.\mp\frac{\nu h}{\omega^2}\left[\cos\beta + \sum_{n=2,4,\dots}(-1)^{\frac{n}{2}+1}\frac{2\alpha n}{\pi(n^2-1)^2}\sin n\beta\right]\right\}A_{1/2} \pm \left\{\left[1 - \left(\frac{\nu}{\omega}\right)^2\right][\sin 2\beta\cos\Delta\gamma \pm \cos 2\beta \sin\Delta\gamma] \right.\\ &\left. + 2\frac{\nu h}{\omega^2}[\cos 2\beta \cos\Delta\gamma \mp \sin 2\beta\sin\Delta\gamma]\right\}A_1 = \pm\frac{q_0}{\omega^2}(\sin 2\beta\cos 2\gamma_{1/2} \pm \cos 2\beta \sin 2\gamma_{1/2}), \qquad (2'),\ (2'') \end{aligned}\right. \tag{7}$$

where

$$\Delta\gamma = 2\gamma_{1/2} - \gamma_1. \tag{8}$$

For convenience of discussion, equations (7) have been additionally numbered as per notation of points in Fig. 1. The superscripts refer to equations (1′), (2′) and the subscripts to equations (1″), (2″).

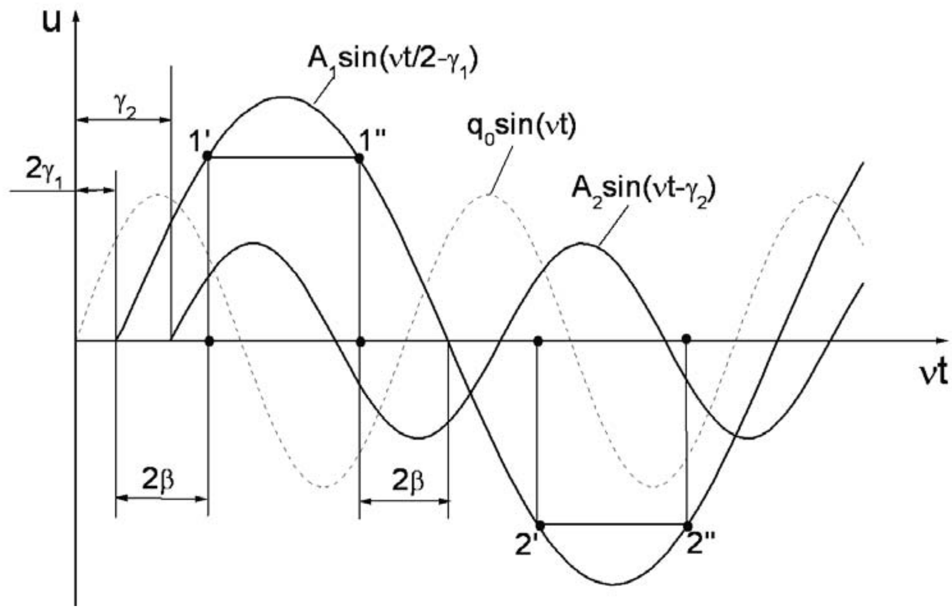

Fig. 1. Fundamental harmonics of the vibration process.

For averaging the trigonometric functions of angle β over the interval of its possible variation (from β_0 to $\pi/2$), we replace them with the average values,

$$(\sin\beta)_{av} = \frac{2\cos\beta_0}{\pi - 2\beta_0}, \qquad (\sin 2\beta)_{av} = \frac{1+\cos 2\beta_0}{\pi - 2\beta_0},$$
$$(\cos\beta)_{av} = \frac{2(1-\sin\beta_0)}{\pi - 2\beta_0}, \qquad (\cos 2\beta)_{av} = \frac{2-\sin 2\beta_0}{\pi - 2\beta_0}. \tag{9}$$

Considering that the amplitudes of higher harmonics ($n > 2$) of the free vibration spectrum are small, equations (7) take into account only the second harmonic amplitude of which is equal to $2\alpha/9\pi$. Then, using at the algebraic sum of equations [(1′) + (1″)] – [(2′) + (2″)] we find the lower-harmonic relative amplitude $\overline{A}_{1/2} = A_{1/2}/A_1$,

$$\overline{A}_{1/2} = \frac{\alpha(2-\sin 2\beta_0)\sin\Delta\gamma}{2\left[(2-\alpha) - \frac{1}{2}\left(\frac{\nu}{\omega}\right)^2\right]\cos\beta_0 - \frac{\alpha^2}{\pi}[\pi - 2\beta_0 + 0.22222(2-\sin 2\beta_0)]}, \tag{10}$$

while in the case of the algebraic sum of equations [(1′) – (2′)] – [(1″) – (2″)] we have

$$\overline{A}_{1/2} = \frac{\alpha(1+\cos 2\beta_0)\cos\Delta\gamma}{4(1-\sin\beta_0)\frac{\nu h}{\omega^2}}. \tag{11}$$

We equate expressions (10) and (11) and arrive at the formula for the phase displacement difference,

$$\tan\Delta\gamma = \frac{1+\cos 2\beta_0}{4(1-\sin\beta_0)(2-\sin 2\beta_0)\frac{\nu h}{\omega^2}}\left\{2\left[(2-\alpha) - \frac{1}{2}\left(\frac{\nu}{\omega}\right)^2\right]\cos\beta_0 - \frac{\alpha^2}{\pi}[\pi - 2\beta_0 + 0.22222(2-\sin 2\beta_0)]\right\}. \tag{12}$$

To use the above expressions one should know the value β_0 meeting the condition $\sin\beta_0 \approx A_1/A_{1/2}$. Let us consider the previously presented expressions for $\beta_0 = 0$:

$$\overline{A}_{1/2} = \frac{2\alpha \sin\Delta\gamma}{2\left[(2-\alpha) - \frac{1}{2}\left(\frac{\nu}{\omega}\right)^2\right] - 1.141741\alpha^2} \equiv \frac{\alpha\cos\Delta\gamma}{2\frac{h\nu}{\omega^2}}, \tag{13}$$

$$\tan\Delta\gamma = \frac{2\left[(2-\alpha) - \frac{1}{2}\left(\frac{\nu}{\omega}\right)^2\right] - 1.141741\alpha^2}{4\frac{h\nu}{\omega^2}}. \tag{14}$$

We preset the values of α and $h/\omega = \delta/2\pi$ (δ is the logarithmic decrement of free vibration) and determine $\Delta\gamma$ and $\overline{A}_{1/2}$ for the case of tuned subresonance $\nu/\omega = 4\sqrt{1-\alpha}/(1+\sqrt{1-\alpha})$ using formulas (13) and (14). With the approximate value of $\overline{A}_{1/2}$ available, we calculate $\beta_0 = \arcsin(1/\overline{A}_{1/2})$ and then find $\Delta\gamma$ again by (12) and $\overline{A}_{1/2}$ by (10) or (11). To further refine the $\overline{A}_{1/2}$ value, β_0 has to be determined such that $\sin\beta_0$ should be equal to the resultant $\overline{A}_{1/2}$ value, and so forth.

Calculated Results. Figure 2 shows the functions of the relative amplitude $\overline{A}_{1/2}(\alpha)$ as calculated by formulas (13) and (14) with $\nu = 2\omega_0$ for various values of the logarithmic decrement δ, while Fig. 3 gives the $\overline{A}_{1/2}(\delta)$ functions for various α values. As an example, the dash-and-dotted lines in Fig. 2 indicate the $\overline{A}_{1/2}(\alpha)$ functions for $\delta = 0.01$ and 0.005, which were obtained using (10) and (11) during the determination of β_0 from the condition $\beta_0 \approx \arcsin(1/\overline{A}_{1/2})$. Clearly, expressions (13) are suitable for practical applications, which makes calculations much simpler.

With the ratio $\alpha/\delta \le 10$ the relative amplitude $\overline{A}_{1/2}$ turns out to be directly proportional to the value of α for a given δ value and inversely proportional to δ for a given α. These $\overline{A}_{1/2}$ regions are shown with solid lines in Fig. 2.

Analysis of the results obtained reveals that for the case of tuned resonance with $\alpha/\delta < 10$ the functions $\overline{A}_{1/2} = f(\alpha, \delta)$ are adequately described – as in the case of a weak resonance – by the formula

$$\overline{A}_{1/2} = \frac{\pi}{2}\frac{\alpha}{\delta}, \tag{15}$$

with the coefficient of proportionality being the same.

For the determination of the lower-harmonic relative amplitude $\overline{A}_{1/2}$ in subharmonic resonance for a specific structural element, it is necessary to find α which depends on the type, relative size and location of a normal-rupture crack as well as on the relative dimenions and vibration mode of the structural element. For example, for a beam of rectangular cross section with a single transverse edge crack the parameter α is given by [4]

$$\alpha = \frac{D(h/l,\, x_{cr})H_1(\gamma)}{1 + D(h/l,\, x_{cr})H_1(\gamma)},$$

where

$$D(h/l,\, x_{cr}) = 2\pi\frac{bh^2 S_1 P^2(x_{cr})}{S_2^2\int_0^l P^2(x)dx}, \qquad H_1(\gamma) = \int_0^\gamma \gamma F_1^2(\gamma)d\gamma,$$

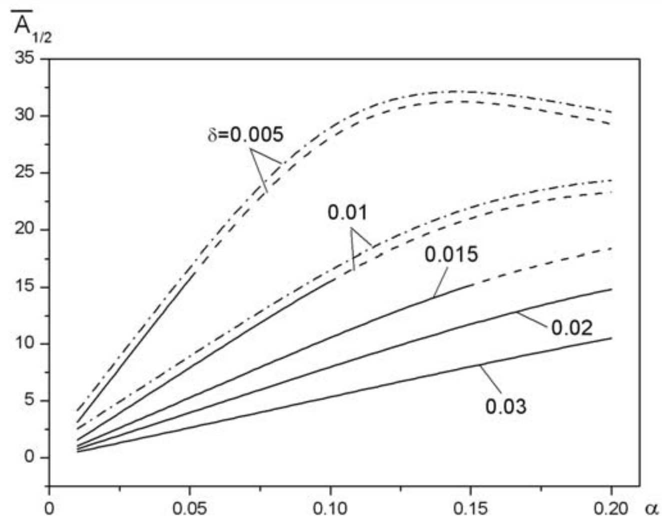

Fig. 2

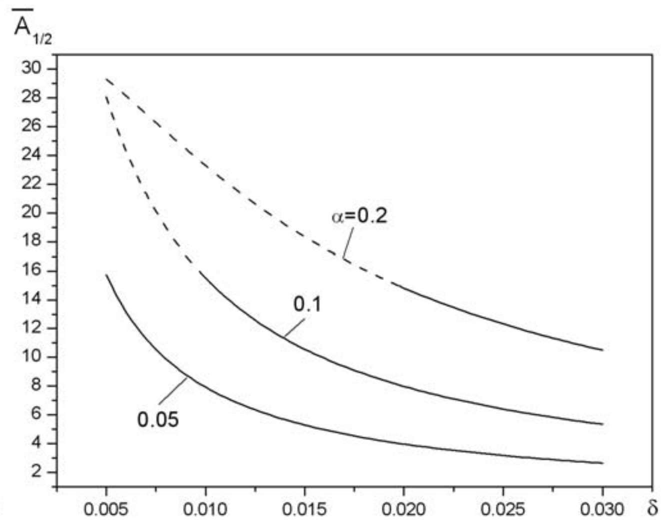

Fig. 3

Fig. 2. Lower-harmonic relative amplitude $\overline{A}_{1/2}$ as a function of α with an excitation frequency $\nu = 2\omega_0$ for various δ values: solid and dashed lines – calculation by formulas (13) and (14); dash-and-dotted lines – calculation by (10), (11), and (12) with β_0 determined from the condition $\sin\beta_0 \approx \arcsin(1/\overline{A}_{1/2})$.

Fig. 3. The functions of the lower-harmonic relative amplitude $\overline{A}_{1/2}$ vs. the logarithmic decrement δ, which were calculated by (13) and (14) for various α values.

where $P(x)$ is the axial force in longitudinal vibrations or the bending moment in bending vibrations of a beam, S_1 is the cross-sectional area in longitudinal vibrations or the axial moment of inertia in bending vibrations, S_2 is cross-sectional area in longitudinal vibrations or section modulus in bending vibrations, b and h are the width and height of the beam cross section, l is the beam length, x_{cr} is the coordinate of the cracked section, γ is the crack relative depth, and $F_1(\gamma)$ is the dimensionless function of the crack relative depth, which is involved in the expression for the normal stress intensity factor.

Using the data on $F_1(\gamma)$ function as provided in [5], we arrive at the following expressions:
for longitudinal vibrations

$$H_1(\gamma) = 0.6272\gamma^2 - 0.17248\gamma^3 + 5.92134\gamma^4 - 10.70538\gamma^5 + 31.56845\gamma^6 - 67.47602\gamma^7$$

$$+139.12342\gamma^8 - 146.6824\gamma^9 + 92.35521\gamma^{10},$$

and for bending vibrations

$$H_1(\gamma) = 0.6295\gamma^2 - 1.0472\gamma^3 + 4.602\gamma^4 - 9.9752\gamma^5 + 20.2948\gamma^6 - 32.9933\gamma^7$$

$$+47.0408\gamma^8 - 40.6933\gamma^9 + 19.6\gamma^{10}.$$

The value of $D(h/l,\ x_{cr})$ depends on the relative height of the beam cross section (h/l), crack location (x_{cr}), beam vibration mode (i) and, for an example, for a cantilever beam in longitudinal vibration, is given by

$$D(h/l,\ x_{cr}) = 4\frac{\pi h}{l}\cos^2\left[\frac{\pi(2i-1)}{2l}x_{cr}\right],$$

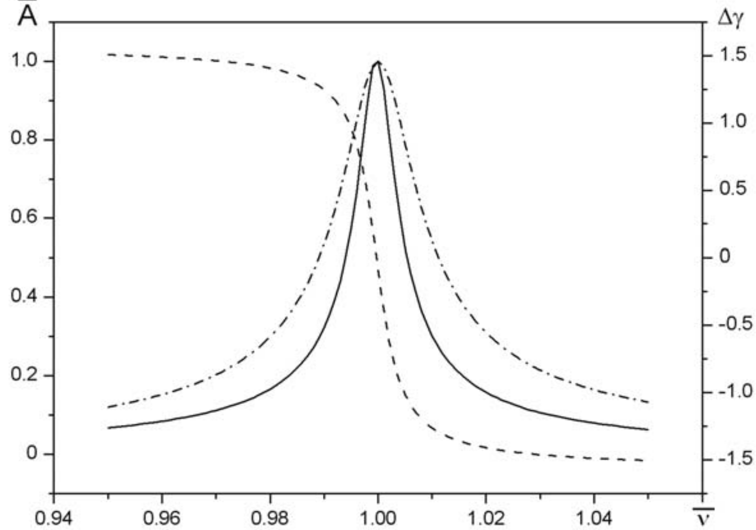

Fig. 4. Frequency response (solid lines) and phase response (dashed line) functions in the subharmonic resonance region, which were determined by formulas (13) and (14), with $\alpha = 0.08$ and $\delta = 0.02011$.

and in flexural vibration by

$$D(h/l,\ x_{cr}) = 24\frac{\pi h}{l}\left[S(k_i x_{cr}) - \frac{S(k_i l)}{T(k_i l)}T(k_i x_{cr})\right]^2,$$

where $S(kx)$ and $T(kx)$ are Krylov's functions,

$$S(kx) = \frac{(\cosh kx + \cos kx)}{2}, \qquad T(kx) = \frac{(\sinh kx + \sin kx)}{2},$$

$$k_1 l = 1.875, \qquad k_2 l = 4.694, \qquad k_3 l = 7.855,$$

for $i > 3$ we have

$$k_i l = \frac{\pi(2i-1)}{2l}.$$

The $D(h/l,\ x_{cr})$ function is constant for the beam vibration mode at hand and given values of h/l and x_{cr}, and the trend of the dependence of $\overline{A}_{1/2}$ on the relative crack depth is governed by the dependence of the $H_1(\gamma)$ function on γ and may deviate significantly from a linear function.

Using expressions (10)–(12) or (13), (14) we can determine the amplitude-response and phase-response characteristics of the system under consideration in the subharmonic resonance region.

By way of illustration, Fig. 4 shows the above functions for $\alpha = 0.08$ and $\delta = 0.02011$. For comparison, the dash-and-dotted line represents the amplitude-response characteristic for the principal resonance region. The functions have been plotted in relative coordinates $\overline{\overline{A}} = \overline{A}_{1/2}(\nu)/\overline{A}_{1/2}(\nu = 2\omega_0)$ and $\overline{\nu} = \nu/2\omega_0$ for a subharmonic resonance and $\overline{A} = A(\nu)/A(\nu = \omega_0)$ and $\overline{\nu} = \nu/\omega_0$ for the principal resonance.

Reliability Assessment of the Proposed Procedure. To assess the reliability of calculated results, we compare them with the data of numerical solution as obtained by the acceleration averaging method [6, 7].

For the case of a tuned subharmonic resonance with $\delta = 0.00503$, Fig. 5 shows the $\overline{A}_{1/2}(\alpha)$ function calculated by (13) and that plotted by the numerical solution data. It is evident that for $\alpha < 0.1$, which corresponds to

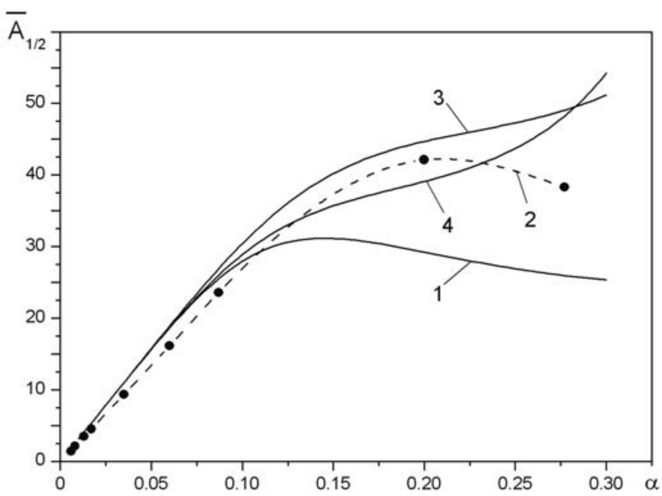

Fig. 5

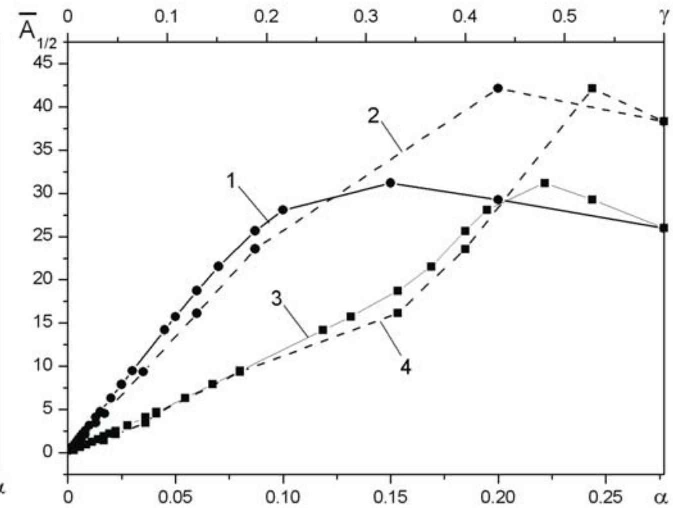

Fig. 6

Fig. 5. The lower-harmonic relative amplitude as a function of α with $\delta = 0.00503$ under subharmonic resonance: (*1*) by formula (13); (*2*) by the data of numerical solution; (*3*) by formula (13) with the coefficient at α^2 being equal to 1; (*4*) by formula (13) with the coefficient being equal to $(2-\alpha)/2 + 0.141471$.

Fig. 6. The functions of the lower-harmonic relative amplitude $\overline{A}_{1/2}$ vs. α (*1*, *2*) and relative crack depth γ (*3*, *4*) in subharmonic resonance: (*1*, *3*) calculations; (*2*, *4*) numerical solution.

$\alpha/\delta < 20$, there is a fairly good agreement between the functions. Also, this confirms the existence of almost linear dependence of the lower-harmonic relative amplitude on the α parameters in this α/δ range, with the coefficient of proportionality deviating slightly from the calculated one.

However, as the α parameter increases the discrepancy between the analytical and numerical solutions grows considerably. Apparently, this is attributable to a possible change in relative amplitude of the harmonic $A_2 \cos 2(\nu t/2 - \gamma_{1/2})$ at high α values. Specifically, taking $A_2 = 0$, which requires that the coefficient 1.141471 at α^2 should be replaced with 1.0 in formulas (13) and (14), we have the relationship shown with curve *3* in Fig. 5. It is possible that the value of the constant component may undergo a change with increasing α. In particular, setting $A_0 = A_{1/2}\,\alpha(2-\alpha)/2\pi$ in place of $A_0 = A_{1/2}\,\alpha/\pi$ [1], which makes the coefficient at α^2 equal to 0.141471 + $(2-\alpha)/2$, we arrive at the function $\overline{A}_{1/2}(\alpha)$ shown with curve *4* in Fig. 5. This explanation makes sense, as the values of the constant component A_0 and the second-harmonic amplitude A_2 in the free vibration spectrum were obtained in [3] by the asymptotic method of nonlinear mechanics, which assumes a low value of the α parameter.

To assess the change in trend of $\overline{A}_{1/2}$ function when switching from the α parameter to the crack relative depth γ, Fig. 6 gives, as an example, curves of $\overline{A}_{1/2}$ vs. α and γ for the case of $\delta = 0.00503$. The initial curve *1* was calculated by formula (13) using (14), while curve *2* was plotted by the data of numerical solution. For curves *3* and *4* the data on the ratio between α and γ were taken for the case of longitudinal vibration of a beam of rectangular cross-section, $h/l = 0.13333$ and $x_{cr}/l = 0.2$ [8].

Analysis of the numerical solution data for other values of the logarithmic decrement δ has demonstrated that for the range of stable values of $\overline{A}_{1/2}$ above unity the results of the proposed approximate calculations with $\alpha/\delta < 20$ have been verified as well. The amplitude $\overline{A}_{1/2}$ is directly proportional to α for a given value of δ and inversely proportional to δ at a given value of α. Furthermore, the numerical solution is adequately described by a unified relation $\overline{A}_{1/2} = K\alpha/\delta$ with the coefficient of proportionality $K \cong 1.34$ which is 14.7% lower than the calculated one.

Determination of Absolute Values of Amplitude and Phase Difference for Individual Harmonics. As in [1, 2], we assume that the first-harmonic amplitude $A_1 \sin(\nu t - \gamma_1)$ corresponds to a solution for forced vibrations of a linear system with the natural frequency of a body with a closing crack – formula (3),

$$\frac{A_i}{q_0} = \frac{1}{\omega_0^2 \sqrt{\left[1 - \left(\frac{\nu}{\omega_0}\right)^2\right]^2 + \frac{1}{4}\left(\frac{\delta}{\pi}\right)^2 \left(\frac{\nu}{\omega_0}\right)^2}}, \tag{16}$$

while the phase difference γ_1 is determined from the balance of input energy ΔW_q and absorbed energy ΔW_h per cycle of vibration with a period of $2\pi/\omega_0$.

For the present case of single-harmonic excitation and viscous friction we have

$$\Delta W_q = \pi q_0 \left(\sin\gamma_1 - \frac{2\alpha}{9\pi} \bar{A}_{1/2} \cos 2\gamma_{1/2} \right) A_1, \tag{17}$$

$$\Delta W_h \approx 2\pi h \nu \left\{ 1 + \frac{1}{4}[1 + 0,02\alpha^2]\bar{A}_{1/2}^2 \right\} A_1^2. \tag{18}$$

Unlike the weak resonance case, where $A_{1/2} < A_1$, expression (18) includes also the second harmonic of the free vibration spectrum, i.e., $2\alpha A_{1/2} \cos(\nu t - 2\gamma_{1/2})/9\pi$, thus giving rise to the term $0.02\alpha^2$.

From the condition $\Delta W_q = \Delta W_h$ in view of (16) and (8), (14) we derive an equation for γ_1,

$$(1 - 0.07074\alpha \sin\Delta\gamma)\sin\gamma_1 - 0.07074\alpha\cos\Delta\gamma\cos\gamma_1 = 2h\nu\left[1 + \frac{(1+0.02\alpha^2)\bar{A}_{1/2}^2}{4}\right]\frac{A_1}{q_0},$$

where A_1/q_0 is calculated from formula (16). With $\Delta\gamma$ and γ_1 available according to (8) we determine the value $\gamma_{1/2} = (\Delta\gamma + \gamma_1)/2$.

CONCLUSIONS

1. We have discussed an approximate method for calculating vibration parameters of an elastic body with a closing crack, which is modeled by a single-mass system with an asymmetric bilinear characteristic of the restoring force, in the region of a 1/2-order subharmonic resonance.

2. Analytical expressions have been derived for the determination of the main vibrodiagnostic indicator of a crack – the lower-harmonic relative amplitude $\bar{A}_{1/2}$ – in the subharmonic resonance region.

3. The results of calculation of the vibrodiagnostic parameter $\bar{A}_{1/2}$ are in good agreement with the data of numerical solution for $\alpha/\delta < 20$.

4. With the ratio $\alpha/\delta \le 10$ the lower-harmonic relative amplitude $\bar{A}_{1/2}$ has been found to be directly proportional to the parameter of nonlinearity of a vibrating system α and inversely proportional to the logarithmic decrement δ of the system's vibrations.

5. Function $\bar{A}_{1/2}(\alpha, \delta)$ is described by a unified formula $\bar{A}_{1/2} = \frac{\pi}{2}\frac{\alpha}{\delta}$ with an accuracy sufficient for practical applications.

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