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VIBRODIAGNOSTIC PARAMETERS OF FATIGUE DAMAGE IN RECTANGULAR PLATES. PART 1. A PROCEDURE OF DETERMINATION OF DAMAGE PARAMETERS

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We describe a procedure of approximate analytical determination of fatigue damage parameters of rectangular plates. As an initial damage characteristic we apply the relative value of the plate potential-strain-energy variation due to availability of a Mode I crack, and, based on this value, we find relations for determination of the plate natural frequency variation, as well as parameters of distortion of monoharmonic oscillations in the modes of main resonance and superharmonic resonance of the 2nd order.

Keywords: rectangular plate, fatigue damage, Mode I crack, closing crack, vibradiagnostic damage parameters, principal and superharmonic resonances.

Introduction. Among numerous studies of vibration of cracked bodies, as applied to the problems of fatigue-damage vibrodiagnostic of such bodies, the vibration of plates has received comparatively little attention.

Investigators usually consider changes in natural modes and frequencies of vibration as vibrodiagnostic parameters of damage, with primary emphasis on the determination of plates' natural frequencies as a more sensitive damage indicator. Specifically, Cornwell et al. [1] noted that the strain energy distribution method [2] as a modification of the experimental modal method applied in vibration tests of a $430 \times 450 \times 9$-mm plate using 31 accelerometers revealed only the notches of more than 50 mm in length. The findings [3–6] suggested a poor damage sensitivity of changes of vibration modes. However, it should be mentioned that a change of natural frequency is not always an adequately sensitive damage indicator. In particular, the tests of aluminum plates (measuring $250 \times 225 \times 2.5$ mm) with through-the-thickness notches 12 mm in length showed that the maximum change of frequency in five vibration modes did not exceed 1% [7]. Analysis of other vibrodiagnostic parameters, such as distortion of monoharmonicity of a vibration process and occurrence of sub- and superharmonic resonances [8–11], was applied to plates in [12] only.

Just some of the above-mentioned works consider rectangular plates mainly with through-the-thickness edge or central cracks/notches. The known design-theoretical studies of vibration of rectangular plates consider both the finite-element numerical solutions [13] and numerical-analytical ones, which are based on the Levi–Nadai solutions with setting up of first- and second-kind Fredholm integral equations including singularity of crack-tip stresses [14] and use the finite Fourier transformation of discontinuous functions. In order to solve this problem, some version of the discrete method is used [15]. It consists in splitting a plate into a few subregions according to the crack pattern, choosing a set of shape functions for each subregion, setting up a continuity matrix for the whole region of the plate and an equation for eigenvalues by minimizing the energy functional [16].

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There have been a few studies known which are dedicated to investigation of vibration of a plate with a surface crack. In particular, Plakhtienko [12] considered vibrations of a rectangular plate with surface cracks simulated by some wedge-shaped notches with right-angled tip and with a depth equal to the crack depth. The notch width was a priory taken equal to double crack depth. The equations of vibration are set up on the basis of the Ostrogradskii–Hamilton principle and the Ritz method. There is also a different approach to modeling a crack, which was applied by Anifantis et al. [17] when studying vibration of an annular plate. An edge surface crack of constant depth is simulated by local distributed compliance determined both by finite-element method (FEM) as well as by using the energy relationships of linear fracture mechanics and the known expressions for Mode I stress intensity factor as derived for a prismatic beam with an edge crack.

The numerical-analytical solutions proposed therein fail to explicitly yield relationships for a change of natural frequency of a plate as a function of the relative size and location of a crack and relative dimensions and vibration mode of the plate. This is attributable to the crack simulation method used and to the fact that the problem for eigenvalues should be considered by incorporating a change of the plate's vibration mode in the presence of a crack.

The present study considers approximate analytical methods of determination of main vibrodiagnostic parameters of fatigue damage in rectangular plates with various edge grip conditions, vibration modes, and Mode I crack types, neglecting any change of vibration mode in the case of crack opening displacement. This assumption is quite reasonable for opening displacement of small cracks, as was demonstrated by the example of prismatic bars [18, 19]. This is essential from the standpoint of vibrodiagnostics of early stages of fatigue damage in plate-shaped structural members.

This paper covers the main principles of the procedure of determination of damage parameters.

A Procedure of Determination of a Relative Energy Characteristic of Damage in a Plate with a Crack. Let us consider a homogeneous rectangular plate of uniform thickness t, which contains shallow cracks in the sections parallel to the plane yOz (Fig. 1).

As with bar-shaped elements [8, 9, 20], the presence of small cracks is integrally assessed in terms of a relative change of the plate stiffness,

$$\alpha = \frac{D - D_c}{D}, \tag{1}$$

which in this particular case is represented by cylindrical stiffness

$$D = \frac{Et^3}{12(1-\nu^2)}, \tag{2}$$

where D_c is the cylindrical stiffness of a cracked plate during deformation under such conditions that the crack or a portion thereof lies in a zone of nominal tensile stress in bending, σ_x; D_c is further expressed in terms of the parameter α,

$$D_c = (1-\alpha)D. \tag{3}$$

Considering the small size of the crack, we disregard its effect on any change of the plate's vibration mode which is taken as

$$w(x, y) = AX(x)Y(y), \tag{4}$$

where $X(x)$ and $Y(y)$ are the beam functions that correspond to beams with similar edge grip conditions.

Analyzing a plate with or without a crack or with a crack being closed, with the value and distribution pattern of internal forces (bending moments M_x, M_y and torque M_{xy}) being equal, we set up the energy balance equation

$$\Pi_c = \Pi_0 + \Delta\Pi_c, \tag{5}$$

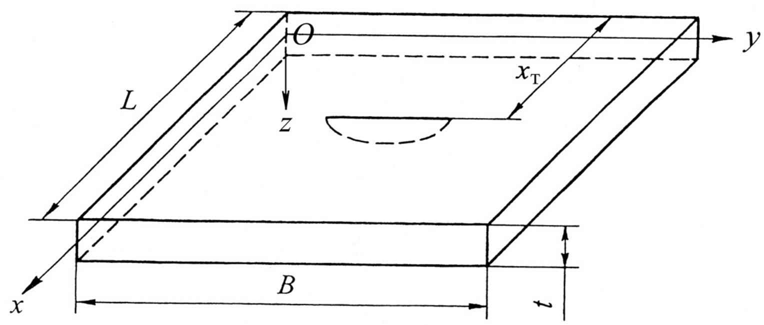

Fig. 1. Schematic representation of a homogeneous rectangular plate of uniform thickness t with a linear shallow crack.

where Π_0 is the potential energy of an intact plate,

$$\Pi_0 = \frac{1}{2D}\int_0^L\int_0^B (M_x^2 + M_y^2 + M_{xy}^2)dxdy = \frac{D}{2}\int_0^L\int_0^B\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]dxdy, \tag{6}$$

Π_c is the strain energy in a plate with an open crack, which is determined in terms of Π_0 and α as

$$\Pi_c = \frac{1}{2(1-\alpha)D}\int_0^L\int_0^B (M_x^2 + M_y^2 + M_{xy}^2)dxdy = \frac{1}{1-\alpha}\Pi_0, \tag{7}$$

$\Delta\Pi_c$ is the strain energy increment due to a crack-induced increase in the plate compliance and is to be determined with a specified value and distribution pattern of internal forces.

As mentioned above, the assumption that the pattern of distribution of internal forces is the same for an open and closed cracks enables us to neglect any change in the plate vibration mode in the case of an open crack. This assumption is acceptable for relatively small cracks, especially taking into account that the known experimental studies of the influence of damage on the vibration mode of plates [1, 3–6] demonstrate a poor sensitivity of modal changes to damage.

The energy balance equation (5) in view of (7) yields

$$\alpha = \frac{\Delta\Pi_c/\Pi_0}{1+\Delta\Pi_c/\Pi_0} = \frac{\kappa}{1+\kappa},$$

where

$$\kappa = \frac{\Delta\Pi_c}{\Pi_0} \tag{8}$$

is a convenient energy characteristic of damage in a plate, which will be used for further analysis.

We disregard any displacements of the surface crack, i.e. consider them to be Mode I cracks. Therefore, the determination of the parameter κ is reduced to finding a potential energy increment $\Delta\Pi_c$ for a given value and pattern of distribution of the bending moment in this particular case,

$$M_x = D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \tag{9}$$

which governs the value of nominal bending stresses σ_c in the sections where the crack is located ($x = x_c$),

$$\sigma_c(y, z) = 12 M_{x=x_c}(y) \frac{z}{t^3}. \tag{10}$$

In view of the relationship between the potential energy derivative for a unit-thickness layer along the crack and the stress intensity factor (this relationship results from the consideration of two-dimensional problem of a crack in a loaded linearly elastic plate-like body [21]), an expression for the potential energy increment due to the existence of a normal rupture flat crack of area S and contour Γ located parallel to the plane zOy in the plate can be written, with some assumption, as follows:

$$\Delta\Pi_c = \frac{1-\nu^2}{E} \iint\limits_{(S)} K_1^2 dS = \frac{t^3}{12D} \iint\limits_{(S)} K_1^2 \delta\vec{\rho} \cos\theta d\Gamma, \tag{11}$$

where K_1 is the normal stress intensity factor which is a function of the nominal maximum bending stress $\sigma_c(y)$ in the crack section ($x = x_c$), as calculated for an intact plate, and a function of shape, size, and coordinates of the crack front (of the contour Γ) point under consideration; $\delta\vec{\rho}$ is the vector of possible displacement of the crack front points; θ is the angle between $\delta\vec{\rho}$ and a normal to the crack front.

Given the expression for the stress intensity factor K_1 for a particular type of crack, and in view of formulas (6), (8), (10), (11) the value of κ can be determined for the crack size and location under study, for the specified plate dimensions, grip conditions, vibration modes:

$$\kappa = \frac{\dfrac{t^3}{6D^2} \iint\limits_{(S)} [K_1(\sigma_c)]^2 \delta\vec{\rho} \cos\theta d\Gamma}{\int_0^L \int_0^B \left[\left(\dfrac{\partial^2 w}{\partial y^2} \right)^2 + \left(\dfrac{\partial^2 w}{\partial y^2} \right)^2 + 2\nu \dfrac{\partial^2 w}{\partial x^2} \dfrac{\partial^2 w}{\partial y^2} + 2(1-\nu) \left(\dfrac{\partial^2 w}{\partial x \partial y} \right)^2 \right] dxdy}. \tag{12}$$

A Procedure of Determination of Vibrodiagnostic Parameters of Damage in a Plate. Based on determination of a plate's natural frequency in a given vibration mode in terms of the ratio between the reduced stiffness K and reduced mass M of an equivalent system with one degree of freedom, considering that the reduced stiffness of a plate with an open crack is given by $K_c = (1-\alpha)K$, and similar to (3), we have the following expression for the natural frequency of a plate with an open crack:

$$\omega_c = \sqrt{1-\alpha}\,\omega_0 = \frac{1}{\sqrt{1+\kappa}} \omega_0, \tag{13}$$

where ω_0 is the natural frequency of an intact plate, which is equal to $\sqrt{K/M}$.

Under the deformation conditions such that the crack is alternately open in one half-cycle and closed in the other, the natural frequency of this nonlinear system with an asymmetric bilinear elastic characteristic is determined from the known relationship [22] which, in view of formula (13), becomes

$$\omega_{ccl} = \frac{2\omega_0\omega_c}{\omega_0 + \omega_c} = \frac{2\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}} \omega_0 = \frac{2}{1+\sqrt{1+\kappa}} \omega_0. \tag{14}$$

Thus, the problem of determination of natural frequencies of a plate with an open crack (ω_c) and a periodically closing crack (ω_{ccl}) is reduced to finding the characteristic κ (12) for the given mode of vibration.

The values of a relative change of natural frequency of a plate are conveniently used as vibrodiagnostic parameters of the plate damage induced by a Mode I crack

$$\beta_c = \frac{\omega_0 - \omega_c}{\omega_0} = \frac{\sqrt{1+\kappa} - 1}{\sqrt{1+\kappa}} \tag{15}$$

in the case of open cracks, and

$$\beta_{ccl} = \frac{\omega_0 - \omega_{ccl}}{\omega_0} = \frac{\sqrt{1+\kappa} - 1}{\sqrt{1+\kappa} + 1} \tag{16}$$

in the case of periodically closing cracks.

Figure 2 shows the relative changes of natural frequency of the plate, β_c and β_{ccl}, as a function of κ. Noteworthy is the following potential feature. In case there is one closing crack, the value κ to be determined corresponds to the vibration half-cycle where the crack is open, i.e., $\sigma_c(y) > 0$, while in the other alternate half-cycle where the crack is closed and the plate acts as an intact one, we have $\kappa = 0$. On the other hand, in the case of more than one crack, the characteristic κ can have either different or equal values in alternate half-cycles (±), $\kappa_{(+)}$ and $\kappa_{(-)}$, depending on the crack location as determined by the value of coordinate x_c, and on the vibration mode of interest.

In this case, the plate natural frequencies in the respective half-cycles are given by

$$\omega_{c(+)} = \frac{1}{\sqrt{1+\kappa_{(+)}}}\omega_0, \qquad \omega_{c(-)} = \frac{1}{\sqrt{1+\kappa_{(-)}}}\omega_0, \tag{17}$$

while the plate natural frequency per cycle is determined by

$$\omega^*_{ccl} = \frac{2\omega_{c(+)}\omega_{c(-)}}{\omega_{c(+)} + \omega_{c(-)}} = \frac{2\omega_0}{\sqrt{1+\kappa_{(+)}} + \sqrt{1+\kappa_{(-)}}} \tag{18}$$

and the respective relative change in natural frequency of the plate is estimated as

$$\beta^*_{ccl} = \frac{\omega_0 - \omega^*_{ccl}}{\omega_0} = 1 - \frac{2}{\sqrt{1+\kappa_{(+)}} + \sqrt{1+\kappa_{(-)}}}. \tag{19}$$

With $\omega_{c(-)} = \omega_0$, i.e., where all of the cracks are closed in the alternate (–) half-cycle, we have $\kappa_{(+)} = \kappa$, $\kappa_{(-)} = 0$, and formulas (17), (18), (19) will accordingly become (13), (14), (16).

However, some publications stated that the change of natural frequency of structural members as noted in practice is inadequately sensitive to the presence of small cracks. It is commonly accepted that a reliable damage diagnostic is possible if a change of natural frequency exceeds 5% [23], i.e., with $\kappa > 0.1$–0.2.

Furthermore, the use of natural frequencies of a plate as a vibrodiagnostic damage indicator requires a cumbersome procedure of certification of initial frequency values for an object to be inspected, with strictly reproducible grip conditions.

Therefore, there is a need to assess the potential effectiveness of sensitive vibrodiagnostic damage indicators such as the parameters of vibration harmonicity distortion resulting from nonlinearity of a vibrating system due to the presence of periodically closing cracks, i.e., where the values κ differ in alternate cycles [$\kappa_{(+)} \neq \kappa_{(-)}$].

The most representative parameters of the harmonicity distortion as determined during the harmonic analysis of a measured vibration process of an elastic body with a closing crack is the value of a ratio of constant component A_0 and second-harmonic amplitude A_2 to the fundamental-harmonic amplitude A_1. These parameters are revealed

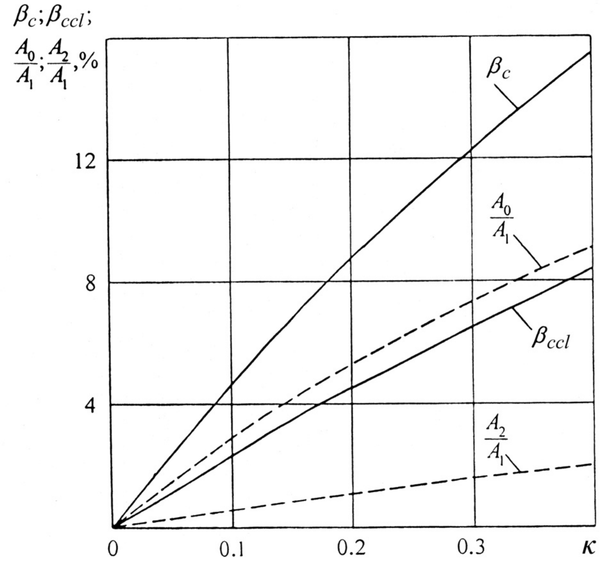

Fig. 2. The influence of the plate damage parameter κ on the relative change of natural frequencies β_c and β_{ccl} and relative values of constant component A_0/A_1 and second-harmonic amplitude A_2/A_1 in the principle resonance.

both in the principal resonance (excitation frequency $\upsilon = \omega_{ccl}$) [8] and especially in the 2nd-order superharmonic resonance ($\upsilon = 0.5\omega_{ccl}$) [9] and can be expressed in terms of κ.

A study of vibration in a given natural mode for a plate with a closing crack is brought down to analyzing vibration of an equivalent system with one degree of freedom with an asymmetric bilinear characteristic of restoring force [12]. The forced vibration of such an essentially nonlinear system [24] can be described by the equation [20]

$$\frac{d^2u}{dt^2} + 2h\frac{du}{dt} + \omega_0^2[1 - 0.5\alpha(1 + \operatorname{sign} u)]u = q_0 \sin \upsilon t, \tag{20}$$

where h is a coefficient representing dissipative properties of the vibrating system.

The complexity of an analytical solution of this equation explains why the known publications provide no expressions for explicit definition of the solution parameters. The method of setting up the solution in Fourier series, which yields approximate solutions with an accuracy required, involves usually rather intricate and cumbersome sets of transcendental equations which are far from being convenient to analyze and require numerical solution [12, 24, 25].

Therefore, for approximate estimation of such main vibrodiagnostic parameters as the A_0/A_1 and A_2/A_1 ratios we use the explicitly established dependence of these ratios on the parameter α in the principle resonance [20] and weak ($A_2 < A_1$) 2nd-order superharmonic resonance [26].

Thus, in view of the relationship $\alpha = \kappa/(1+\kappa)$ the relative value of the constant component in both the principal and superharmonic resonances is given by

$$\frac{A_0}{A_1} \approx \frac{\kappa}{\pi(1+\kappa)}, \tag{21}$$

the relative value of the second-harmonic amplitude in the principle resonance ($\upsilon = \omega_{ccl}$) by

$$\frac{A_2}{A_1} \approx \frac{2\kappa}{9\pi(1+\kappa)} \tag{22}$$

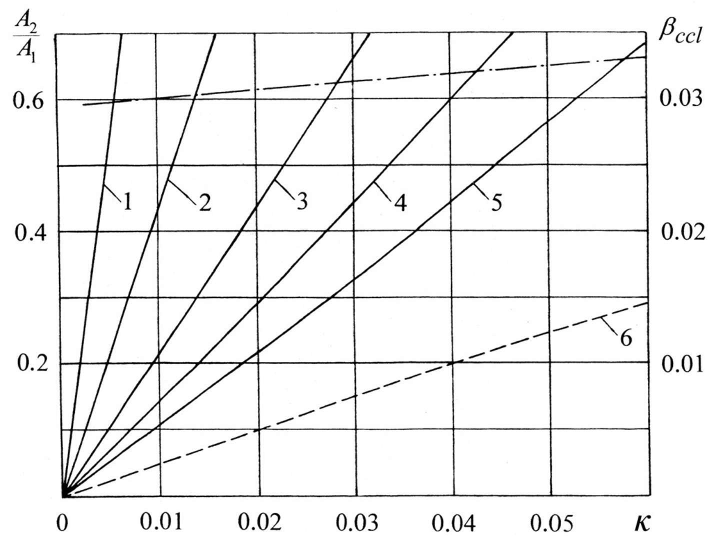

Fig. 3. The effect of the plate damage parameter κ on the second-harmonic amplitude ratio A_2/A_1 in the case of the 2nd-order superharmonic resonance for various values of the coefficient h/ω_0 representing the plate damping capacity (solid lines) and on the relative change of natural frequency of the plate (dashed line): (*1*) $h/\omega_0 = 0.001$; (*2*) $h/\omega_0 = 0.0025$; (*3*) $h/\omega_0 = 0.005$; (*4*) $h/\omega_0 = 0.0075$; (*5*) $h/\omega_0 = 0.01$; (*6*) β_{ccl}.

and in the 2nd-order superharmonic resonance ($\upsilon = 0.5\omega_{ccl}$) by

$$\frac{A_2}{A_1} \approx \frac{\kappa(1+\sqrt{1+\kappa})[(8-\pi)(1+\kappa)-\pi]}{32(1+\kappa)^2}\frac{\omega_0}{h}. \tag{23}$$

Formula (23) was derived for the case of a weak resonance and therefore its use with an adequate accuracy is restricted by the condition $\kappa < \frac{5.5h/\omega_0}{1-5.5h/\omega_0}$, where h/ω_0 can be expressed in terms of the logarithmic decrement of vibration δ, $\frac{h}{\omega_0} = \frac{\delta}{2\pi}$. When the above restriction is satisfied, the difference between the values calculated by (23) and the results of numerical solution of the differential equation (20) for a limiting value of κ does not exceed 5%.

For the case where the nonlinearity is governed by a difference between the parameter κ in alternate half-cycles $[\kappa_{(+)} \neq \kappa_{(-)}]$ the equivalent value $\alpha = \frac{\kappa_{(+)} - \kappa_{(-)}}{1+\kappa_{(+)}}$ and accordingly the parameter κ in formulas (18) through (20) should be taken equal to

$$\kappa = \frac{\kappa_{(+)} - \kappa_{(-)}}{1+\kappa_{(-)}}. \tag{24}$$

For comparative assessment of representativeness of the change of natural frequency of an elastic body with cracks and the main parameters of distortion of monoharmonicity of the vibration process in the principle resonance, relationships (21) and (22) are shown by dashed lines in Fig. 2. It is evident that in the principal resonance the monoharmonicity distortion parameters are somewhat less representative in comparison to the level of the natural frequency change. However, an absolutely different trend is observed in the superharmonic resonance. For the case of the 2nd-order superharmonic resonance, the dependence of the second-harmonic relative amplitude on the damage parameter κ is shown by solid lines for various values h/ω_0 that represent dissipative properties of the vibrating system in the range of logarithmic decrement $\delta_h = 0.628$–6.28%, while the dashed line indicates the function of the relative change of natural frequency of the elastic body (β_{ccl}) with a crack. The dash-and-dot line in Fig. 3 corresponds to a limit value of the parameter κ, which for a given value δ, allows one to use formula (23) with a good reliability, i.e., corresponds to the condition $\kappa = \dfrac{2.25\delta}{\pi - 2.25\delta}$.

In the case of the superharmonic resonance, the monoharmonicity distortion parameter A_2/A_1 is seen to be more than one order of magnitude more significant, in comparison to the change of natural frequencies, and it can be considered as the most sensitive damage indicator.

Conclusion. Based on the energy relationships of linear fracture mechanics and the results of solving a differential equation of forced vibration of an elastic body with an asymmetric bilinear characteristic of restoring force, we have presented herein a procedure of approximate analytical determination of the relative energy characteristic and vibrodiagnostic parameters of damage in rectangular plates due to the presence of Mode I cracks.

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