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COMPARATIVE ANALYSIS OF NONLINEAR RESONANCES OF A MECHANICAL SYSTEM WITH UNSYMMETRICAL PIECEWISE CHARACTERISTIC OF RESTORING FORCE

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We present results of a numerical comparative analysis of superharmonic resonances of the order 2/1–1/5 and subharmonic resonance of the order 1/2 of a mechanical single degree-of-freedom vibrating system with unsymmetrical characteristic of restoring force at different ratios of system rigidities in half-cycles and under the conditions of considerable change in vibration damping level in the system.

Keywords: closing crack, sub- and superharmonic resonance, vibration spectrum, diagnostic signs of damage.

Introduction. In the case of cyclic deformation of an elastic body, fatigue crack has the property of opening in the extension half-cycle and closing in the compression half-cycle (closing crack). At the instants of crack closure and opening, an abrupt change in the rigidity of the body takes place, which gives rise to a considerable nonlinearity of its dynamic behavior and, as a consequence, to the so-called nonlinear effects, to which sub- and superharmonic (nonlinear) resonances belong, as well as to nonlinearity of the vibrational response of the mechanical system (displacement, velocity, acceleration, strain, etc.) in resonant regimes of vibration.

Periodic change in the rigidity of a mechanical system gives rise to a number of difficulties, which arise in the analytical solution of the problem of its forced vibrations. Approximate analytical solutions [1–7] are limited by simplifying assumptions of the properties of vibrating system and consider one or two nonlinear resonances, which makes it impossible to perform a comparative analysis of super- and subharmonic resonances of different order. The order of nonlinear resonance is determined by the number of super- or subharmonic in the vibration spectrum, whose amplitude increases monotonically as its frequency approaches the eigenfrequency of the vibrating system and reaches a maximum when these frequencies coincide. The superficial resemblance of the phenomenon to resonance determined its name – nonlinear resonance.

In [5–9] consider superresonance of second order and [10, 11] subharmonic resonance of the order $f_c / p = 1/2$ (f_c is vibration eigenfrequency of a cracked body and p is driving-force frequency). Since super-resonance vibrations of second and third order of a single degree-of-freedom system had been investigated in [1, 2] without regard to damping, it appears to be impossible to compare the amplitudes of these nonlinear resonances.

In [12–14], the crack opening-closure process was determined by driving-force frequency. This model describes incorrectly the behavior of vibrating system in the cases of strong superharmonic resonances, where the higher-harmonic amplitudes reach the values at which the system changes its rigidity more than once within vibration period [15]. Taking this phenomenon into account is one of the main difficulties in the analytical study of superharmonic resonances.

Numerical solutions of the problem of forced vibrations of a bar with closing crack by the action of a harmonic concentrated force, which were obtained by electric simulation [16–20] or by finite-element simulation [15, 21–23], showed its vibrations to be characterized by the presence of superharmonic resonance of second order, whose amplitude is by an order of magnitude smaller than that of main resonance. It has also been found that under

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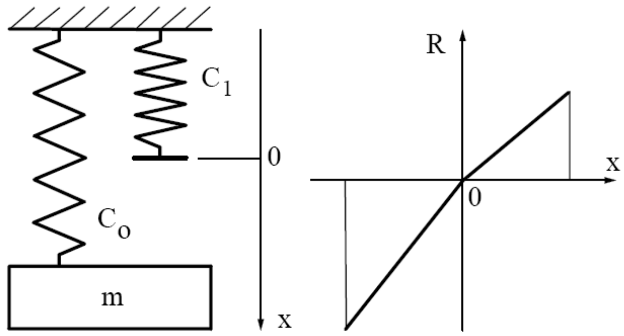

Fig. 1. Model of a cracked body.

superharmonic resonance, there is a considerable nonlinearity of the vibration process, which is characterized by a large second harmonic amplitude in the vibration spectrum. Moreover, in [16, 17] superresonances of the orders $f_c/p = 3/1$ and 4/1 and subresonance of the order $f_c/p = 1/2$ were detected, and it was shown in particular that superharmonic resonance of second order occurs at much lower crack depth values than subharmonic resonance. In the above works, the possibility of change in damping level as a result of crack initiation was not taken into account [24]. At the same time, the suppressing influence of damping on nonlinear effects was pointed out, e.g., in [5, 25, 26]. It follows that when studying the interrelation between damage parameters and nonlinear-effect parameters, the change in damping level in the system because of crack initiation and growth must be taken into account.

The aim of the work was computational investigation and comparative analysis of the peculiarities of the manifestation of nonlinear resonances of a vibrating system with unsymmetrical piecewise characteristic of restoring force at different damping levels in the system and sensitivity of nonlinear effects under nonlinear resonances of different orders to the presence of a damage, such as closing crack, in the mechanical system.

Cracked-Body Model. A relatively simple and easy-to-grasp model of elastic body with a closing crack in the form of a single degree-of-freedom mechanical vibrating system with unsymmetrical piecewise characteristic of restoring force (Fig. 1) and linear viscous friction is employed. In some cases, this model allows one to describe with accuracy sufficient for engineering practice the dynamic behavior of complex constructions [27].

Forced vibrations of the above system are described by the differential equation:

$$\frac{d^2x}{dt^2} + 2\alpha_m \frac{dx}{dt} + \frac{\omega^2}{C} R(x) = q_0 \sin pt. \tag{1}$$

Here α_m is viscous friction coefficient, ω is the angular vibration eigenfrequency of an undamaged body, q_0 is driving-force amplitude per unit of generalized mass, C is the generalized rigidity of a system that models an uncracked body of mass m or in the vibration half-cycle when the crack is closed, and $R(x)$ is the restoring force, the equations of which are of the form:

$$R(x) = \begin{cases} C_o x, & x \geq 0, \\ Cx, & x < 0, \end{cases} \tag{2}$$

where C_o is the system rigidity in the vibration half-cycle when the crack is open. Using the symbols adopted in Fig. 1, we have $C = C_o + C_1$.

An exact analytical solution of Eq. (1) can be obtained for the case of free vibrations of the system under consideration without regard for damping [28] or with allowance for linear viscous friction [29]. For the case of forced vibrations of the system, approximate analytical solutions have been obtained with due regard for damping [3, 4, 7], which are limited by a number of simplifying assumptions of the properties of the vibrating system.

In the present work, Eq. (1) was solved numerically by the Newmark method [30], which imposes no restrictions on the nonlinearity level of vibrating system and allows one to obtain solutions with high accuracy [31].

TABLE 1. Amplitude Ratio of Main Resonance and Nonlinear Resonances

C_o/C	$\gamma = a/h$	$S_{1/1}/S_{1/2}$	$S_{1/1}/S_{2/1}$	$S_{1/1}/S_{3/1}$	$S_{1/1}/S_{4/1}$	$S_{1/1}/S_{5/1}$
0.998	0.0122	– / –	421.5 / –	– / –	574.7 / –	– / –
0.991	0.0500	637.7 / –	280.2 / 45.5	550.8 / –	494.1 / 58.2	595.2 / –
0.983	0.1000	381.1 / –	233.1 / 43.0	548.2 / –	445.5 / 57.5	594.4 / –
0.965	0.2000	194.0 / –	179.6 / 36.5	525.3 / 54.9	412.0 / 55.4	581.0 / –
0.913	0.4000	77.3 / 62.9	124.0 / 27.7	380.1 / 52.9	387.9 / 48.7	504.3 / 57.7
0.723	0.6000	21.9 / 21.2	70.0 / 17.7	277.8 / 34.4	361.4 / 39.0	314.7 / 46.1

Note. Over the line are given the values corresponding to $\delta_\alpha = 0.5\%$ and under the line the values corresponding to $\delta_\alpha = 5.0\%$.

The main problem of adequate modeling of a cracked body is a justified choice of the rigidity ratio of vibrating system, C_o/C. In what follows, such a choice was made on the basis of experimental data on the variation of the longitudinal vibration resonance frequency of a steel specimen with a massive weight at the end as a function of fatigue edge crack depth [31]. The rigidity ratio of a vibrating system at corresponding values of relative crack depth $\gamma = a/h$ (a is crack depth and h is specimen cross-section height) is presented in Table 1.

The spectral analysis of vibrations of the system under investigation was performed by means of Fourier series of the form:

$$x = \frac{a_0}{2} + \sum_{k=1}^{\infty} A_k \sin(k\omega_c t + \varphi_k), \tag{3}$$

where $A_k = \sqrt{a_k^2 + b_k^2}$, $\varphi_k = \arctan(a_k/b_k)$, and ω_c is the angular vibration eigenfrequency of a cracked body.

Results of Calculations. The amplitude-frequency characteristics (AFC) of the vibrating system under consideration have been obtained in a wide frequency range at the damping level which corresponded to the value of the logarithmic vibration decrement of the system, $\delta_\alpha = 0.5\%$. The dynamicity coefficient β was determined as the ratio of the sustained forced vibration amplitude of damaged system to the static deflection of undamaged system.

Figure 2 shows as an example AFCs at two rigidity ratio values. As can be seen, when the system rigidity changes substantially ($C_o/C = 0.913$), strong super-resonances of second-fifth order and a sub- resonance of the order $f_c/p = 1/2$ occur (Fig. 2a). In this case, the amplitudes of sub- and superresonance of second order exceed those of superresonances of third-fifth order by a factor of over 3.1–4.8. The unusual shape of the AFCs of nonlinear resonances of odd orders and the fact that these resonant modes do not practically manifest themselves at small relative change in rigidity (Fig. 2b), which corresponds to a crack of an area of 5% of cross-section, are noteworthy. The change in the amplitude of subresonance of the order $f_c/p = 1/2$ and superresonances of second and fourth order relative to the vibration amplitude of undamages system is still significant and is 2.9-, 1.7- and 1.4-fold, respectively.

At all vibrating system damage levels under consideration, the vibration amplitudes under nonlinear resonances are by one or two orders of magnitude smaller than the vibration amplitude under main resonance (Table 1). The difference between the amplitudes decreases with increasing damage level and vibration damping level in the system, but remains considerable (note that damping affects only slightly the vibration amplitude ratio under main and subharmonic resonance). The absolute value of the vibration amplitude under nonlinear resonances is low, and reliable detection of nonlinear resonance by its change may turn out to be problematic. Therefore, the change in vibration amplitude in the vicinity of nonlinear resonances is of little use for damage diagnostics.

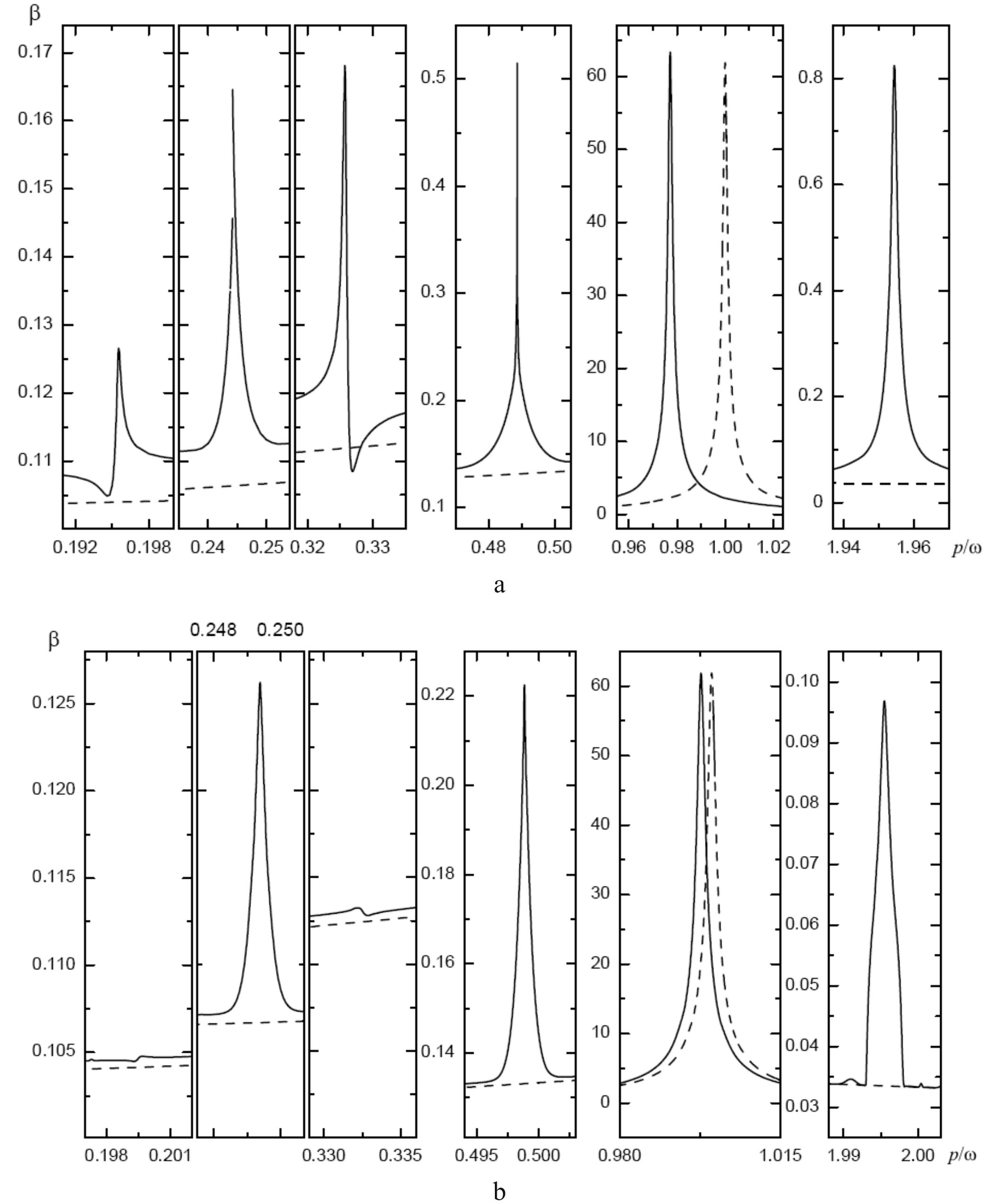

Fig. 2. Amplitude-frequency characteristics of a damaged (solid lines) and an undamaged (dashed lines) vibrating system: (*1*) $C_o/C = 0.913$, $\gamma = 0.4$, $\delta_\alpha = 0.5\%$; (*2*) $C_o/C = 1.0$, $\gamma = 0$, $\delta_\alpha = 0.5\%$; (*3*) $C_o/C = 0.991$, $\gamma = 0.05$, $\delta_\alpha = 0.5\%$.

The possibility of exciting exact resonances and hence of practical application of nonlinear effects as diagnostic signs of damage depends on the width of resonance curves. As is known, in the case of sharp resonance, a high discreteness of driving-force frequency variation is required for the accurate determination of resonance curve parameters.

Resonance curves of the same shape as in Fig. 2 have been obtained for the values of system rigidity ratio listed in Table 1. The width of the resonance curves was determined directly from AFCs at an amplitude that was equal to the half amplitude of main resonance ($\Delta_{1/1}$), subresonance of the order $f_c/p = 1/2$ ($\Delta_{1/2}$) and super-resonances of second-fifth order ($\Delta_{2/1}$, $\Delta_{3/1}$, $\Delta_{4/1}$, and $\Delta_{5/1}$, respectively). The results of the measurement are listed in Table 2.

TABLE 2. Width of Resonance Curves ($\delta_\alpha = 0.5\%$)

$\gamma = a/h$	$\Delta_{1/2}$	$\Delta_{1/1}$	$\Delta_{2/1}$	$\Delta_{3/1}$	$\Delta_{4/1}$	$\Delta_{5/1}$
0.0122	–	0.00277	0.00069	–	0.00030	–
0.0500	0.00235	0.00279	0.00132	–	0.00063	–
0.1000	0.00238	0.00276	0.00124	–	0.00065	–
0.2000	0.00245	0.00277	0.00055	–	0.00088	–
0.4000	0.00256	0.00281	0.00020	0.00122	0.00170	0.00128
0.6000	0.00306	0.00302	0.00008	–	–	–

As can be seen, the widths of the resonance curves under main and subharmonic resonances are purely comparable. In both cases, there is a tendency to some increase in the width of AFCs with crack growth. A more complex dependence exists under superharmonic resonance of second order: the width of AFCs increases with decrease in crack size to $\gamma = 0.05$, but remains a factor of 2.1–37.8 smaller than under main resonance. This conclusion agrees qualitatively with the results obtained for a single degree-of-freedom system, but at higher C_o/C values and a higher damping level [5]. Thus, it is more difficult to excite an exact superharmonic resonance of second order than main resonance or subharmonic resonance of the order $f_c/p = 1/2$.

In contrast to superharmonic resonance of second order, the width of resonance curve increases in direct proportion to the crack size under superharmonic resonance of fourth order (Table 2). In this case, its width remains small for small and medium cracks (we adopted the following conventional classification of cracks by their size: $\gamma \le 0.1$, small cracks; $0.1 < \gamma \le 0.3$, medium cracks; $\gamma > 0.3$, large cracks).

Due to the special shape of resonance curves under odd superharmonic resonances, it appears to be impossible to accurately determine their width. The data obtained for these regimes of vibration at $C_o/C = 0.913$ are, strictly speaking, conventional; it can be concluded, however, that the width of AFCs under nonlinear resonances of third and fifth order is approximately half as large as that under main resonance.

As is evident from Table 3, as the crack becomes larger the frequencies of main resonance and nonlinear resonances of the system decrease monotonically (to 8% in the crack size range in question). This is also evidenced by AFC shift towards lower frequencies under main resonance (Fig. 2).

In [5] it was shown that the frequency of exact superresonance of second order is somewhat lower than the frequency at which the second harmonic, which predominates in the vibration spectrum under the above conditions, reaches the highest value. Therefore, in addition to the exact frequency value of main resonance and nonlinear resonances, the frequencies at which the amplitude of the predominant harmonic in the vibration spectrum reaches a maximum have been calculated (Table 4). The subharmonic whose frequency is half as high as driving-force frequency is predominant in the spectrum of subharmonic resonance of the order $f_c/p = 1/2$, and the second-fifth harmonics are predominant in the spectrum of superresonances of second-fifth order, respectively.

A comparison of the data listed in Tables 3 and 4 showed that the frequencies of exact resonances really differ, though only slightly, from the frequencies of the amplitude maxima of the predominant harmonies: under subresonance, this difference reaches 0.0008% and under superresonances of second, third, fourth and fifth order 0.0031, 0.18, 0.069, and 0.052%, respectively. The frequencies of exact superresonances of second-fourth order are lower than those of the amplitude maxima of the predominant harmonics, and the frequencies of exact subresonance of the order $f_c/p = 1/2$ and exact superresonance of fifth order are higher. The above analysis did not take account of data for the rigidity ratio $C_o/C = 0.723$ since due to a considerable nonlinearity of the vibrating system, AFCs have a break under superresonance of fourth order (Fig. 2a). It is known [31] that the second-harmonic amplitude in the vicinity of main resonance has no extremum: it decreases monotonically when the driving-force frequency increases. Therefore, the data relating to main resonance are not given in Table 4.

The occurrence of nonlinear resonances is a qualitative sign of the presence of a damage, such as fatigue crack, and the amplitudes of these resonances allow one to judge the damage level. It is known that vibration processes are essentially nonharmonic (are distorted nonlinearly) under nonlinear resonances; this can also be utilized for the qualitative and quantitative assessment of damage. The cause of considerable nonlinear distortions of

TABLE 3. Frequencies of Main Resonance and Nonlinear Resonances ($\delta_\alpha = 0.5\%$)

$\gamma = a/h$	$\omega_{1/2}$	$\omega_{1/1}$	$\omega_{2/1}$	$\omega_{3/1}$	$\omega_{4/1}$	$\omega_{5/1}$
0.0122	–	0.999486	0.499723	–	0.249817	–
0.0500	1.995469	0.997724	0.498835	0.331933	0.249384	0.199607
0.1000	1.991416	0.995695	0.497823	0.331503	0.248878	0.199126
0.2000	1.982175	0.991079	0.495514	0.330163	0.247705	0.198261
0.4000	1.954481	0.977238	0.488592	0.325693	0.244171	0.195489
0.6000	1.838169	0.919077	0.459532	0.306334	0.230217	0.184278

Note. The subscripts on ω denote the order of nonlinear resonance.

TABLE 4. Frequencies of the Maxima of the Predominant Harmonics in Vibration Spectra under Main Resonance and Nonlinear Resonances ($\delta_\alpha = 0.5\%$)

$\gamma = a/h$	$\omega_{1/2}$	$\omega_{2/1}$	$\omega_{3/1}$	$\omega_{4/1}$	$\omega_{5/1}$
0.0122	–	0.499723	–	0.249822	–
0.0500	1.995468	0.498844	0.332542	0.249385	0.199530
0.1000	1.991411	0.497832	0.331865	0.248882	0.199077
0.2000	1.982159	0.495522	0.330326	0.247738	0.198154
0.4000	1.954361	0.488607	0.325724	0.244339	0.195387
0.6000	1.836734	0.459524	0.307142	0.230503	0.184714

Note. The dashes denote that the corresponding nonlinear resonances are absent.

vibrations under nonlinear resonances is that in these regimes, a harmonic whose frequency coincides with the main-resonance frequency appears in the vibration spectrum. The amplitude of this harmonic is comparable to or in large excess over the fundamental harmonic amplitude under exact nonlinear resonance. Under superharmonic resonances of the order $f_c/p = 2/1,\ 3/1,\ 4/1$, etc., the second, third, fourth, etc, harmonics, respectively, and under subharmonic resonance of the order $f_c/p = 1/2$, the first subharmonic prevail in the vibration spectrum. Therefore, the ratio of the predominant-harmonic amplitude in the vibration spectrum to the fundamental harmonic amplitude may be used as a sign of damage, neglecting other components of the spectrum.

Let us confine ourselves to analysis of three nonlinear resonances which are of the greatest practical interest: subharmonic resonance of the order $f_c/p = 1/2$ and superharmonic resonances of the orders $f_c/p = 2/1$ and $3/1$. Let us use the ratio of the predominant-harmonic amplitude in the vibration spectrum ($A_{1/2}$, $A_{2/1}$, and $A_{3/1}$) to the first-harmonic amplitude (A_1) and the ratio of the amplitude of exact nonlinear resonance ($S_{1/2}$, $S_{2/1}$, and $S_{3/1}$) to the forced-vibration amplitude of undamaged system at the same frequency (S) as diagnostic signs of damage.

As follows from Fig. 3, under subharmonic resonance, crack growth in the range $0.1 < \gamma \le 0.6$ leads to a considerable increase (of almost two orders of magnitude) in both diagnostic signs of damage in question. Under superharmonic resonances of second and third order, the analogous changes are by one and two orders of magnitude, respectively, smaller. Whereas under superharmonic resonance of second order the diagnostic signs of damage change materially when there are medium and large cracks, as is the case with subharmonic resonance, under superharmonic resonance of third order, they do so only when there are large cracks.

The dependences of relative changes in predominant harmonics and vibration amplitudes on the crack size in corresponding regimes of vibration are qualitatively identical. This means that the amplitude of exact nonlinear resonances is mainly determined by the amplitude of the predominant harmonic in the vibration spectrum, whose frequency coincides with the main-resonance frequency. The fractional increase in nonlinear-resonance amplitude is quantitatively somewhat larger than that of the predominant harmonics. The reason of this is that both characteristics were determined at exact-resonance frequencies, which, as was shown above, do not coincide with the frequencies at which the amplitudes of the predominant harmonics reach maximum values.

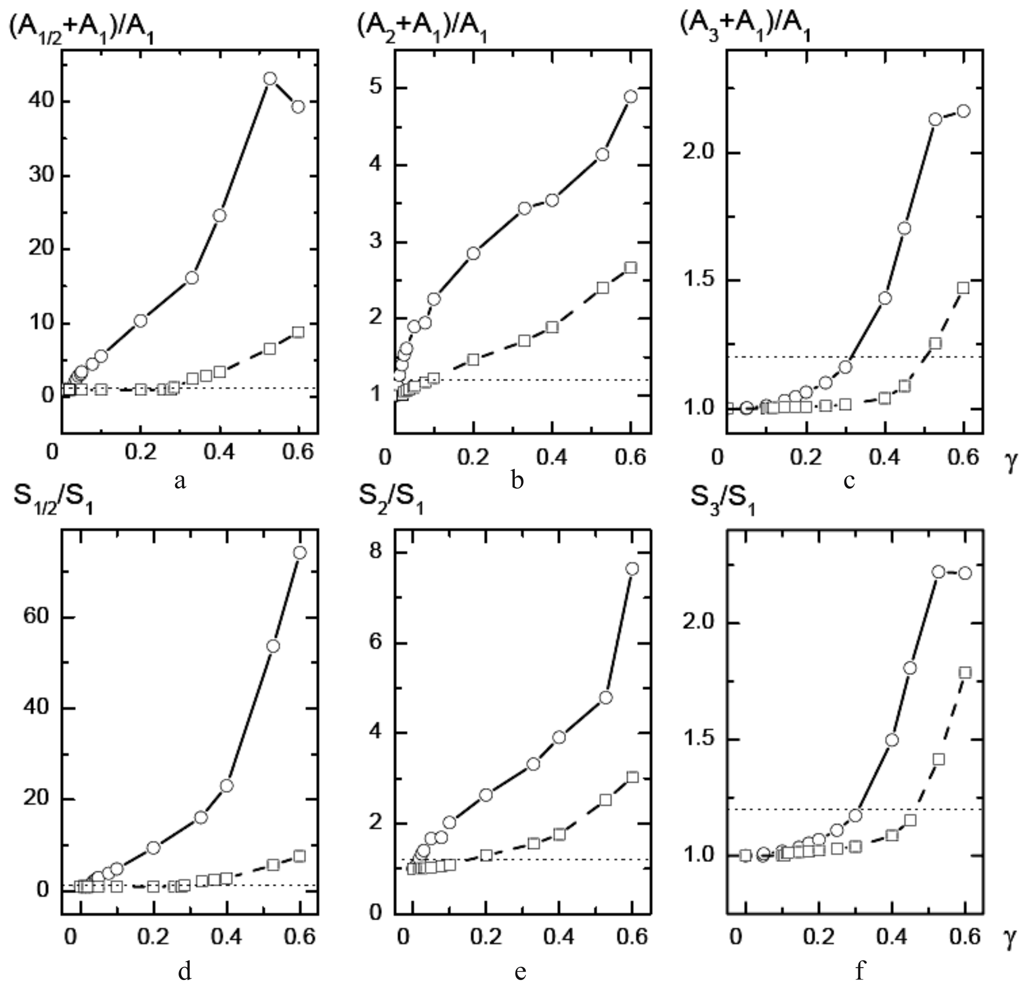

Fig. 3. Plots of relative change in predominant harmonic (a–c) and resonance amplitude (d–f) against damage size under subharmonic resonance of the order 1/2 (a, d) and superharmonic resonance of the orders 2/1 (b, e) and 3/1 (c, f). (Here and in Fig. 4: solid lines, $\delta_\alpha = 0.5\%\%$; dashed lines, $\delta_\alpha = 5.0\%$.)

Damping reduces the sensitivity of all above diagnostic signs to the presence of a damage. The increase of an order of magnitude in the vibration decrement of the system decreases the subresonance amplitude by a factor of up to 10, the amplitude of superresonance of second order by a factor of up to 2.5, and the amplitude of superresonance of third order by a factor of up to 1.6, which is due to a corresponding decrease in predominant-harmonic amplitude in the vibration spectrum. In this case, the trend of the plots of the signs of damage against relative crack size does not change qualitatively.

Since the diagnostics of small damages is of the greatest practical interest, Fig. 4 shows the same plots as Fig. 3, but at smaller relative crack size. As can be seen, there is a qualitative difference between sub- and superharmonic resonances. Subharmonic resonance occurs when the crack reaches a definite value, which agrees with the results of [16]. For example, at damping levels of $\delta_\alpha = 0.5$ and 5.0%, a subharmonic appears in the vibration spectrum at crack sizes of $\gamma = 0.03$ and 0.28 respectively. Further crack growth leads to a considerable increase in subharmonic. The increase in the amplitudes of the predominant harmonics with increasing crack size occurs at a lower rate under superharmonic resonances, but is observed from the instant of crack nucleation. The absolute value of this increase under superharmonic resonance of third order is negligible in the region of small racks. At the same time, the rate of change of the second-harmonic amplitude in the region of small cracks under superharmonic resonance of second order and at low damping level is sufficiently high for practical application.

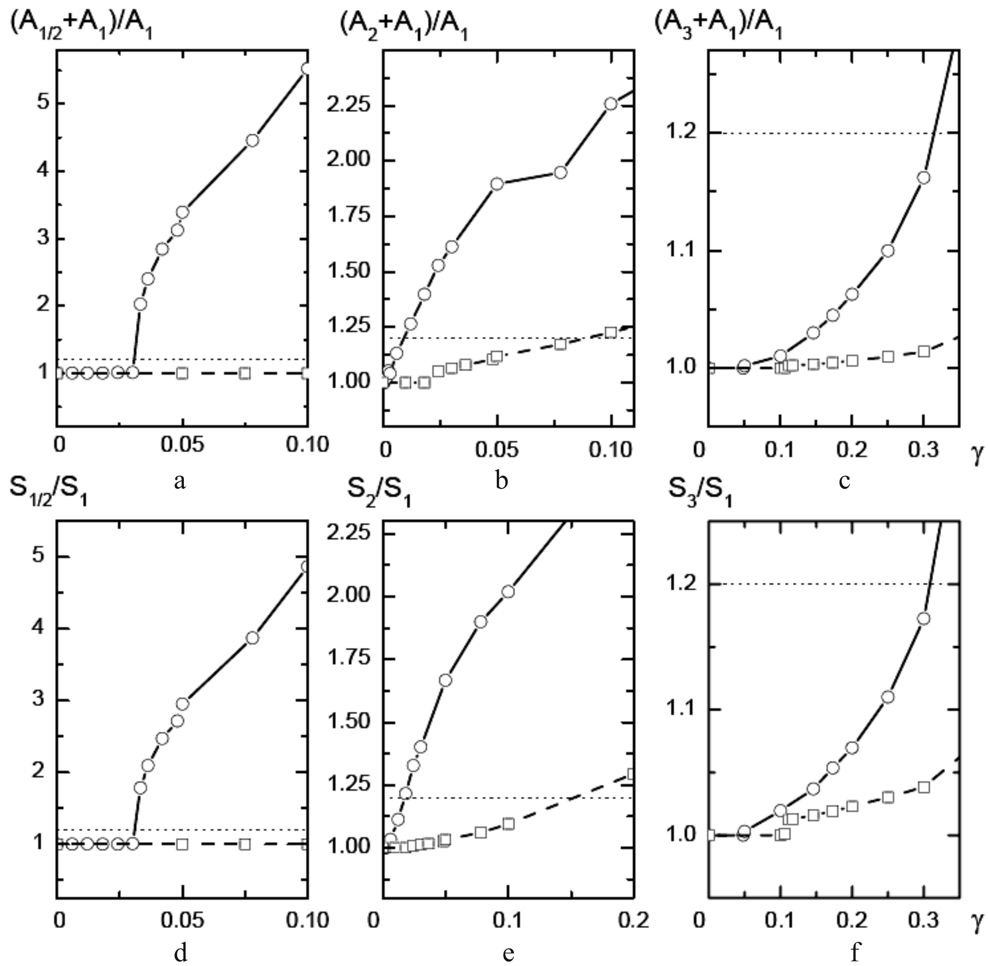

Fig. 4. Plots of relative change in predominant harmonic (a–c) and resonance amplitude (d–f) against damage size under subharmonic resonance of the order 1/2 (a, d) and superharmonic resonance of the orders 2/1 (b, e) and 3/1 (c, f) in the region of small cracks.

Reliable experimental recording of the diagnostic sign of damage depends on the precision and resolution of measuring equipment and on the noise level of electronic devices since vibration-based diagnosis of damages is usually performed at low vibration levels. These factors determine to one extent or another the detection limit of any damage diagnosis technique, i.e., the minimal damage size that can be reliably detected.

In the further analysis it is assumed conventionally that the relative determination error of one damage characteristic or another is 20%. It follows that damage can be reliably detected if the change in diagnostic sign exceeds this value. In Figs. 3 and 4, the assumed error is indicated by dashed lines. As can be seen, allowance for the experimental-determination error of the sign of damage reduces the sensitivity of both diagnosis methods. Whereas under subharmonic resonance of the order $f_c/p=1/2$ and superharmonic resonance of the order $f_c/p=2/1$ at a low vibration damping level in the system this reduction is relatively small, under superharmonic resonance of the order $f_c/p=3/1$ it is so large that it is hardly expedient to use this regime of vibration in practice.

The minimal damage size that is detected by both diagnosis methods with allowance for measurement error depends largely upon the vibration damping level in the system. Table 5 lists results of calculations of minimal crack sizes which were determined at 20% change in the predominant harmonic in the vibration spectrum under subharmonic resonance of the order $f_c/p=1/2$ and superharmonic resonances of the order $f_c/p=2/1$ and $3/1$ and at a change of two orders of magnitude in vibration damping level in the system. An analysis of the results of calculations showed superharmonic resonance of second order to be more sensitive to the presence of damage. For

TABLE 5. Minimal Crack Sizes Calculated from the Variation of the Predominant Harmonic in the Vibration Spectrum under Subresonance of the Order 1/2 and Superresonances of the Orders 2/1 and 3/1

δ_α, %	$\gamma_{1/2}$	$\gamma_{2/1}$	$\gamma_{3/1}$	$\gamma_{1/2}/\gamma_{2/1}$	$\gamma_{3/1}/\gamma_{2/1}$
0.101	0.013	0.0018	0.162	7.22	90.00
0.503	0.033	0.0090	0.314	3.67	34.89
1.005	0.069	0.0183	0.384	3.77	20.98
2.011	0.120	0.0363	0.440	3.31	12.12
5.027	0.285	0.0895	0.503	3.18	5.62
10.053	0.420	0.1732	0.575	2.42	3.32

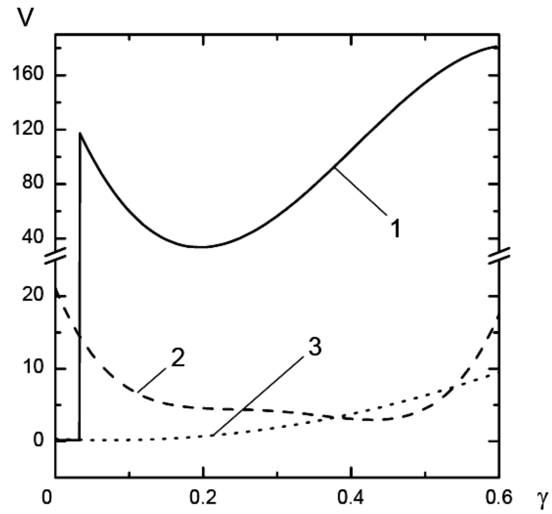

Fig. 5. Rate of change of plots of predominant harmonics under subharmonic resonance of the order 1/2 (*1*) and superharmonic resonance of the orders 2/1 (*2*) and 3/1 (*3*) against damage size on unit increment of damage($\delta_\alpha = 0.5\%$).

instance, at the lowest vibration damping level studied in the system ($\delta_\alpha = 0.101\%$), the sensitivity of nonlinear distortions of vibrations under superharmonic resonance of second order is 7.2 times as high as under subharmonic resonance and 90.0 times as high as under superharmonic resonance of third order. This difference decreases with increasing damping level in the system since increase in damping reduces most greatly the sensitivity of superharmonic resonance of second order. For example, when the vibration decrement of the system increases by a factor of 100, the sensitivity of superharmonic resonance of second order decreases by a factor of 96, that of subharmonic resonance of the order $f_c/p = 1/2$ by a factor of 32, and that of superharmonic resonance of third order by a factor of 3.5.

An important practical characteristic of diagnosis methods is, in addition to damage sensitivity, the power of change in the function describing the dependence of the diagnostic sign on the damage size, $F(\gamma)$, on unit increment of damage size:

$$V = \frac{\partial F(\gamma)}{\partial \gamma}. \quad (4)$$

As is seen from Fig. 5, the diagnosis method based on the variation of the predominant harmonic under subresonance of the order $f_c/p = 1/2$ is most sensitive to crack growth. The main disadvantage of this method is that it is insensitive to the presence of crack before it reaches a certain size, which is directly dependent on the vibration damping level in the system. The threshold of complete insensitivity of this method is the higher, the higher the damping level in the system. From Table 5 it follows that at a damping level of $\delta_\alpha = 2.0\%$, the above method is insensitive to the presence of small cracks.

We obtained that the change in the predominant harmonic under superharmonic resonance of the order $f_c / p = 3/1$ in comparison with that of the order $f_c / p = 2/1$ is more pronounced when the damage size changes no corroboration of the conclusion [16]. From Fig. 5 it follows that the rate of change of diagnostic sign under superharmonic resonance of the order $f_c / p = 2/1$ in the range of small and medium cracks is higher than under superharmonic resonance of the order $f_c / p = 3/1$.

Conclusions. A peculiarity of vibrations of a mechanical system with unsymmetrical piecewise characteristic of restoring force is the presence of a number of nonlinear resonances whose amplitudes are much smaller than the main-resonance amplitude. The width of AFC of odd superharmonic resonances is comparable to that of main resonance. The width of amplitude-frequency characteristic of even superharmonic resonances is much smaller than that of main resonance, in view of which the excitation of exact superharmonic resonances of even orders becomes problematic.

The most sensitive characteristic to the presence of damages of medium and large size is the nonlinearity of system vibration under subharmonic resonance, which does not occur in the case of small damages. At the same time, superharmonic resonance of second order is observed from the instant of crack nucleation, and its intensity at low damping level is sufficiently high for the diagnosis of small cracks.

When 20% measurement error of the predominant-harmonic amplitude in the vibration spectrum is allowed for, the sensitivity of superharmonic resonance of second order to the presence of a damage is by one or two orders of magnitude higher than that of subharmonic resonance of the order $f_c / p = 1/2$ and superharmonic resonance of third order. When the damping level in the system increases by two orders of magnitude, the difference between the sensitivities of the above nonlinear resonances decreases, but the sensitivity of superharmonic resonance of second order remains higher than that of the other nonlinear resonances.

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