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ANALYSIS OF BOUNDARY-VALUE PROBLEMS DESCRIBING THE NON-ISOTHERMAL PROCESSES OF ELASTOPLASTIC DEFORMATION TAKING INTO ACCOUNT THE LOADING HISTORY

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We discuss the theory and approximate methods for solving boundary-value problems of thermoplasticity in a quasi-static formulation when the process of non-isothermal elastoplastic deformation of a body is a sequence of equilibrium states. In this case, the stress-strain state depends on the loading history, and the process of inelastic deformation is to be observed over the whole time interval being studied. The boundary-value problem is stated as a non-linear operator equation in the Hilbertian space. The conditions that provide the existence, uniqueness and continuous dependence of the generalized solution on the applied loads and initial strains are defined. A convergence of the methods of elastic solutions and variable elastic parameters is studied to solve the boundary-value problems describing the non-isothermal processes of active loading taking into account the initial strains dependent on the deformation history and heating.

Keywords: theory of plasticity, stress and strain deviators, non-isothermal processes, elastoplastic deformation, simple loading, thermomechanical surface, boundary-value problem, iterative methods, convergence, accuracy.

Introduction. In the investigation of non-isothermal processes of elastoplastic deformation, the stress, strain and displacement components at each loading stage are determined by solving a set of non-linear equations for increments in the sought-for values during a loading stage. To solve the non-linear boundary-value problem stated in increments, approximate methods are used with the aid of which the problem of thermoplasticity at each stage of loading is reduced to a sequential solution of auxiliary linear problems. In this case, the accuracy satisfying the resolving equations for complete values of stresses, strains and displacements are to be controlled, since the boundary-value problem is solved approximately for the increments in these values, and therefore, in their summation, an error can be accumulated [1]. In addition, the use of the constitutive relationships in increments assumes a higher degree of smoothness of the approximating functions for stress–strain diagrams since, for the computing process to be stable, it is necessary to provide the continuity of tangential moduli. It should also be taken into account that in the statement of the boundary-value problem in increments, the loading stage duration should be sufficiently short. Therefore, the use of numerical methods for solving the problem in a three-dimensional statement can result in unacceptable computation costs.

An alternative approach consists in integrating the equation of state for a loading stage in order to obtain a set of resolving equations for complete stress, strain and displacement components, and not in increments [1, 2]. This makes it possible to avoid difficulties connected with computation of tangential moduli from stress–strain diagrams and accumulation of errors in the numerical solution of the problem in increments, which contributes to the computing process stability [1]. In this case, the duration of the loading stage can be sufficiently long if, within the stage of loading, the deformation of all points of the body occurs along the trajectories close to rectilinear. In the cases where the loading trajectory is a broken line composed of rectilinear segments, the solution of the

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boundary-value problem can be obtained at enlarged time steps, which considerably reduces the computation cost in the numerical modeling.

It is apparent that in devising the efficient approximate methods for solving the problems of thermoelasticity, it is necessary to have clear information on the conditions of existence and properties of accurate solutions for the problem under consideration. It may be considered that the boundary-value problem is stated correctly if the existence and uniqueness of its solution in a certain class of functions are proved, the solution stability with respect to small perturbations of initial data and its continuous dependence on external effects in the process of loading is established.

This paper presents the results of analysis of the boundary-value problem of thermoplasticity that describes the processes of deformation along the trajectories consisting of rectilinear segments of the broken line or close to them. The emphasis is on a generalized statement and the investigation of the convergence of approximate methods for solving the boundary-value problem.

Main Statements of the Phenomenological Model. Let $\sigma(t)=(\sigma_{ij}(t))$ $(1\le i,\ j\le 3)$ be the stress tensor presented in the form of two components: $\sigma(t)=\sigma_S(t)+\sigma_D(t)$, with $\sigma_S(t)$ being the spherical tensor and $\sigma_D(t)$ the stress deviator. By analogy with the stress tensor, the small strain tensor $\varepsilon(t)=(\varepsilon_{ij}(t))$ $(1\le i,\ j\le 3)$ allows the expansion of the form $\varepsilon(t)=\varepsilon_S(t)+\varepsilon_D(t)$, with $\varepsilon_S(t)$ being the spherical tensor and $\varepsilon_D(t)$ the strain deviator. The solution of the non-isothermal elastoplastic problem is based on the following main statements.

It is assumed that the variation in the body volume over the whole range of stresses and strains is of an elastic character, i.e., there exists a linear dependence between $\sigma_S(t)$ and $\varepsilon_S(t)$:

$$\varepsilon_S(t) = \frac{1}{k_0(T(t))}\sigma_S(t) + \varepsilon_S^T(t), \tag{1}$$

where $k_0(T(t))$ is the modulus of the total volumetric expansion dependent on the temperature $T(t)$ and $\varepsilon_S^T(t)$ is the tensor of unconstrained thermal strains.

The total strain deviator $\varepsilon_D(t)$ will be conditionally presented as a sum of elastic $\varepsilon_D^e(t)$ and plastic $\varepsilon_D^p(t)$ components:

$$\varepsilon_D(t) = \varepsilon_D^e(t) + \varepsilon_D^p(t). \tag{2}$$

The elastic component of the strain deviator is defined by the generalized Hook law which can be presented for an isotropic body as

$$\varepsilon_D^e(t) = \frac{1}{2G_0(T(t))}\sigma_D(t), \tag{3}$$

where $G_0(T(t))$ is the initial shear modulus dependent on temperature in the general case.

Using relations (3) we obtain

$$\bar{\sigma}(t) = 3G_0(T(t))\bar{\varepsilon}^e(t), \tag{4}$$

where $\bar{\sigma}(t)$ and $\bar{\varepsilon}^e(t)$ are the intensities of the stress $\sigma_D(t)$ and strain $\varepsilon_D(t)$ deviators defined by the following relationships:

$$\bar{\sigma}(t) = \sqrt{\frac{3}{2}}\,||\sigma_D(t)||, \qquad \bar{\varepsilon}^e(t) = \sqrt{\frac{2}{3}}\,||\varepsilon_D^e(t)||. \tag{5}$$

Here and below, the scalar product $(\cdot,\cdot)$ induced by the convolution of the corresponding tensors and the norm $||\cdot||$ associated with this scalar product are used.

The plastic component of the strain deviator is defined on the basis of the plastic yielding law [3, 4] associated with the von Mises yield surface [3]:

$$d\varepsilon_D^p(t) = \frac{3}{2}\,\frac{\overline{d\varepsilon}^p(t)}{\bar{\sigma}(t)}\,\sigma_D(t), \tag{6}$$

where $\overline{d\varepsilon}^p(t)$ is the intensity of the increments in the plastic strains,

$$\overline{d\varepsilon}^p(t) = \sqrt{\frac{2}{3}}\,||d\varepsilon_D^p(t)||. \tag{7}$$

We note that in consideration of the processes of deformation along the trajectories of small curvature, equation (6) can be derived based on the isotropy postulate and the principle of lag formulated by Il'yushin in [5] and experimentally justified for wide classes of materials at room and elevated temperatures.

Thus, the constitutive equations describing the non-isothermal processes of elastoplastic deformation consist of a condition for an elastic variation of the volume (1) and relationships (2), (3), (6) which are equivalent to the Prandtl–Reiss equations of state [6, 7] and are of the following form:

$$d\varepsilon_D(t) = d\left(\frac{1}{2G_0(T(t))}\,\sigma_D(t)\right) + \frac{3}{2}\,\frac{\overline{d\varepsilon}^p(t)}{\bar{\sigma}(t)}\,\sigma_D(t). \tag{8}$$

In using equations (8), the whole process of loading is divided into the time steps in such a way that the instants of time, which make distinction between the stages of loading and unloading, coincide as closely as possible with the time instants of variation in the deformation process direction from loading to unloading, and vice versa.

Let us integrate expression (8) over a loading stage. As a result, at the end of the mth loading stage we obtain

$$\varepsilon_D(t_m) - \varepsilon_D(t_{m-1}) = \frac{1}{2G_0(T(t_m))}\,\sigma_D(t_m) - \frac{1}{2G_0(T(t_{m-1}))}\,\sigma_D(t_{m-1}) + \frac{3}{2}\int_{t_{m-1}}^{t_m} \sigma_D(t)\,\frac{\overline{d\varepsilon}^p(t)}{\bar{\sigma}(t)}. \tag{9}$$

Using relationships (3) we find

$$\varepsilon_D(t_m) - \varepsilon_D^p(t_{m-1}) = \frac{1}{2G_0(T(t_m))}\,\sigma_D(t_m) + \frac{3}{2}\int_{t_{m-1}}^{t_m} \sigma_D(t)\,\frac{\overline{d\varepsilon}^p(t)}{\bar{\sigma}(t)}. \tag{10}$$

At the end of the mth loading stage, the plastic strain deviator $\varepsilon_D^p(t_m)$ is determined using the formula

$$\varepsilon_D^p(t_m) = \frac{3}{2}\int_{t_{m-1}}^{t_m} \sigma_D(t)\,\frac{\overline{d\varepsilon}^p(t)}{\bar{\sigma}(t)} + \varepsilon_D^p(t_{m-1}). \tag{11}$$

Denote the increment in the plastic strains at the end of the mth loading stage by $\Delta_m\varepsilon_D^p$:

$$\Delta_m\varepsilon_D^p = \varepsilon_D^p(t_m) - \varepsilon_D^p(t_{m-1}). \tag{12}$$

Then, according to formulas (11) and (12) we have

$$\Delta_m\varepsilon_D^p = \frac{3}{2}\int_{t_{m-1}}^{t_m} \sigma_D(t)\,\frac{\overline{d\varepsilon}^p(t)}{\bar{\sigma}(t)}. \tag{13}$$

Let $s = s(t)$ be the arc length of the plastic strain path determined by the expression given below:

$$s(t) = \int_0^t ||d\varepsilon_D^p(t')|| = \int_0^t || \frac{d\varepsilon_D^p(t')}{dt'} || dt'. \tag{14}$$

Suppose that at the loading stage, the arc length of the plastic strain trajectory increases monotonically during the deformation process. Since during the time dt, the arc length $s = s(t)$ gets the increment

$$ds(t) = \frac{ds(t)}{dt} dt = ||d\varepsilon_D^p(t)|| > 0, \tag{15}$$

then, taking into account the designations $s_{m-1} = s(t_{m-1})$ and $s_m = s(t_m)$, expression (13) can be presented in the following form:

$$\Delta_m \varepsilon_D^p = \int_{s_{m-1}}^{s_m} \frac{\sigma_D(s)}{||\sigma_D(s)||} ds. \tag{16}$$

Let us integrate (16) using the formula of rectangles. As a result, we get

$$\Delta_m \varepsilon_D^p \approx \frac{\sigma_D(s_m)}{||\sigma_D(s_m)||} \int_{s_{m-1}}^{s_m} ds. \tag{17}$$

The error of this formula satisfies the estimate

$$\left| \int_{s_{m-1}}^{s_m} \frac{\sigma_D(s)}{||\sigma_D(s)||} ds - \frac{\sigma_D(s_m)}{||\sigma_D(s_m)||} \int_{s_{m-1}}^{s_m} ds \right| \le \frac{1}{2} |s_m - s_{m-1}|^2 \max_{s_{m-1} \le s \le s_m} \left| \frac{d}{ds} \frac{\sigma_D(s)}{||\sigma_D(s)||} \right|. \tag{18}$$

Since the small strains are considered, the estimate $|s_m - s_{m-1}|^2 << 1$ is valid. Moreover, if within the loading stage, the directing stress deviator $\sigma_D(s)/||\sigma_D(s)||$ varies sufficiently smoothly with reference to the argument s, it seems likely that the use of formula (17) does not introduce a great error.

Suppose that a simple loading is realized at each stage [5]. Considering that under simple loading $\sigma_D(s)/||\sigma_D(s)|| = \text{const}$, expression (13) takes the following form:

$$\Delta_m \varepsilon_D^p = \frac{3}{2} \frac{\sigma_D(t_m)}{\bar{\sigma}(t_m)} \int_{t_{m-1}}^{t_m} d\bar{\varepsilon}^p(t). \tag{19}$$

In accordance with (10), (19) we obtain

$$\varepsilon_D(t_m) - \varepsilon_D^p(t_{m-1}) = \frac{1}{2G(t_m)} \sigma_D(t_m), \tag{20}$$

$$\Delta_m \varepsilon_D^p = \frac{1}{2} \left(\frac{1}{G(t_m)} - \frac{1}{G_0(T(t_m))} \right) \sigma_D(t_m), \tag{21}$$

where $G(t_m)$ is the scalar function defined by the following expression:

$$\frac{1}{G(t_m)} = \frac{1}{G_0(T(t_m))} + \frac{3}{\overline{\sigma}(t_m)} \int_{t_{m-1}}^{t_m} \overline{d\varepsilon}^p(t). \tag{22}$$

Based on (12), (20), and (21), we write

$$\sigma_D(t_m) = 2G(t_m)(\varepsilon_D(t_m) - \varepsilon_D^p(t_{m-1})), \tag{23}$$

$$\varepsilon_D^p(t_m) = \varepsilon_D(t_m) - \frac{1}{2G_0(T(t_m))}\sigma_D(t_m). \tag{24}$$

Moreover, using relationships (21) and (22) we find

$$\int_{t_{m-1}}^{t_m} \overline{d\varepsilon}^p(t) = \overline{\Delta_m \varepsilon}^p = \left(\frac{1}{G(t_m)} - \frac{1}{G_0(T(t_m))}\right)\frac{\overline{\sigma}(t_m)}{3}, \tag{25}$$

where $\overline{\Delta_m \varepsilon}^p$ is the intensity of the deviator of the increments in the plastic strains at the end of the mth loading stage

$$\overline{\Delta_m \varepsilon}^p = \sqrt{\frac{2}{3}}\, ||\Delta_m \varepsilon_D^p||. \tag{26}$$

Thus, the constitutive relationships describing the non-isothermal simple processes of elastoplastic deformation can be presented in the following way:

$$\sigma(t_m) = k_0(T(t_m))(\varepsilon_S(t_m) - \xi_S(t_m)) + 2G(t_m)(\varepsilon_D(t_m) - \xi_D(t_m)), \tag{27}$$

where $\xi(t_m) = (\xi_{ij}(t_m))$, $1 \le i, j \le 3$ is the initial strain tensor that corresponds to the spherical unconstrained thermal strain tensor $\varepsilon_S^T(t_m)$ and the plastic strain deviator $\varepsilon_D^p(t_{m-1})$

$$\xi(t_m) = \varepsilon_S^T(t_m) + \varepsilon_D^p(t_{m-1}). \tag{28}$$

The plastic strains at the end of the mth loading stage are determined from relationships (24) which, in view of equations (23), can be written in the following way:

$$\varepsilon_D^p(t_m) = \left(1 - \frac{G(t_m)}{G_0(T(t_m))}\right)(\varepsilon_D(t_m) - \varepsilon_D^p(t_{m-1})) + \varepsilon_D^p(t_{m-1}). \tag{29}$$

Denote the deviator of the active strains $\varepsilon_D^a(t_m)$ that occur in the body element in addition to the plastic strains by $\varepsilon_D^p(t_{m-1})$:

$$\varepsilon_D^a(t_m) = \varepsilon_D(t_m) - \varepsilon_D^p(t_{m-1}). \tag{30}$$

Then, relationships (23) and (29) can be presented as

$$\sigma_D(t_m) = 2G(t_m)\varepsilon_D^a(t_m), \tag{31}$$

$$\Delta_m \varepsilon_D^p = \left(1 - \frac{G(t_m)}{G_0(T(t_m))}\right) \varepsilon_D^a(t_m). \tag{32}$$

By substituting the components of the stress deviator (31) into the expression of the stress intensity (5) we obtain

$$G(t_m) = \frac{\bar{\sigma}(t_m)}{3\bar{\varepsilon}^a(t_m)}, \tag{33}$$

where $\bar{\varepsilon}^a(t_m)$ is the intensity of the active strain deviator $\varepsilon_D^a(t_m)$ which is equal to

$$\bar{\varepsilon}^a(t_m) = \sqrt{\frac{2}{3}}\,||\varepsilon_D^a(t_m)||. \tag{34}$$

Using relationships (4), (25), (33) we find

$$\bar{\varepsilon}^e(t_m) = \frac{G(t_m)}{G_0(T(t_m))}\bar{\varepsilon}^a(t_m), \quad \overline{\Delta_m \varepsilon^p} = \left(1 - \frac{G(t_m)}{G_0(T(t_m))}\right)\bar{\varepsilon}^\alpha(t_m), \tag{35}$$

whence it follows

$$\bar{\varepsilon}^a(t_m) = \bar{\varepsilon}^e(t_m) + \overline{\Delta_m \varepsilon^p}. \tag{36}$$

The Odquist parameter $q(t_m)$ that characterizes the accumulated plastic strain at the end of the *m*th loading stage is calculated from the following relationships:

$$q(t_m) = \int_0^{t_m} \overline{d\varepsilon^p} = \int_0^{t_{m-1}} \overline{d\varepsilon^p} + \int_{t_{m-1}}^{t_m} \overline{d\varepsilon^p} = q(t_{m-1}) + \overline{\Delta_m \varepsilon^p} = \sum_{k=1}^{m} \overline{\Delta_k \varepsilon^p}. \tag{37}$$

Equations (31), (32) are characterized by the functional dependence

$$\bar{\sigma} = \Psi(\bar{\varepsilon}^a, T), \tag{38}$$

which is specified on the basis of the following equation of the instantaneous thermomechanical surface:

$$\bar{\sigma} = f(\bar{\varepsilon}, T), \tag{39}$$

where by the strain $\bar{\varepsilon}$ is meant the pure force component, i.e., the total strain minus pure thermal one. It is assumed that functional dependence (39) is independent of the hydrostatic pressure, stress deviator type and is found from the simple tension test data for cylindrical specimens.

Note that functional dependence (39) describes the material elastoplastic deformation taking into account the hardening by the time of the beginning of the loading stage. The argument in this equation is in effect the active strain, namely, the total strain minus the initial plastic strain. Therefore, the value of the accumulated plastic strain by the beginning of the loading stage is taken as a measure of the hardening. In other words, under re-loading and subsequent loadings, equation (38) takes into account the dependence of the generalized stress–strain curves on the value of the accumulated plastic strain. On this basis, we present functional dependence (38) in a more general form:

$$\bar{\sigma} = \Psi(\bar{\varepsilon}^a, q, T), \tag{40}$$

where the parameter that characterizes the material hardening by the beginning of the current loading stage is taken as an additional argument q. Here, the dependence of the hardening parameter q on the deformation process takes into account the loading history. The simplest assumption about the hardening character is that the value of the accumulated plastic strain, i.e., the Odqvist parameter is taken as a measure of hardening. For isothermal processes, equation (40) can be interpreted as a surface of deformation with initial hardening. For non-isothermal processes, it describes many thermomechanical processes as a function of the hardening value. For fixed values of the hardening parameter q, Eq. (40) can be interpreted as an instantaneous thermomechanical surface with initial hardening.

To specify functional dependence (40), we use equation of a thermomechanical surface (39). In uniaxial tension of a specimen, the total strain $\bar{\varepsilon}$ is related to the active strain $\bar{\varepsilon}^a$ by the relationship: $\bar{\varepsilon} = \bar{\varepsilon}^a + q$, where q is the initial plastic strain. Then, taking into account the linear dependence at the elastic portion of deformation we obtain

$$\Psi(\bar{\varepsilon}^a, q, T) = \begin{cases} 3G_0(T)\bar{\varepsilon}^a, & \bar{\varepsilon}^a \le \bar{\varepsilon}_p(q, T), \\ f(\bar{\varepsilon}^a + q, T), & \bar{\varepsilon}^a > \bar{\varepsilon}_p(q, T), \end{cases} \tag{41}$$

where $\bar{\varepsilon}_p(q, T)$ is the strain corresponding to the instant limit of proportionality $\bar{\sigma}_p(q, T)$ dependent on the accumulated plastic strain q and temperature T.

Since the dependence between $\bar{\sigma}_p(q, T)$ and $\bar{\varepsilon}_p(q, T)$ is taken to be linear, we obtain the following equation for determining $\bar{\varepsilon}_p(q, T)$:

$$f(\bar{\varepsilon}_p(q, T) + q, T) = 3G_0(T)\bar{\varepsilon}_p(q, T). \tag{42}$$

In the active process of loading starting from the natural non-deformed state, it should be assumed that $\bar{\varepsilon}^a = \bar{\varepsilon}$ and $q = 0$. In addition, $\bar{\sigma}_p(0, T)$ and $\bar{\varepsilon}_p(0, T)$ are the limits of proportionality determined from equation of a thermomechanical surface (39). Then, using relationships (33) and (41) we have

$$G(t_1) = \begin{cases} G_0(T(t_1)), & \bar{\varepsilon}(t_1) \le \bar{\varepsilon}_p(0, T(t_1)), \\ \dfrac{f(\bar{\varepsilon}(t_1), T(t_1))}{3\bar{\varepsilon}(t_1)}, & \bar{\varepsilon}(t_1) > \bar{\varepsilon}_p(0, T(t_1)), \end{cases} \tag{43}$$

with the value of $\bar{\varepsilon}_p(0, T(t_1))$ being determined by the following expression:

$$\bar{\varepsilon}_p(0, T(t_1)) = \frac{\bar{\sigma}_p(0, T(t_1))}{3G_0(T(t_1))}. \tag{44}$$

Under unloading and re-loading, we have

$$G(t_m) = \begin{cases} G_0(T(t_m)), & \bar{\varepsilon}^a(t_m) \le \bar{\varepsilon}_p(t_m), \\ \dfrac{f(\bar{\varepsilon}^a(t_m) + q(t_{m-1}), T(t_m))}{3\bar{\varepsilon}^a(t_m)}, & \bar{\varepsilon}^a(t_m) > \bar{\varepsilon}_p(t_m), \end{cases} \tag{45}$$

where the quantity $\bar{\varepsilon}_p(t_m)$ is the root of the equation

$$f(\bar{\varepsilon}_p(t_m) + q(t_{m-1}), T(t_m)) = 3G_0(T(t_m))\bar{\varepsilon}_p(t_m). \tag{46}$$

Let us assume that for the fixed temperature $T(t_m)$, the piece-wise linear approximation $f(\bar{\varepsilon}(t_m), T(t_m))$ is used as a function of strains $\bar{\varepsilon}(t_m)$. For this purpose, the whole interval of variation in $\bar{\varepsilon}(t_m)$ is divided into segments $[\bar{\varepsilon}_{n-1}(t_m), \bar{\varepsilon}_n(t_m)]$ and within each of them the following linear interpolation is given:

$$f(\bar{\varepsilon}(t_m),\, T(t_m)) = f(\bar{\varepsilon}_{n-1}(t_m),\, T(t_m)) + 3g_n(T(t_m))(\bar{\varepsilon}(t_m) - \bar{\varepsilon}_{n-1}(t_m)), \tag{47}$$

where $g_n(T(t_m))$ is the linear hardening module on the segment $[\bar{\varepsilon}_{n-1}(t_m),\, \bar{\varepsilon}_n(t_m)]$,

$$g_n(T(t_m)) = \frac{f(\bar{\varepsilon}_n(t_m),\, T(t_m)) - f(\bar{\varepsilon}_{n-1}(t_m),\, T(t_m))}{3(\bar{\varepsilon}_n(t_m) - \bar{\varepsilon}_{n-1}(t_m))}. \tag{48}$$

Based on formulas (46)–(48), we obtain the relationships by using which the value of $\bar{\varepsilon}_p(t_m)$ is determined:

$$\bar{\varepsilon}_p(t_m) = \bar{\varepsilon}_*(t_m) - q(t_{m-1}), \qquad \bar{\varepsilon}_*(t_m) = \frac{\bar{\sigma}_*(t_m)}{3G_*(t_m)}, \qquad \bar{\varepsilon}_{n-1}(t_m) \le \bar{\varepsilon}_*(t_m) \le \bar{\varepsilon}_{n-1}(t_m), \tag{49}$$

with

$$\begin{aligned} &\bar{\sigma}_*(t_m) = f(\bar{\varepsilon}_{n-1}(t_m),\, T(t_m)) + 3[G_0(T(t_m))q(t_{m-1}) - g_n(T(t_m))\bar{\varepsilon}_{n-1}(t_m)], \\ &G_*(t_m) = G_0(T(t_m)) - g_n(T(t_m)). \end{aligned} \tag{50}$$

For the steady-state creep, functional dependence (40) can be approximately determined with the help of the creep diagrams obtained for fixed values of stresses and temperature by way of plotting isochronic curves of creep [1, 8]:

$$\bar{\sigma} = f(\bar{\varepsilon},\, T(t),\, t). \tag{51}$$

Using the isochronic curves of creep (51), we present functional dependence (40) in the following form:

$$\bar{\sigma} = \Psi(\bar{\varepsilon}^a,\, q,\, T(t),\, t), \tag{52}$$

where by the plastic (inelastic) strain is meant the irreversible strain including both creep strain and instantaneous plastic strain. In this case, the value of the accumulated irreversible strain before the beginning of the loading stage is taken as the hardening parameter q. Thus, the solution of the viscoplastic problem is reduced to that of the elastoplastic one wherein the generalized stress–strain curves of the material are dependent on the value of the accumulated irreversible strain, time and temperature [1, 8].

Generalized Statement of the Boundary-Value Problem. Let the body under consideration occupy the region $\Omega \subset R^3$ and have a regular boundary. We will consider the vector functions that describe the displacement of the body points $u(t)$ as elements of the functional set U. Denote by X the set of admissible tensor functions for the stresses $\sigma(t)$, total $\varepsilon(t)$ and initial $\xi(t)$ strains. We assume that U and X are the Hilbertian spaces with the scalar products $(\cdot,\cdot)_U$ and $(\cdot,\cdot)_X$, respectively. Denote by U^* the space conjugate to U and determine $\langle \rho(t),\, v\rangle$ as a value of the continuous linear functional $\rho(t) \in U^*$ at the element $v \in U$. Then, in the investigation of non-isothermal processes of elastoplastic deformation in the quasi-static statement, the generalized boundary-value problem can be presented by the following set of equations:

$$\begin{aligned} &(\varepsilon(t),\, \eta)_X = (Bu(t),\, \eta)_X, \qquad \forall \eta \in X, \\ &(\sigma(t),\, \chi)_X = (D(\varepsilon(t),\, \xi(t),\, t)(\varepsilon(t) - \xi(t)),\, \chi)_X, \qquad \forall \chi \in X, \\ &(\sigma(t),\, Bv)_X = \langle \rho(t),\, v \rangle, \qquad \forall v \in U, \end{aligned} \tag{53}$$

where B is the continuous linear differential operator acting from the space U into the space X, i.e., the operator for calculating small strains from the specified displacements, D is the nonlinear operator that maps X into itself and establishes the interrelation between stresses and strains, and $\rho(t) \in U^*$ is the linear functional associated with the work of loads applied to the body at possible displacements $v \in U$.

The operator $D: X \to X$ is determined with the help of mapping

$$\eta(t),\ \zeta(t),\ \mu(t) \in X \to D((\eta(t),\ \zeta(t)),\ t)(\mu(t) - \zeta(t))$$

$$= k_0(T(t))(\mu_S(t) - \zeta_S(t)) + 2G(\bar{\varepsilon}^a(\eta(t),\ \zeta(t)),\ T(t),\ t)(\mu_D(t) - \zeta_D(t)), \tag{54}$$

where $G(\bar{\varepsilon}^a,\ T(t),\ t) = \Psi(\bar{\varepsilon}^a,\ T(t),\ t) / 3\bar{\varepsilon}^a$ is the secant shear modulus and $\bar{\varepsilon}^a$ is the intensity of the active strain deviator,

$$\eta(t),\ \zeta(t) \to \bar{\varepsilon}^a(\eta(t),\ \zeta(t)) = \sqrt{\frac{2}{3}}\,||\eta_D(t) - \zeta_D(t)||. \tag{55}$$

To study the conditions of existence and uniqueness of solution of the boundary-value problem, we present the set of equations (53) in the form of one nonlinear operator equation for displacements:

$$A(u(t),\ \xi(t),\ t) = \rho(t) \quad \text{in} \quad U^*, \qquad u(t) \in U, \tag{56}$$

where $A: U \to U^*$ is the nonlinear operator of plasticity theory determined with the help mapping:

$$A(u(t),\ \xi(t),\ t) \colon v \in U \to (\sigma(u(t),\ \xi(t),\ t),\ \varepsilon(v))_X$$

$$= (D(Bu(t),\ \xi(t),\ t)(Bu(t) - \xi(t)),\ Bv)_X = \langle A(u(t),\ \xi(t),\ t),\ v \rangle. \tag{57}$$

If the operator $A: U \to U^*$ has the properties of strong monotonicity and the Lipschitz continuity, i.e., there exist real numbers m, M, and M_1 such that

$$\begin{cases} \langle A(v,\zeta) - A(w,\zeta),\ v - w \rangle \geq m\,||v - w||_U^2, & \forall v, w \in U, \\ ||A(v,\zeta) - A(w,\zeta)||_{U^*} \leq M\,||v - w||_U, & \forall v, w \in U, \\ ||A(v,\zeta) - A(v,\chi)||_{U^*} \leq M_1\,||\zeta - \chi||_X, & \forall \zeta, \chi \in X, \end{cases} \tag{58}$$

then the solution of operator equation (56) exists and is unique, and also is continuously dependent on the applied loads $\rho(t) \in U^*$ and initial strains $\xi(t) \in X$ [9].

Let us determine the non-linear operator $\Phi: X \to X$ with the help of mapping:

$$\Phi\colon \eta,\ \zeta \in X \to \Phi(\eta,\ \zeta) = D(\eta,\ \zeta)(\eta - \zeta), \tag{59}$$

which associates the result of the action of $D(\eta, \zeta)$ on $(\eta - \zeta)$, i.e., the element $D(\eta, \zeta)(\eta - \zeta) \in X$, with the arbitrary elements $\eta, \zeta \in X$ and operator $\eta, \zeta \to D(\eta, \zeta)$.

Let the mapping $\Phi: X \to X$ be Frechet differentiable at each point (η, ζ), i.e., there exist the linear operators Φ'_ε and Φ'_ξ such that

$$\lim_{\mu \to 0} \frac{||\Phi(\eta + \mu,\ \zeta) - \Phi(\eta,\ \zeta) - \Phi'_\varepsilon(\eta,\ \zeta)\mu||_X}{||\mu||_X} = 0,$$
$$\lim_{\chi \to 0} \frac{||\Phi(\eta,\ \zeta + \chi) - \Phi(\eta,\ \zeta) - \Phi'_\xi(\eta,\ \zeta)\chi||_X}{||\chi||_X} = 0, \tag{60}$$

where $\Phi'_\varepsilon(\eta, \zeta)\mu = d\Phi((\eta, \zeta), (\mu, 0))$ is the Frechet differential of mapping $\eta \to \Phi(\eta, \zeta)$ onto the increment $(\mu, 0)$, $\Phi'_\varepsilon(\eta, \zeta)$ is the derivative of the Frechet operator Φ at the point (η, ζ), $\Phi'_\xi(\eta, \zeta)\chi = d\Phi((\eta, \zeta); (0, \chi))$ is the Frechet differential of mapping $\zeta \to \Phi(\eta, \zeta)$ onto the increment $(0, \chi)$, and $\Phi'_\xi(\eta, \zeta)$ is the derivative of the Frechet operator Φ at the point (η, ζ).

Lemma. If $\Phi : X \to X$ is the continuously differentiable mapping and the operators $\Phi'_\varepsilon(\eta, \zeta)$ and $\Phi'_\xi(\eta, \zeta)$ satisfy the following conditions:

$$\exists\, m > 0:\ (\Phi'_\varepsilon(\eta, \zeta)\mu, \mu)_X \geq m\,||\mu||_X^2\,, \qquad \forall\, \eta, \zeta, \mu \in X, \tag{61}$$

$$\exists\, M > 0:\ ||\Phi'_\varepsilon(\eta, \zeta)\mu||_X \leq M||\mu||_X\,, \qquad \forall\, \eta, \zeta, \mu \in X, \tag{62}$$

$$\exists\, M_1 > 0:\ ||\Phi'_\xi(\eta, \zeta)\chi||_X \leq M_1||\chi||_X\,, \qquad \forall\, \eta, \zeta, \chi \in X, \tag{63}$$

then the operator $A : U \to U^*$ determined from relationship (57) is strictly monotone and Lipschitz continuous.

◀ Let $\eta = Bv$ and $\mu = Bw$ for any $v, w \in U$. Then, according to (57) and (59) for arbitrary $v, w \in U$, we have

$$\langle A(v, \zeta) - A(w, \zeta),\ v - w \rangle = (\Phi(\eta, \zeta) - \Phi(\mu, \zeta),\ \eta - \mu)_X\,, \tag{64}$$

from which, using the formula of finite increments [10] and inequality (61), we get

$$\langle A(v, \zeta) - A(w, \zeta),\ v - w \rangle$$

$$= \int_0^1 (\Phi'_\varepsilon(p\eta + (1-p)\mu, \zeta)(\eta - \mu), \eta - \mu)_X\, dp \geq m\,||\eta - \mu||_X^2 = m\,||v - w||_U^2\,, \qquad \forall\, v, w \in U, \qquad \forall \zeta \in X. \tag{65}$$

Moreover, by definition of the norm in the space U^* we have

$$||A(v, \zeta) - A(w, \zeta)||_{U^*} = \sup_{h \in U} \frac{|\langle A(v, \zeta) - A(w, \zeta), h \rangle|}{||h||_U} = \sup_{h \in U} \frac{|(\Phi(\eta, \zeta) - \Phi(\mu, \zeta), Bh)_X|}{||Bh||_X}, \qquad \forall\, v, w \in U, \tag{66}$$

and, consequently, on the basis of the Cauchy–Bunyakovskii–Schwarz inequality, the finite increment formula and inequality (63) we find

$$||A(v, \zeta) - A(w, \zeta)||_{U^*} \leq ||\Phi(\eta, \zeta) - \Phi(\mu, \zeta)||_X$$

$$= \int_0^1 ||\Phi'_\varepsilon(p\eta + (1-p)\mu, \zeta)(\eta - \mu)||_X\, dp \leq M\,||\eta - \mu||_X = M\,||v - w||_U\,, \qquad \forall\, v, w \in U, \qquad \forall \zeta \in X. \tag{67}$$

Using the finite increment formula and inequality (63) we obtain

$$||A(v, \zeta) - A(v, \chi)||_{U^*} \leq ||\Phi(\eta, \zeta) - \Phi(\eta, \chi)||_X$$

$$= \int_0^1 ||\Phi'_\xi(\eta,\ p\zeta + (1-p)\chi)(\zeta - \chi)||_X\, dp \leq M_1\,||\zeta - \chi||_X\,, \qquad \forall\, v \in U, \qquad \forall \zeta, \chi \in X. \tag{68}$$

Based on inequalities (65), (67), (68) we conclude that $A : U \to U^*$ is the strictly monotone and Lipschitz continuous operator possessing the strictly monotone and Lipschitz continuous inverse operator $A^{-1} : U^* \to U$. ▶

In order to prove the existence and uniqueness of solution to the boundary-value problem stated in the form of nonlinear operator equation (56), we make some assumptions about the functional dependence $\bar{\sigma} = f(\bar{\varepsilon}, T)$ that describes the instantaneous thermomechanical surface.

We assume that for all $\bar{\varepsilon}$, perhaps, except for a finite number of the isolated points, the function $\bar{\sigma} = \bar{\sigma}(\bar{\varepsilon})$ describing the material stress–strain curve satisfies the following conditions:

$$0 < \bar{g}_1 \le \bar{g}(\bar{\varepsilon}) \le G(\bar{\varepsilon}) \le G_0 < \infty. \tag{69}$$

Inequalities (69) are written for isothermal conditions and assume a simple geometrical interpretation. For all values of $\bar{\varepsilon}$, the tangential modulus

$$\bar{g}(\bar{\varepsilon}) = \frac{1}{3}\frac{d\bar{\sigma}(\bar{\varepsilon})}{d\bar{\varepsilon}} \tag{70}$$

is strictly positive and does not exceed the secant modulus $G(\bar{\varepsilon}) = \bar{\sigma}(\bar{\varepsilon})/3\bar{\varepsilon}$ which, in its turn, does not exceed the initial shear modulus G_0.

If we introduce the temperature T into the functional dependence $\bar{\sigma} = \bar{\sigma}(\bar{\varepsilon})$ as a second argument, we obtain the equation of instantaneous thermomechanical surface $\bar{\sigma} = f(\bar{\varepsilon}, T)$. For nonisothermal processes, inequalities (69) can be presented in a more general form:

$$0 < \min_T \bar{g}_1(T) \le \bar{g}(\bar{\varepsilon}, T) \le G(\bar{\varepsilon}, T) \le \max_T G_0(T) < \infty. \tag{71}$$

Furthermore, the equation $\Psi = \Psi(\bar{\varepsilon}^a, q, T)$ describing the instantaneous thermomechanical surface with initial hardening q, is defined by relationships (41) and thus, on the basis of inequalities (71), we obtain

$$0 < \min_T \bar{g}_1(T) \le \bar{g}(\bar{\varepsilon}^a, q, T) \le G(\bar{\varepsilon}^a, q, T) \le \max_T G_0(T) < \infty. \tag{72}$$

The body under consideration can be nonuniform and its elastic and plastic properties can depend on the coordinates $x \in \Omega$. We assume that the functions $x \to \Psi(x, \bar{\varepsilon}^a, q, T)$, $q \to \Psi(x, \bar{\varepsilon}^a, q, T)$, and $T \to \Psi(x, \bar{\varepsilon}^a, q, T)$ are measurable and $x \to G(x, \bar{\varepsilon}^a, q, T)$ is measurable and bounded on Ω for all $\bar{\varepsilon}^a$, q, and T. At all points of the region Ω, perhaps, except for the set of measure zero, the function $\bar{\varepsilon}^a \to \Psi(x, \bar{\varepsilon}^a, q, T)$ is continuous and has a bounded partial derivative $\partial\Psi(x, \bar{\varepsilon}^a, q, T)/\partial\bar{\varepsilon}^a$ that satisfies the conditions of (72).

Theorem. If the equation $\Psi = \Psi(\bar{\varepsilon}^a, q, T)$ describing the instantaneous thermomechanical surface with initial hardening q satisfies the conditions of (74), the operator $\Phi: X \to X$ defined by relationship (61) satisfies the conditions of the lemma, i.e., there exist such real numbers as m, M, and M_1 for which inequalities (61)–(63) are fulfilled, and in this case

$$m = 2\,\text{vrai}\min_{x\in\Omega}\min_T \bar{g}_1(x, T), \qquad M_1 = M = \text{vrai}\max_{x\in\Omega}\max_T k_0(x, T). \tag{73}$$

◀ The above assumptions about the properties of the function $\Psi = \Psi(\bar{\varepsilon}^a, q, T)$ provide the Frechet differentiability of the operator $D(\eta, \zeta)$. According to (59) and the rules of differentiation of complex mappings, we have

$$\begin{aligned} &d\Phi((\eta, \zeta); (\mu, 0)) = \Phi'_\varepsilon(\eta, \zeta)\mu = dD((\eta, \zeta); (\mu, 0))(\eta - \zeta) + D(\eta, \zeta)\mu, \quad \forall \eta, \zeta, \mu \in X, \\ &d\Phi((\eta, \zeta); (0, \chi)) = \Phi'_\xi(\eta, \zeta)\chi = dD((\eta, \zeta); (0, \chi))(\eta - \zeta) - D(\eta, \zeta)\chi, \quad \forall \eta, \zeta, \chi \in X, \end{aligned} \tag{74}$$

where $dD((\eta, \zeta); (\mu, 0))$ is the Frechet differential of mapping $\eta \to D(\eta, \zeta)$ at the point (η, ζ) on the increment $(\mu, 0)$ and $dD((\eta, \zeta); (0, \chi))$ is the Frechet differential of mapping $\zeta \to D(\eta, \zeta)$ on the increment $(0, \chi)$.

Based on (54) for arbitrary $\eta, \zeta, \mu, \chi, \lambda \in X$, we have

$$dD((\eta, \zeta); (\mu, 0))\lambda = 2\frac{dG(\bar{\varepsilon}^a)}{d\bar{\varepsilon}^a} d\bar{\varepsilon}^a((\eta, \zeta); (\mu, 0))\lambda_D,$$
$$dD((\eta, \zeta); (0, \chi))\lambda = 2\frac{dG(\bar{\varepsilon}^a)}{d\bar{\varepsilon}^a} d\bar{\varepsilon}^a((\eta, \zeta); (0, \chi))\lambda_D, \quad (75)$$

where $d\bar{\varepsilon}^a((\eta, \zeta); (\mu, 0))$ is the differential of mapping $\eta \to \bar{\varepsilon}^a(\eta, \zeta)$ on the increment $(\mu, 0)$ and $d\bar{\varepsilon}^a((\eta, \zeta); (0, \chi))$ is the Frechet differential of mapping $\zeta \to \bar{\varepsilon}^a(\eta, \zeta)$ on the increment $(0, \chi)$.

Using relationship (55) we find

$$d\bar{\varepsilon}^a((\eta, \zeta); (\mu, 0)) = \sqrt{\frac{2}{3}}\frac{(\eta_D - \zeta_D, \mu_D)}{||\eta_D - \zeta_D||},$$
$$d\bar{\varepsilon}^a((\eta, \zeta); (0, \chi)) = \sqrt{\frac{2}{3}}\frac{(\zeta_D - \eta_D, \chi_D)}{||\eta_D - \zeta_D||}, \quad (76)$$

and therefore expressions (75) can be presented in the form

$$dD((\eta, \zeta); (\mu, 0))\lambda = 2\sqrt{\frac{2}{3}}\frac{dG(\bar{\varepsilon}^a)}{d\bar{\varepsilon}^a}\frac{(\eta_D - \zeta_D, \mu_D)}{||\eta_D - \zeta_D||}\lambda_D,$$
$$dD((\eta, \zeta); (0, \chi))\lambda = 2\sqrt{\frac{2}{3}}\frac{dG(\bar{\varepsilon}^a)}{d\bar{\varepsilon}^a}\frac{(\zeta_D - \eta_D, \chi_D)}{||\eta_D - \zeta_D||}\lambda_D. \quad (77)$$

Then, in view of equalities

$$\frac{dG(\bar{\varepsilon}^a)}{d\bar{\varepsilon}^a} = \frac{1}{\bar{\varepsilon}^a}\left(\frac{d\Psi(\bar{\varepsilon}^a)}{d\bar{\varepsilon}^a} - \frac{\Psi(\bar{\varepsilon}^a)}{\bar{\varepsilon}^a}\right) = \frac{\bar{g}(\bar{\varepsilon}^a) - G(\bar{\varepsilon}^a)}{\bar{\varepsilon}^a} \quad (78)$$

based on formulas (55), (74), (77), and (78) we obtain

$$\Phi'_\varepsilon(\eta, \zeta)\mu = 2(\bar{g} - G)\frac{(\eta_D - \zeta_D, \mu_D)}{||\eta_D - \zeta_D||^2}(\eta_D - \zeta_D) + D(\eta, \zeta)\mu,$$
$$\Phi'_\xi(\eta, \zeta)\chi = 2(G - \bar{g})\frac{(\eta_D - \zeta_D, \chi_D)}{||\eta_D - \zeta_D||^2}(\eta_D - \zeta_D) - D(\eta, \zeta)\chi. \quad (79)$$

Therefore, for arbitrary $\eta, \zeta, \mu, \chi, \lambda \in X$, we have

$$(\Phi'_\varepsilon(\eta, \zeta)\mu, \lambda) = 2(\bar{g} - G)\frac{(\eta_D - \zeta_D, \mu_D)(\eta_D - \zeta_D, \lambda_D)}{||\eta_D - \zeta_D||^2} + k_0(\mu_S, \lambda_S) + 2G(\mu_D, \lambda_D), \quad (80)$$

$$(\Phi'_\xi(\eta, \zeta)\chi, \lambda) = 2(G - \bar{g})\frac{(\eta_D - \zeta_D, \chi_D)(\eta_D - \zeta_D, \lambda_D)}{||\eta_D - \zeta_D||^2} - k_0(\chi_S, \lambda_S) - 2G(\chi_D, \lambda_D). \quad (81)$$

Using relationships (80) and (81) we get

$$
\begin{aligned}
&(\Phi'_\varepsilon(\eta,\zeta)\mu,\lambda)=(\mu,\Phi'_\varepsilon(\eta,\zeta)\lambda), \quad \forall\,\eta,\zeta,\mu,\lambda\in X,\\
&(\Phi'_\xi(\eta,\zeta)\chi,\lambda)=(\chi,\Phi'_\xi(\eta,\zeta)\lambda), \quad \forall\,\eta,\zeta,\chi,\lambda\in X,
\end{aligned}
\tag{82}
$$

from which it follows that $\Phi'_\varepsilon(\eta,\zeta)$ and $\Phi'_\xi(\eta,\zeta)$ are the self-adjoint operators for all $\eta,\ \zeta\in X$.

On the basis of equation (82) for arbitrary $\eta,\ \zeta,\ \mu\in X$, we find

$$
(\Phi'_\varepsilon(\eta,\zeta)\mu,\mu)=2(\bar g-G)\frac{(\eta_D-\zeta_D,\mu_D)^2}{\|\eta_D-\zeta_D\|^2}+k_0\|\mu_S\|^2+2G\|\mu_D\|^2 . \tag{83}
$$

According to the Cauchy–Bunyakovskii–Schwarz inequality, we have

$$
|(\eta_D-\zeta_D,\mu_D)|\le\|\eta_D-\zeta_D\|\,\|\mu_D\|. \tag{84}
$$

In addition, in accordance with the conditions of (72), the inequality $\bar g-G\le 0$ is fulfilled, and thus, in view of (84), from equation (83) it follows

$$
(\Phi'_\varepsilon(\eta,\zeta)\mu,\mu)\ge k_0\|\mu_S\|^2+2\bar g\|\mu_D\|^2\ge 2\bar g\|\mu\|^2 , \tag{85}
$$

which leads to inequality (61) with the constant $m=2\,\underset{x\in\Omega}{\operatorname{vrai\,min}}\,\min_T \bar g_1(x,T)$.

To prove inequality (62), we note that the operator $\Phi'_\varepsilon(\eta,\zeta)$ is self-adjoint and positive for all $\eta,\ \zeta\in X$ and therefore its norm is defined by expression:

$$
\|\Phi'_\varepsilon(\eta,\zeta)\|_X=\sup_{\mu\in X}\frac{(\Phi'_\varepsilon(\eta,\zeta)\mu,\mu)_X}{\|\mu\|_X^2}. \tag{86}
$$

Using equality (83) for arbitrary $\eta,\ \zeta,\ \mu\in X$, we get

$$
(\Phi'_\varepsilon(\eta,\zeta)\mu,\mu)\le k_0\|\mu_S\|^2+2G\|\mu_D\|^2\le k_0\|\mu\|^2 , \tag{87}
$$

which leads to inequality (62) with the constant $M=\underset{x\in\Omega}{\operatorname{vrai\,max}}\,\max_T k_0(x,T)$.

To prove inequality (63), we note that the operator $\Phi'_\xi(\eta,\zeta)$ is self-adjoint but not positive and therefore its norm is defined by the expression:

$$
\|\Phi'_\xi(\eta,\zeta)\|_X=\sup_{\chi\in X}\frac{|(\Phi'_\xi(\eta,\zeta)\chi,\chi)_X|}{\|\chi\|_X^2}. \tag{88}
$$

On the basis of equality (81) for arbitrary $\eta,\ \zeta,\ \chi\in X$, we find

$$
(\Phi'_\xi(\eta,\zeta)\chi,\chi)=2(G-\bar g)\frac{(\eta_D-\zeta_D,\chi_D)^2}{\|\eta_D-\zeta_D\|^2}-k_0\|\chi_S\|^2-2G\|\chi_D\|^2 , \tag{89}
$$

from which, making use of the inequality $G-\bar g\ge 0$, we obtain

$$|(\Phi'_\xi(\eta, \zeta)\chi, \chi)_X| \le k_0 ||\chi_S||^2 + 2G||\chi_D||^2 \le k_0 ||\chi||^2, \tag{90}$$

which leads to inequality (63) with the constant $M_1 = M$. ▶

Corollary. From the properties of the operator $\Phi: X \to X$ that are established by the theorem, the results of the lemma and general data on the strictly monotone and Lipschitz-continuous operators $A: U \to U^*$ follows the one-valued solvability of operator equation (56) and also the continuous dependence of the generalized solution $u(t)$ on the applied loads $\rho(t) \in U^*$ and initial strains $\xi(t) \in X$.

Iterative Methods for Solving Boundary-Value Problems of Thermoplasticity. Let us consider the generalized method of elastic solutions [11] in which the solution $u \in U$ at each loading step is constructed as a limit of the sequence $\{u^k\}_{k=1}^{\infty} \in U$ of solutions for auxiliary linear problems. With this purpose in mind, we introduce into consideration the linear self-adjoint operator Q acting in the space X. Then, there exist two real positive numbers q_1 and q_2 such that

$$q_1 ||\mu||_X^2 \le (Q\mu, \mu)_X \le q_2 ||\mu||_X^2, \qquad \forall \mu \in X, \tag{91}$$

and therefore, the operator $Q: X \to X$ can be used for construction of the scalar product $(\cdot,\cdot)_Q$ and norm $||\cdot||_Q$ in the space X that is equivalent to the main norm of this space, i.e., the norm $||\cdot||_X$:

$$(\eta, \mu)_Q = (Q\eta, \mu)_X, \qquad ||\eta||_Q = (\eta, \eta)_Q^{1/2}, \qquad \forall \eta, \mu \in X. \tag{92}$$

In the method of elastic solutions, the sequence of linear approximations $\{u^k\}_{k=1}^{\infty} \in U$ is constructed in the form of the following iterative procedure:

$$(Bu^{k+1}, Bv)_Q = (Bu^k, Bv)_Q - \alpha[(\Phi(Bu^k, \xi), Bv)_X - \rho(v)], \qquad \forall v \in U, \tag{93}$$

where $\alpha > 0$ is the numerical parameter introduced to control the convergence, which can be varied from iteration to iteration.

By comparing equations (56) and (93), we obtain

$$(Bu^{k+1} - Bu, Bv)_Q = (Bu^k - Bu, Bv)_Q - \alpha\,(Q^{-1}[\Phi(Bu^k, \xi) - \Phi(Bu, \xi)], Bv)_Q, \qquad \forall v \in U, \tag{94}$$

from which, for $v = u^{k+1} - u \in U$, it follows that

$$||Bu^{k+1} - Bu||_Q \le ||Bu^k - Bu - \alpha Q^{-1}[\Phi(Bu^k, \xi) - \Phi(Bu, \xi)]||_Q. \tag{95}$$

Or else, the last inequality can be written in the following way:

$$||\varepsilon^{k+1} - \varepsilon||_Q \le ||\varepsilon^k - \varepsilon - \alpha Q^{-1}[\Phi(\varepsilon^k, \xi) - \Phi(\varepsilon, \xi)]||_Q. \tag{96}$$

Let us introduce into consideration the nonlinear operator $T_\alpha(\eta, \xi)$ acting in the space X and determined with the help of mapping:

$$\eta, \xi \in X \to T_\alpha(\eta, \xi) = \eta - \alpha Q^{-1}\Phi(\eta, \xi). \tag{97}$$

Then, by using the formula of finite increments, inequality (96) is transformed in the following manner:

$$||\varepsilon^{k+1} - \varepsilon||_Q \le \sup_{\eta \in X} ||T'_\alpha(\eta, \xi)||_Q \, ||\varepsilon^k - \varepsilon||_Q, \tag{98}$$

where $T'_\alpha(\eta, \xi)$ is the value of the operator T_α derivative at the point (η, ξ). Taking into account that Q is the linear operator, we get

$$\mu \in X \to T'_\alpha(\eta, \xi)\mu = \mu - \alpha Q^{-1}\Phi'_\varepsilon(\eta, \xi)\mu, \tag{99}$$

and therefore, for arbitrary $\mu, \ \chi \in X$, the following relationship is met:

$$(T'_\alpha(\eta, \xi)\mu, \chi)_Q = (Q\mu, \chi)_X - \alpha(\Phi'_\varepsilon(\eta, \xi)\mu, \chi)_X . \tag{100}$$

The expression in the right-hand side of (100) is symmetrical with respect to $\mu, \ \chi \in X$, i.e.,

$$(T'_\alpha(\eta, \xi)\mu, \chi)_Q = (\mu, T'_\alpha(\eta, \xi)\chi)_Q, \tag{101}$$

and therefore, $T'_\alpha(\eta, \xi)$ is the self-adjoint operator for all $\eta, \xi \in X$ with respect to the scalar product $(\cdot,\cdot)_Q$. The norm of the operator $T'_\alpha(\eta, \xi)$ is determined by the expression

$$||T'_\alpha(\eta, \xi)||_Q = \sup_{\mu \in X} \frac{|(T'_\alpha(\eta, \xi)\mu, \mu)_Q|}{||\mu||_Q^2}. \tag{102}$$

Using relationships (92), (100), and (102) we obtain

$$||T'_\alpha(\eta, \xi)||_Q = \sup_{\mu \in X}\left|1 - \alpha\frac{(\Phi'_\varepsilon(\eta, \xi)\mu, \mu)_X}{(Q\mu, \mu)_X}\right|. \tag{103}$$

We assume that there exist such real positive numbers γ_{01} and γ_{02} at which the inequalities

$$\gamma_{01}(Q\mu, \mu)_X \le (\Phi'_\varepsilon(\eta, \zeta)\mu, \mu)_X \le \gamma_{02}(Q\mu, \mu)_X \tag{104}$$

are fulfilled for any $\eta, \ \zeta, \ \mu \in X$.

If inequalities (104) are used for estimating (103), we obtain

$$\sup_{\eta \in X} ||T'_\alpha(\eta, \xi)||_Q \le q(\alpha) = \max(|1 - \alpha\gamma_{01}|, |1 - \alpha\gamma_{02}|), \tag{105}$$

and therefore, the condition $0 < \alpha < 2/\gamma_{02}$ provides the convergence of the method of elastic solutions for any initial approximation $u^0 \in U$.

The optimum value of α_{opt} is the solution of the equation

$$1 - \alpha_{opt}\gamma_{01} = \alpha_{opt}\gamma_{02} - 1 \tag{106}$$

and is calculated from the formula

$$\alpha_{opt} = \frac{2}{\gamma_{02} + \gamma_{01}}. \tag{107}$$

By substituting the value of α_{opt} into (105), we obtain

$$\sup_{\eta \in X} ||T'_{\alpha_{opt}}(\eta, \xi)||_Q \le q(\alpha_{opt}) = \frac{\gamma_{02} - \gamma_{01}}{\gamma_{02} + \gamma_{01}} < 1. \tag{108}$$

Thus, we arrive at the inequalities using which it is possible to estimate the maximum rate of convergence of the iteration process:

$$||\varepsilon^k - \varepsilon||_Q \le q(\alpha_{opt})||\varepsilon^{k-1} - \varepsilon||_Q \le q^k(\alpha_{opt})||\varepsilon^0 - \varepsilon||_Q . \tag{109}$$

Based on inequalities (91) and (109), we have the estimate that characterizes the maximum rate of convergence of the iteration process for strains:

$$||\varepsilon^k - \varepsilon||_X \le \sqrt{\frac{q_2}{q_1}}\, q^k(\alpha_{opt})\, ||\varepsilon^0 - \varepsilon||_X . \tag{110}$$

A similar estimate can be obtained for stresses. Actually, in view of inequalities (61) and (62), we find

$$m||\varepsilon^k - \varepsilon||_X \le ||\sigma^k - \sigma||_X \le M||\varepsilon^k - \varepsilon||_X , \tag{111}$$

and therefore, based on inequalities (110), (111), we obtain the optimum estimate of the rate of convergence for stresses:

$$||\sigma^k - \sigma||_X \le \frac{M}{m}\sqrt{\frac{q_2}{q_1}}\, q^k(\alpha_{opt})\, ||\sigma^0 - \sigma||_X . \tag{112}$$

Inequalities (110) and (112) make it possible to find the convergence of the elastic solution method irrespective of the choice of the initial approximation at a rate of the geometrical progression.

The convergence of the method for isothermal processes of active loading was first proved in [12], for non-isothermal ones in [13], however, without taking into account the initial strains dependent on the deformation process.

Note. Let $Q = D_0$, where D_0 is the linear operator that corresponds to the moduli of elasticity $k_0(x, T)$ and $G_0(x, T)$. Then for the method of elastic solutions, the following a priori estimates take place:

$$\begin{cases} q_1 = 2\,\underset{x\in\Omega}{\text{vrai}\min}\,\min_T G_0(x, T) > 0, \\ q_2 = \underset{x\in\Omega}{\text{vrai}\max}\,\max_T k_0(x, T) < \infty, \\ \gamma_{01} = \underset{x\in\Omega}{\text{vrai}\min}\,\min_T \dfrac{\bar{g}_1(x, T)}{G_0(x, T)} > 0, \\ \gamma_{02} = 1. \end{cases} \tag{113}$$

Let us consider the other, no less popular method for solving elastoplastic problems using successive approximations, namely, the method of variable elastic parameters that possesses a higher rate of convergence as compared with the method of elastic solutions. The convergence of this method was proved in [15, 16] with a restrictive assumption about a relative variation if the secant modulus of shear of a material. The proof and evaluation of the convergence of the method of variable elastic parameters for less rigid constraints are obtained in [17], however, without taking into account the initial strains dependent on the deformation process.

In the generalized method of variable elastic parameters, the sequence of linear approximations $\{u^k\}_{k=1}^{\infty} \in U$ is constructed at each loading stage in the form of the following iteration procedure:

$$(D(Bu^k, \xi)Bu^{k+1}, Bv)_X = (D(Bu^k, \xi)Bu^k, Bv)_X - \alpha[(\Phi(Bu^k, \xi), Bv)_X - \rho(v)], \qquad \forall v \in U, \tag{114}$$

where $\alpha > 0$ is the numerical parameter introduced to control the convergence, which can be varied in the process of iterations.

The iteration process of (114) can be interpreted as a method of corrections:

$$(D(Bu^k, \xi) B\omega^k, Bv)_X = (\Phi(Bu^k, \xi), Bv)_X - \rho(v), \qquad \forall v \in U, \qquad u^{k+1} = u^k - \alpha\omega^k, \tag{115}$$

where $\omega^k \in U$ is the correction for the $(k+1)$th iteration.

In the consideration of the convergence of the method of variable parameters of elasticity, we present the iteration process in Eq. (115) in the form of the operator equation for displacements. To this end, we write the correction equation:

$$\Lambda(u^k, \xi)\omega^k = A(u^k, \xi) - \rho \quad \text{in} \quad U^*, \tag{116}$$

where $\Lambda\colon U \to U^*$ is the nonlinear operator determined by the mapping

$$\Lambda(u, \xi)v : w \in U \to (D(Bu, \xi) Bv, Bw)_X, \qquad \forall v \in U. \tag{117}$$

Taking into account the properties of the operator $D : X \to X$, we conclude that $\Lambda : U \to U^*$ is the symmetrical coercive bounded operator possessing the bounded coercive inverse operator $\Lambda^{-1}\colon U^* \to U$ and therefore, the expression for the correction $\omega^k \in U$ can be presented in the form

$$\omega^k = (\Lambda(u^k, \xi))^{-1}(A(u^k, \xi) - \rho). \tag{118}$$

Thus, we obtain the following equation for displacements:

$$u^{k+1} = u^k - \alpha(\Lambda(u^k, \xi))^{-1}(A(u^k, \xi) - \rho). \tag{119}$$

According to (119), the element $u \in U$ is the fixed point of the operator $\Gamma_\alpha\colon U \to U$ determined by the mapping

$$\Gamma_\alpha\colon v \in U \to \Gamma_\alpha(v, \xi, \rho) = v - \alpha(\Lambda(v, \xi))^{-1}(A(v, \xi) - \rho). \tag{120}$$

Let the mapping $\Gamma_\alpha\colon U \to U$ has the fixed point $u \in U$ and is Frechet differentiable at this point. Moreover, there exists a norm that is equivalent to the fundamental norm of the space U for which the following condition is fulfilled:

$$\| \Gamma'_\alpha(u, \xi) \| \le q(\alpha) < 1. \tag{121}$$

Then, according to the Ostrovskii theorem [18] for an arbitrary initial approximation $u^0 \in U$ which is sufficiently close to $u \in U$, the sequence $\{u^k\}_{k=1}^{\infty} \in U$ constructed using the iteration process of (119) converges to the point $u \in U$ at a rate of geometric progression, with the estimation of the rate of convergence being characterized by the following inequality:

$$\| u^k - u \| \le q^k(\alpha) \| u^0 - u \|. \tag{122}$$

We note that the presence of the fixed point $u \in U$ of mapping (120) is attained owing to the existence of the unique solution of operator equation (56), and the above assumptions about the properties of the operator $\Phi : X \to X$ provide the Frechet differentiability of the operator $\Gamma_\alpha\colon U \to U$.

According to (120) and the rules of differentiation of complex mappings, the Frechet differential of the operator $\Gamma_\alpha\colon U \to U$ at the point $v \in U$ on the increment $w \in U$ has the form

$$w \to \Gamma'_\alpha(v, \xi, \rho) w = w - \alpha(\Lambda(v, \xi))^{-1} A'(v, \xi) w + \alpha(\Lambda(v, \xi))^{-1}(\Lambda'(v, \xi) w)(\Lambda(v, \xi))^{-1}(A(v, \xi) - \rho). \tag{123}$$

Since $u \in U$ is the fixed point of the operator $\Gamma_\alpha: U \to U$, we find

$$v \to \Gamma'_\alpha(u, \xi) v = v - \alpha (\Lambda(u, \xi))^{-1} A'(u, \xi) v, \qquad \forall v \in U, \tag{124}$$

where $A'(u, \xi) v$ is the Frechet operator differential A at the point $u \in U$ on the increment $v \in U$

$$A'(u, \xi) v\colon w \to (\Phi'_\varepsilon(Bu, \xi) Bv, Bw)_X = \langle A'(u, \xi) v, w \rangle. \tag{125}$$

We make some remarks concerning the properties of the operator Λ. Since $\Lambda: U \to U^*$ is the symmetrical coercive bounded operator, there exist two real positive numbers q_1 and q_2 such that

$$q_1 ||w||_U^2 \le \langle \Lambda(v, \xi) w, w \rangle \le q_2 ||w||_U^2, \qquad \forall v, w \in U. \tag{126}$$

Then the operator $\Lambda: U \to U^*$ can be used for constructing the scalar product $(\cdot,\cdot)_\Lambda$ and norm $||\cdot||_\Lambda$ in the space U, which is equivalent to the fundamental norm of this space, i.e., the norm $||\cdot||_U$:

$$(v, w)_\Lambda = \langle \Lambda(u, \xi) v, w \rangle, \qquad ||v||_\Lambda = (v, v)_\Lambda^{1/2}, \qquad \forall v, w \in U. \tag{127}$$

Let us show that the operator $\Gamma'_\alpha(u, \xi)$ satisfies the condition of (121) for the introduced metric $||\cdot||_\Lambda$. Since the relationship

$$(\Gamma'_\alpha(u, \xi) v, w)_\Lambda = \langle \Lambda(u, \xi) v, w \rangle - \alpha \langle A'(u, \xi) v, w \rangle \tag{128}$$

is fulfilled for arbitrary $v, w \in U$, we arrive at the following equation:

$$(\Gamma'_\alpha(u, \xi) v, w)_\Lambda = (v, \Gamma'_\alpha(u, \xi) w)_\Lambda, \qquad \forall v, w \in U, \tag{129}$$

from which it follows that $\Gamma'_\alpha(u, \xi)$ is the self-adjoint operator for the scalar product $(\cdot,\cdot)_\Lambda$ and therefore, its norm is determined by the expression

$$||\Gamma'_\alpha(u, \xi)||_\Lambda = \sup_{v \in U} \frac{|(\Gamma'_\alpha(u, \xi) v, v)_\Lambda|}{||v||_\Lambda^2}. \tag{130}$$

Using relationships (127), (128), and (130) we obtain

$$||\Gamma'_\alpha(u, \xi)||_\Lambda = \sup_{v \in U} \left| 1 - \alpha \frac{\langle A'(u, \xi) v, v \rangle}{\langle \Lambda(u, \xi) v, v \rangle} \right|. \tag{131}$$

We assume that for all $\eta, \zeta, \mu \in X$, there exist such real positive numbers γ_1 and γ_2 for which the following inequalities are fulfilled:

$$\gamma_1 (D(\eta, \zeta)\mu, \mu)_X \le (\Phi'_\varepsilon(\eta, \zeta)\mu, \mu)_X \le \gamma_2 (D(\eta, \zeta)\mu, \mu)_X. \tag{132}$$

Then, based on (117), (125), (132) for arbitrary $v, w \in U$, we have

$$\gamma_1 \langle \Lambda(v, \xi) w, w \rangle \le \langle A'(v, \xi) w, w \rangle \le \gamma_2 \langle \Lambda(v, \xi) w, w \rangle. \tag{133}$$

According to (131) and (133) we find

$$||\Gamma'_\alpha(u, \xi)||_\Lambda \le q(\alpha) = \max(|1 - \alpha\gamma_1|, |1 - \alpha\gamma_2|). \tag{134}$$

Thus, the condition $q(\alpha)<1$ will be fulfilled only in the case if $\alpha \in (0, 2/\gamma_2)$. In this case, in view of inequalities (126), the estimate that characterizes the rate of convergence of the following iteration process:

$$||u^k - u||_U \le \sqrt{\frac{q_2}{q_1}}\, q^k(\alpha)\, ||u^0 - u||_U \tag{135}$$

is true.

Based on (135), it is easy to determine that the minimum estimate

$$||\Gamma'_{\alpha_{opt}}(u, \xi)||_\Lambda \le q(\alpha_{opt}) = \frac{\gamma_2 - \gamma_1}{\gamma_2 + \gamma_1} < 1 \tag{136}$$

is attained for

$$\alpha_{opt} = \frac{2}{\gamma_1 + \gamma_2}. \tag{137}$$

The optimum estimates of the rate of convergence for strains and stresses have the form

$$||\varepsilon^k - \varepsilon||_X \le \sqrt{\frac{q_2}{q_1}}\, q^k(\alpha_{opt})\, ||\varepsilon^0 - \varepsilon||_X , \tag{138}$$

$$||\sigma^k - \sigma||_X \le \frac{M}{m}\sqrt{\frac{q_2}{q_1}}\, q^k(\alpha_{opt})\, ||\sigma^0 - \sigma||_X . \tag{139}$$

Inequalities (136), (138) and (139) make it possible to establish the local convergence of the method of variable elastic parameters in the nonisothermal processes of active loading by taking into account the initial strains

Note. For the method of variable elastic parameters, a priori estimates of the following type are true:

$$\begin{cases} q_1 = 2\, \underset{x\in\Omega}{\text{vrai}\min}\, \min_{\bar{\varepsilon}}\, \min_T G(x, \bar{\varepsilon}, T) > 0, \\ q_2 = \underset{x\in\Omega}{\text{vrai}\max}\, \max_T k_0(x, T) < \infty, \\ \gamma_1 = \underset{x\in\Omega}{\text{vrai}\min}\, \min_{\bar{\varepsilon}}\, \min_T \dfrac{\bar{g}(x, \bar{\varepsilon}, T)}{G(x, \bar{\varepsilon}, T)} > 0, \\ \gamma_2 = 1. \end{cases} \tag{140}$$

According to estimates (130) and (140), the geometric ratio in the estimate of the rate of convergence of the method of variable elastic parameters is lower than that in the estimate of the convergence of the method of elastic solutions, i.e., the rate of convergence of the method of variable elastic parameters is higher than that of the method of elastic solutions.

Note that the above given proofs of the convergence of the method of elastic solutions and variable elastic parameters ignore the error of calculations of the initial strains $\xi(t) \in X$ dependent on the deformation process. In fact, the initial strain tensor for each loading step is determined by solving the elastoplastic problem for the previous loading step, and therefore, includes an error due to the approximate solution of operator equation (56) for each loading step. Thus, we derive the equation that takes into account the input data error for the initial strains $\tilde{\xi}$:

$$A(\tilde{u}, \tilde{\xi}) = \rho \quad \text{in} \quad U^*, \quad \tilde{u} \in U. \tag{141}$$

Let us estimate the error $\tilde{u}-u \in U$. Using the first inequality in (58), we have

$$m\|\tilde{u}-u\|_U^2 \le \langle A(\tilde{u},\xi)-A(u,\xi),\ \tilde{u}-u\rangle = \left\langle A(\tilde{u},\xi)-A(\tilde{u},\tilde{\xi}),\ \tilde{u}-u\right\rangle, \tag{142}$$

whence, taking into account the third inequality in (58), we find

$$\|\tilde{u}-u\|_U \le \frac{1}{m}\|A(\tilde{u},\xi)-A(\tilde{u},\tilde{\xi})\|_{U^*} \le \frac{M_1}{m}\|\tilde{\xi}-\xi\|_X . \tag{143}$$

Thus, the following estimate is true:

$$\|\tilde{\varepsilon}-\varepsilon\|_X \le \frac{M_1}{m}\|\tilde{\xi}-\xi\|_X . \tag{144}$$

In fact, for each loading step, instead of operator equation (56), approximate equation (141) is solved and therefore the above given estimates of the rate of convergence of the methods of elastic solutions and variable elastic parameters establish the convergence of the sequential approximations to the solution of operator equation (141). On this basis, we have the estimate

$$\|\varepsilon^k-\tilde{\varepsilon}\|_X \le \sqrt{\frac{q_2}{q_1}}\,q^k\|\varepsilon^0-\tilde{\varepsilon}\|_X . \tag{145}$$

To estimate the error $\varepsilon^k-\varepsilon$, we use the triangle inequality

$$\|\varepsilon^k-\varepsilon\|_X \le \|\varepsilon^k-\tilde{\varepsilon}\|_X + \|\tilde{\varepsilon}-\varepsilon\|_X , \tag{146}$$

from which, taking into account the estimates of (144) and (145), we find

$$\|\varepsilon^k-\varepsilon\|_X \le \sqrt{\frac{q_2}{q_1}}\,q^k\|\varepsilon^0-\tilde{\varepsilon}\|_X + \frac{M_1}{m}\|\tilde{\xi}-\xi\|_X . \tag{147}$$

In addition, according to the triangle inequality and estimate of (144), we have

$$\|\varepsilon^0-\tilde{\varepsilon}\|_X \le \|\varepsilon^0-\varepsilon\|_X + \frac{M_1}{m}\|\tilde{\xi}-\xi\|_X , \tag{148}$$

and therefore, inequality (147) takes the form

$$\|\varepsilon^k-\varepsilon\|_X \le \sqrt{\frac{q_2}{q_1}}\,q^k\|\varepsilon^0-\varepsilon\|_X + \frac{M_1}{m}\left(1+\sqrt{\frac{q_2}{q_1}}\,q^k\right)\|\tilde{\xi}-\xi\|_X . \tag{149}$$

Let us estimate the error $\tilde{\xi}-\xi$, where the element $\tilde{\xi}(t_m)$ is determined by the expression

$$\tilde{\xi}(t_m) = \varepsilon_S^T(t_{m-1}) + (\varepsilon_D^p)^{k_{m-1}}(t_{m-1}). \tag{150}$$

According to (24), (28) and (150) we get

$$\tilde{\xi}(t_m)-\xi(t_m) = (\varepsilon_D^p)^{k_{m-1}}(t_{m-1}) - \varepsilon_D^p(t_{m-1})$$

$$= \varepsilon_D^{k_{m-1}}(t_{m-1}) - \varepsilon_D(t_{m-1}) - \frac{1}{2G_0(T(t_{m-1}))}(\sigma_D^{k_{m-1}}(t_{m-1}) - \sigma_D(t_{m-1})), \tag{151}$$

with

$$\sigma_D^{k_{m-1}}(t_{m-1}) - \sigma_D(t_{m-1}) = \Phi(\varepsilon_D^{k_{m-1}}(t_{m-1}), \tilde{\xi}_D(t_{m-1})) - \Phi(\varepsilon_D(t_{m-1}), \xi_D(t_{m-1})). \tag{152}$$

Let us introduce into consideration the calculation operator P for plastic strains determined according to (24) by the following mapping:

$$\eta_D, \zeta_D \in X \to P(\eta_D, \zeta_D) = \eta_D - \frac{1}{2G_0}\Phi(\eta_D, \zeta_D). \tag{153}$$

Then, in accordance with (151)–(153), we have

$$(\varepsilon_D^p)^{k_{m-1}}(t_{m-1}) - \varepsilon_D^p(t_{m-1}) = P(\varepsilon_D^{k_{m-1}}(t_{m-1}), \tilde{\xi}_D(t_{m-1})) - P(\varepsilon_D(t_{m-1}), \xi_D(t_{m-1})), \tag{154}$$

from which, using the triangle inequality and formula of finite increments, we obtain

$$||\tilde{\xi}(t_m) - \xi(t_m)||_X = ||(\varepsilon_D^p)^{k_{m-1}}(t_{m-1}) - \varepsilon_D^p(t_{m-1})||_X$$

$$\leq \sup_{\eta_D \in X} ||P'_\varepsilon(\eta_D, \tilde{\xi}_D(t_{m-1}))||_X \, ||\varepsilon_D^{k_{m-1}}(t_{m-1}) - \varepsilon_D(t_{m-1})||_X$$

$$+ \sup_{\zeta_D \in X} ||P'_\xi(\varepsilon_D(t_{m-1}), \zeta_D)||_X \, ||\tilde{\xi}(t_{m-1}) - \xi(t_{m-1})||_X, \tag{155}$$

where $P'_\varepsilon(\eta_D, \zeta_D)$ and $P'_\xi(\eta_D, \zeta_D)$ are the derivatives of the operator P at the arbitrary point (η_D, ζ_D) determined in accordance with (153) using the following mappings:

$$\begin{aligned} &\mu_D \in X \to P'_\varepsilon(\eta_D, \zeta_D)\mu_D = \mu_D - \frac{1}{2G_0}\Phi'_\varepsilon(\eta_D, \zeta_D)\mu_D, \\ &\chi_D \in X \to P'_\xi(\eta_D, \zeta_D)\chi_D = \frac{1}{2G_0}\Phi'_\xi(\eta_D, \zeta_D)\chi_D. \end{aligned} \tag{156}$$

Therefore, for arbitrary $\eta, \zeta, \mu, \chi, \lambda \in X$, we have

$$\begin{aligned} &(P'_\varepsilon(\eta_D, \zeta_D)\mu_D, \lambda_D) = \left(1 - \frac{G}{G_0}\right)(\mu_D, \lambda_D) + \frac{G - \bar{g}}{G_0}\frac{(\eta_D - \zeta_D, \mu_D)(\eta_D - \zeta_D, \lambda_D)}{||\eta_D - \zeta_D||^2}, \\ &(P'_\xi(\eta_D, \zeta_D)\chi_D, \lambda_D) = -\frac{G}{G_0}(\chi_D, \lambda_D) + \frac{G - \bar{g}}{G_0}\frac{(\eta_D - \zeta_D, \chi_D)(\eta_D - \zeta_D, \lambda_D)}{||\eta_D - \zeta_D||^2}. \end{aligned} \tag{157}$$

Based on (157) we conclude that $P'_\varepsilon(\eta_D, \zeta_D)$ and $P'_\xi(\eta_D, \zeta_D)$ are the self-adjoint operators for all η_D, $\zeta_D \in X$, and therefore their norm is determined by expressions

$$||P'_\varepsilon(\eta_D, \zeta_D)||_X = \sup_{\mu_D \in X}\frac{|(P'_\varepsilon(\eta_D, \zeta_D)\mu_D, \mu_D)_X|}{||\mu_D||_X^2}, \tag{158a}$$

$$||P'_{\xi}(\eta_D, \zeta_D)||_X = \sup_{\chi_D \in X} \frac{|(P'_{\xi}(\eta_D, \zeta_D)\chi_D, \chi_D)_X|}{||\chi_D||_X^2}. \tag{158b}$$

Taking into account relationships (157) we obtain

$$\begin{aligned}
(P'_{\varepsilon}(\eta_D, \zeta_D)\mu_D, \mu_D) &= \left(1 - \frac{G}{G_0}\right)||\mu_D||^2 + \frac{G - \bar{g}}{G_0}\frac{(\eta_D - \zeta_D, \mu_D)^2}{||\eta_D - \zeta_D||^2}, \\
(P'_{\xi}(\eta_D, \zeta_D)\chi_D, \chi_D) &= \frac{G - \bar{g}}{G_0}\frac{(\eta_D - \zeta_D, \chi_D)^2}{||\eta_D - \zeta_D||^2} - \frac{G}{G_0}||\chi_D||^2,
\end{aligned} \tag{159}$$

from which, for arbitrary η, ζ, μ, $\chi \in X$, the inequalities follow

$$\begin{aligned}
|(P'_{\varepsilon}(\eta_D, \zeta_D)\mu_D, \mu_D)| &\le \left(1 - \frac{\bar{g}}{G_0}\right)||\mu_D||^2, \\
|(P'_{\xi}(\eta_D, \zeta_D)\chi_D, \chi_D)| &\le \frac{G}{G_0}||\chi_D||^2.
\end{aligned} \tag{160}$$

In accordance with (158) and (160) we find

$$||P'_{\varepsilon}(\eta_D, \zeta_D)||_X \le 1 - \gamma_{01}, \qquad ||P'_{\xi}(\eta_D, \zeta_D)||_X \le 1, \tag{161}$$

and therefore, based on inequality (155), we arrive at the estimate

$$||\tilde{\xi}(t_m) - \xi(t_m)||_X \le (1 - \gamma_{01})||\varepsilon^{k_{m-1}}(t_{m-1}) - \varepsilon(t_{m-1})||_X + ||\tilde{\xi}(t_{m-1}) - \xi(t_{m-1})||_X. \tag{162}$$

If inequality (149) is used to estimate the first summand in the right-hand side of (162), then we obtain

$$||\tilde{\xi}(t_m) - \xi(t_m)||_X \le C_1 q^{k_{m-1}}||\varepsilon^0(t_{m-1}) - \varepsilon(t_{m-1})||_X + C_2||\tilde{\xi}(t_{m-1}) - \xi(t_{m-1})||_X, \tag{163}$$

where C_1 and C_2 are the positive constants

$$C_1 = \sqrt{\frac{q_2}{q_1}}(1 - \gamma_{01}), \qquad C_2 = 1 + \frac{M_1}{m}(1 - \gamma_{01} + C_1 q^{k_{m-1}}). \tag{164}$$

Note that in a real iteration process, the number of iterations k_{m-1} is taken such that $q^{k_{m-1}} << 1$ and therefore, it can be assumed that

$$C_2 = 1 + \frac{M_1}{m}(1 - \gamma_{01}) \le \frac{M}{m}. \tag{165}$$

If, in formula (163), each $\tilde{\xi}(t_{m-1}) - \xi(t_{m-1})$ is expressed in terms of the preceding, we obtain the following inequality:

$$||\tilde{\xi}(t_m) - \xi(t_m)||_X \le C_1 \sum_{n=1}^{m-1} C_2^{m-n} q^{k_n}||\varepsilon^0(t_n) - \varepsilon(t_n)||_X. \tag{166}$$

Based on (149) and (166), we arrive at the total error estimate for strains at the end of the loading stage:

$$||\varepsilon^{k_m}(t_m)-\varepsilon(t_m)||_X \le \sqrt{\frac{q_2}{q_1}} \sum_{n=1}^{m} C(t_n)\, q^{k_n}, \tag{167}$$

where $C(t_n)$ are the positive coefficients,

$$\begin{aligned} &C(t_m)=||\varepsilon^0(t_m)-\varepsilon(t_m)||_X, \\ &C(t_n)=(C_2-1)C_2^{m-n}||\varepsilon^0(t_n)-\varepsilon(t_n)||_X, \quad 1\le n\le m-1. \end{aligned} \tag{168}$$

A similar estimate can be obtained for stresses. In fact, using inequalities (62), (63) we find

$$||\sigma^k-\sigma||_X = ||\Phi(\varepsilon^k, \tilde{\xi})-\Phi(\varepsilon, \xi)||_X \le ||\Phi(\varepsilon^k, \tilde{\xi})-\Phi(\varepsilon, \tilde{\xi})||_X + ||\Phi(\varepsilon, \tilde{\xi})-\Phi(\varepsilon, \xi)||_X$$

$$\le \sup_{\eta\in X}||\Phi'_\varepsilon(\eta, \tilde{\xi})||_X\, ||\varepsilon^k-\varepsilon||_X + \sup_{\zeta\in X}||\Phi'_\xi(\varepsilon, \zeta)||_X\, ||\tilde{\xi}-\xi||_X \le M(||\varepsilon^k-\varepsilon||_X + ||\tilde{\xi}-\xi||_X), \tag{169}$$

and therefore, in view of estimates (163), (166), we obtain the estimate of the total error for stresses at the end of the loading step:

$$||\sigma^{k_m}(t_m)-\sigma(t_m)||_X \le M\sqrt{\frac{q_2}{q_1}} \sum_{n=1}^{m} C(t_n)\, q^{k_n}, \tag{170}$$

where $C(t_n)$ are the positive coefficients determined by the following expressions:

$$\begin{aligned} &C(t_m)=||\varepsilon^0(t_m)-\varepsilon(t_m)||_X, \\ &C(t_n)=(C_2-\gamma_{01})C_2^{m-n}||\varepsilon^0(t_n)-\varepsilon(t_n)||_X, \quad 1\le n\le m-1. \end{aligned} \tag{171}$$

Inequalities (167) and (170) make it possible to establish the convergence of the methods of elastic solutions and elastic variable parameters for solving boundary-value problems describing the nonisothermal processes of active loading taking into account the initial strains dependent on the deformation history and heating. In accordance with those estimates, the accuracy of the problem solution for initial loading stages should be such as not to allow the increase of the first coefficients of the total error expansion (167), (170) to influence the accuracy of solution of the elastoplastic problem for subsequent loading stages.

Conclusions. The results of analysis of the generalized boundary-value problem of plasticity that describes the nonisothermal processes of elastoplastic deformation taking into account the loading history have been presented. The boundary-value problem has been stated as a nonlinear operator equation in the Hilbertian space. The conditions which provide the existence, uniqueness and constant dependence of the generalized solution and initial strains have been established. The convergence of the methods of elastic solutions and variable elastic parameters for nonisothermal processes of active loading taking into account the initial strains dependent on the deformation process has been proved.

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