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Resonant MHD Waves in the Solar Atmosphere

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Abstract The linear theory of MHD resonant waves in inhomogeneous plasmas is reviewed. The review starts from discussing the properties of driven resonant MHD waves. The dissipative solutions in Alfvén and slow dissipative layers are presented. The important concept of connection formulae is introduced. Next, we proceed on to non-stationary resonant MHD waves. The relation between quasi-modes of ideal MHD and eigenmodes of dissipative MHD are discussed. The solution describing the wave motion in non-stationary dissipative layers is given. It is shown that the connection formulae remain valid for non-stationary resonant MHD waves. The initial-value problem for resonant MHD waves is considered. The application of theory of resonant MHD waves to solar physics is discussed.

Keywords Plasma · Waves · MHD · Sun

1 Introduction

The last decade has seen an avalanche of observations of MHD waves in the solar atmosphere. It is clear now that MHD waves are ubiquitous in the solar atmosphere. For latest reviews on observations see e.g. Banerjee et al. (2007); De Moortel (2009); Mathioudakis et al. (2010). This has triggered new theoretical research for explaining and interpreting the observed properties. A special point of attention is whether these MHD waves are slow, fast or Alfvén waves. Matters are complicated by the non-uniformity of the plasma in the solar atmosphere. In a uniform plasma of infinite extent the MHD waves can be put in well separated boxes. However, in non-uniform plasmas this clear cut division is often lost as the MHD waves can have mixed properties. The debate on the nature of MHD waves in the solar

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atmosphere has gained new momentum when several groups, e.g. De Pontieu et al. (2007), Okamoto et al. (2007) in space-borne, and Tomczyk et al. (2007) and Jess et al. (2009) in ground-based observations reported the detection of Alfvén waves in the solar atmosphere. Apparently, there is no general consensus about the presence of Alfvén waves in the solar atmosphere. For example Erdélyi and Fedun (2007) highlight the ambiguity in the interpretation of the space-borne observations and show with forward modelling that some of the space-borne observations are more consistent with kink waves, while Van Doorsselaere et al. (2008) argued strongly against the possible presence of Alfvén waves in the solar corona emphasizing that Alfvén waves cannot be anything but torsional. Jess et al. (2009) carried out rigorous tests when interpreting their observations as torsional Alfvén waves. It is very unlikely that Alfvén waves in their pure form as discovered by Alfvén (1942) exist at all in the solar atmosphere. Conventional theory of Alfvén waves requires a uniform plasma of infinite extent with a constant unidirectional magnetic field. Since the plasma structures in the solar atmosphere have a finite extent and are almost as a rule inhomogeneous it would be surprising to discover pure Alfvén waves in the solar atmosphere. In addition the basic characteristic of the ideal Alfvén wave is that the total pressure in the plasma remains constant during the passage of the wave. For inhomogeneous medium, however, the total pressure, in general, couples with the dynamics of the motion, and the assumption of neglect of total pressure perturbations becomes invalid. However, that does not mean that the concept of Alfvén waves is obsolete. In general in an inhomogeneous plasma MHD waves have mixed properties which can be traced back to the properties of the classic slow, fast and Alfvén waves in a homogeneous plasma of infinite extent. The degree to which the classic properties are present in a given MHD wave depends on the background through which the MHD wave propagates. When an MHD wave that starts off as, e.g., a predominantly fast magneto-sonic wave propagates through an inhomogeneous background it can change into an MHD wave with equally strong fast and Alfvén properties and eventually turn into a predominantly Alfvén wave. Here a predominantly fast magneto-sonic wave refers to a wave with the gradient of pressure as the main restoring force and plasma pressure and magnetic pressure being in phase. A predominantly Alfvén wave refers to a wave with the magnetic tension force as its dominant restoring force. A first objective of this review is to make it clear that in non-uniform plasmas MHD waves in general have mixed wave properties and that in non-uniform plasmas they cannot be put in clearly separated boxes as is done for homogeneous plasmas of infinite extent. The second objective is to point out that resonant slow and Alfvén waves are a natural consequence of the non-uniformity of the plasma that supports the MHD waves. The third objective is to point out that because of the MHD wave coupling discrete MHD waves with frequencies in the slow and Alfvén continuous parts of the spectrum are transformed into damped quasi-modes.

Resonant MHD waves are important for the solar atmosphere for several reasons. They have attracted attention because it was realized that resonant absorption of e.g. Alfvén waves is an efficient means for dissipating energy of MHD waves in nonuniform plasmas. It was first studied as means for the supplementary heating of fusion plasmas (see e.g. Tataronis and Grosmann 1973; Grossmann and Tataronis 1973; Chen and Hasegawa 1974a; Hasegawa and Chen 1974) and subsequently proposed as a mechanism for heating magnetic flux tubes in the solar corona by Ionson (1978). Since the original suggestion, resonant absorption has remained a popular mechanism for explaining the heating of the solar corona (see e.g. Wentzel 1979b; Ionson 1985; Hollweg 1988; Hollweg and Yang 1988; Hollweg 1990; Goossens 1991; Poedts and Kerner 1992; Steinolfson and Davila 1993; Ofman et al. 1995; Erdélyi and Goossens 1995). Resonant absorption has been considered as a possible explanation of the observed loss of power of acoustic oscillations in sunspots (e.g. Hollweg 1988;

Lou 1990; Goossens and Poedts 1992; Stenuit et al. 1993; Erdélyi and Goossens 1994; Keppens et al. 1994).

Resonant MHD waves have become important for the transverse oscillations that are observed in coronal loops (Aschwanden et al. 1999, 2002; Nakariakov et al. 1999; Schrijver and Brown 2000; Schrijver et al. 2002; Aschwanden 2006). These transverse oscillations are often triggered by a nearby solar flare and are interpreted as kink MHD waves. Kink refers to azimuthal wave number equal to 1 ($m = 1$). A kink wave is required to explain the observations since MHD waves with their azimuthal wave number equal to 1 are the only modes that displace the axis of the loop and the loop as a whole. A striking property of these transverse waves is their fast damping with damping times of the order of 3–5 periods.

Resonant absorption is a strong contender as a damping mechanism that offers a consistent explanation of this rapid damping (see e.g. Ruderman and Roberts 2002; Goossens et al. 2002a; Van Doorsselaere et al. 2004; Terradas et al. 2006a, 2006b; Aschwanden et al. 2003; Arregui et al. 2007a, 2007b, 2008a; Goossens et al. 2008). Resonant absorption relies on the transfer of energy from a global MHD wave to local resonant Alfvén waves. If this mechanism is indeed operational then this means that the observed transverse oscillations have Alfvénic properties in at least part of the oscillating loop. Time dependent studies of damped coronal loop oscillations due to an initial perturbation with the damping mechanism of resonant absorption were carried out by e.g. Ruderman and Roberts (2002), Terradas et al. (2006a, 2007), Arregui et al. (2007c), and reviewed by Terradas (2009). Recent studies (Morton and Erdélyi 2009, 2010; Morton et al. 2010; Erdélyi et al. 2010) show interesting results as far as MHD damping is concerned. However, more research is needed before definite conclusions can be drawn.

Resonant Alfvén waves are means for transferring energy of footpoint motions into coronal loops (see e.g. De Groof and Goossens 2000, 2002; Goossens and De Groof 2001; De Groof et al. 2002). Resonant MHD waves are important in relation to damped oscillations observed in prominences (see e.g. Arregui et al. 2008b; Soler et al. 2009a) and even in partially ionized plasmas (Soler et al. 2009b). This latter case exemplifies the amazingly robust character of resonant Alfvén MHD waves. Resonant MHD waves can play a role as means for damping global solar oscillations when they interact with the chromospheric magnetic field (see e.g. Čadež et al. 1997; Tirry et al. 1998a; Pintér and Goossens 1999; Van Lommel and Goossens 1999; Van Lommel et al. 2002; Pintér et al. 2007). Resonant MHD waves and instabilities can operate in plumes (see e.g. Tirry et al. 1998b; Andries et al. 2000; Andries and Goossens 2001a, 2001b) or in the magnetopause and magnetotail (see e.g. Lanzerotti et al. 1973; Southwood 1974; Chen and Hasegawa 1974b, 1974c; Inhester 1986; Kivelson and Southwood 1986; Southwood and Kivelson 1986; Smith et al. 1986; Ruderman and Wright 1998; Taroyan and Erdélyi 2002, 2003a, 2003b; Erdélyi and Taroyan 2003) and produce instabilities with the instability thresholds smaller than those in the classical Kelvin-Helmholtz instabilities.

In what follows we shall be concerned both with driven MHD waves and with eigenmodes of the system. The study of driven slow and Alfvén waves involves forced oscillations in dissipative MHD. This means that the time-dependent non-linear equations of dissipative MHD have to be integrated in the presence of a time-varying force term. Most studies have used linear theory of wave motions superimposed on an ideal equilibrium state. Even in the context of linear MHD, often the time integration is circumvented by considering the asymptotic or stationary state of the slow and Alfvén waves. In the asymptotic state all the perturbed quantities oscillate with the same frequency as the incoming wave, so that the time dependence can be removed out of the mathematical formulation. Detailed results based on large-scale numerical simulations of the asymptotic state of Alfvén wave heating were obtained by Grossmann and Smith (1988) in ideal MHD, by Poedts et al. (1989a, 1989b, 1990a, 1990b), in resistive MHD, and by Erdélyi and Goossens (1994, 1995) in visco-resistive MHD. Time-dependent computations of Alfvén wave heating in linear dissipative MHD have been carried out by e.g. Poedts et al. (1990c), Poedts and Kerner (1992), Steinolfson and Davila (1993), Ofman et al. (1995) and Wright and Rickard (1995).

This review paper is organized as follows. In Sect. 2 we discuss the mixed properties of MHD waves in non-uniform plasmas in ideal MHD. We point out that MHD waves in their pure form as they are known for idealized uniform plasmas of infinite extent, hardly ever occur in real situations with inhomogeneous plasmas. Often the phenomenon of MHD waves with mixed properties is referred to as coupling of waves although there is only one wave present which has different properties in different parts of the plasma. However, since the term coupled waves is so popular in the solar physics community and since we are confident that the people who use this term understand its meaning, we shall also use it. In Sect. 3 we turn to resonant Alfvén and slow waves in static equilibria and aim to show that these waves are a natural phenomenon and hard to avoid in inhomogeneous plasmas. Resonant Alfvén and slow waves are first discussed in ideal MHD with a static equilibrium. The main result of those subsections is that, in ideal MHD, resonant Alfvén waves have singular spatial behaviour. In the subsequent two subsections it is shown how the singular behaviour of resonant waves is removed by including dissipation in the system for slow and Alfvén resonances, respectively. In Sect. 4 the properties of resonant waves within a steady plasma are discussed. In Sect. 5 we shall discuss the phenomenon of quasi-modes in non-uniform plasmas, and the initial value problem for resonant waves. Section 6 gives a short summary of this review.

2 Resonant MHD Waves with Mixed Properties

The main aim of this section is to present the relevant equations for linear resonant MHD waves in a static equilibrium and to point out that, in inhomogeneous plasmas, linear resonant MHD waves have mixed properties. The phenomenon of resonant MHD waves with mixed properties or coupling of MHD waves was discussed by Goossens et al. (2002b, 2009), Goossens (2008). It is clearly at work in, e.g., De Groof and Goossens (2000, 2001, 2002), De Groof et al. (2002), Arregui et al. (2003, 2004a, 2004b).

The focus of the present review is on the asymptotic state of driven Alfvén and slow waves in linear MHD and on the resonantly damped quasi-modes in ideal MHD, which are counterparts of eigenmodes in dissipative MHD. In both cases all perturbed quantities can be put proportional to

$$\exp(-i\omega t), \tag{1}$$

where $\omega > 0$ is the frequency of the incoming wave or the external forcing term in a driven problem, while ω is complex with a negative imaginary part in the eigenvalue problem at least for a static plasma. In a steady plasma the wave can become over-stable (see Sect. 4). The asymptotic state is, in principle, only valid for $t \to +\infty$, but in practice it gives an accurate description for $t \gtrsim t_{\mathrm{tran}}$ where $t_{\mathrm{tran}} \propto (R_m)^{1/3}$, R_m is the magnetic Reynolds number, and the coefficient of proportionality depends on the geometrical and physical parameters of the equilibrium state (see Kappraff and Tataronis 1977; Poedts et al. 1990b). In the driven problem the frequency is real and prescribed while, in the eigenvalue problem involving resonantly damped quasi-modes, the frequency is complex and unknown.

The present review is concerned with resonant MHD waves in 1-dimensional magnetic flux tubes. Spatial variations of the equilibrium quantities in different directions affect the waves in different ways. For example in the eigenvalue problem the radial stratification causes damping (see e.g. Hollweg and Yang 1988; Goossens et al. 1992; Ruderman and Roberts 2002; Van Doorsselaere et al. 2004), while axial stratification affects the periods of oscillation (see e.g. Andries et al. 2005a, 2005b, 2009a, 2009b; Arregui et al. 2005; Goossens et al. 2006; Dymova and Ruderman 2005, 2006a, 2006b; McEwan et al. 2006, 2008; Verth and Erdélyi 2008; Ruderman et al. 2008).

It is evident that these two effects should be combined and that there is a need for studying resonant waves in 2-dimensional equilibrium states. However, it is necessary to have a clear understanding of waves in simple 1-D equilibrium configurations before embarking on more complicated resonant MHD wave problems in 2-D equilibrium states.

The steady equilibrium state of the flux tube is idealized as a cylindrically symmetric column of plasma. We use a system of cylindrical coordinates r, φ, z with the z-axis coinciding with the axis of symmetry of the cylinder. The components of the equilibrium magnetic field $\mathbf{B} = (0, B_\varphi, B_z)$, the components of the equilibrium velocity field $\mathbf{v} = (0, v_\varphi, v_z)$, as well as the pressure p and density ρ are functions of the radial coordinate r only. They satisfy the radial force balance equation

$$\frac{d}{dr}\left(p + \frac{B^2}{2\mu_0}\right) = \rho\frac{v_\varphi^2}{r} - \frac{B_\varphi^2}{\mu_0 r}, \qquad B^2 = B_\varphi^2 + B_z^2, \tag{2}$$

where μ_0 is the magnetic permeability of free space. Note that the magnetic surfaces are cylinders $r = \text{const}$.

The linear displacements superimposed on a generally *steady* background state are governed by the linearized versions of the resistive MHD equations,

$$\frac{\partial \rho'}{\partial t} = -\nabla\cdot(\rho\mathbf{v}' + \rho'\mathbf{v}), \tag{3a}$$

$$\rho\left[\frac{\partial \mathbf{v}'}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}' + (\mathbf{v}'\cdot\nabla)\mathbf{v}\right] + \rho'(\mathbf{v}\cdot\nabla)\mathbf{v}' = -\nabla p' + \frac{1}{\mu}(\nabla\times\mathbf{B})\times\mathbf{B}' + \frac{1}{\mu}(\nabla\times\mathbf{B}')\times\mathbf{B}, \tag{3b}$$

$$\rho\left[\frac{\partial T'}{\partial t} + \mathbf{v}\cdot\nabla T'\right] = -\rho\mathbf{v}'\cdot\nabla T - (\gamma-1)\rho T\nabla\cdot\mathbf{v}', \tag{3c}$$

$$\frac{\partial \mathbf{B}'}{\partial t} = \nabla\times(\mathbf{v}'\times\mathbf{B}) + \nabla\times(\mathbf{v}\times\mathbf{B}') + \eta\nabla^2\mathbf{B}, \tag{3d}$$

$$\frac{p'}{p} = \frac{\rho'}{\rho} + \frac{T'}{T}, \tag{3e}$$

where η is the coefficient of magnetic diffusion, and prime denotes an Eulerian perturbation. Equations (3) describe linear motions on a steady background. In the present section we shall confine our attention to linear motions on a static background so that $\mathbf{v} = 0$. This condition will be relaxed in Sect. 4 where issues related to equilibrium flows will be investigated and reviewed in details. We have given the equations for linear motions on a steady background in the present section in order to avoid duplication of equations.

Eventually we shall have to use dissipative MHD to model the resonant behaviour of slow and Alfvén waves as dissipative terms are needed to remove the ideal singularity. Electrical resistivity is a prime candidate in this respect (very similar results are obtained if viscosity is taken to be the dissipation mechanism as was shown by Erdélyi and Goossens (1994, 1995) who considered resonant absorption in an inhomogeneous visco-resistive plasma under solar conditions). However the large values of the viscous and magnetic Reynolds numbers in the solar atmosphere imply that the dissipative terms in the MHD equations are unimportant except in narrow layers. In the case of resonant Alfvén waves the dissipative terms are only important in a narrow layer around the position where the frequency of the wave equals the local Alfvén frequency. A similar principle applies to resonant slow waves where the resonant layer is defined around the position where the frequency of driving waves equals the local slow (or often referred to as cusp) frequency. Outside these narrow layers the MHD waves are accurately described by the equations of ideal MHD.

Since the equilibrium quantities depend on r only we can Fourier-analyze the perturbed quantities with respect to φ and z and prescribe them proportional to

$$\exp[i(m\varphi + k_z z)]. \tag{4}$$

Here m (an integer) and k_z are the azimuthal and axial wave numbers. As the time-dependence $\exp(-i\omega t)$ has already been factored out the perturbed quantities are functions of r only and the partial differential equations are reduced to ordinary differential equations.

The phenomenon of coupling of MHD waves or alternatively the phenomenon of MHD waves with mixed properties and part of the basic physics of resonant slow and Alfvén waves can be understood in the context of linear ideal MHD. In what follows $\boldsymbol{\xi}$ is the Lagrangian displacement and ξ_r is its radial component. In the present configuration the radial direction is the direction normal to the magnetic surfaces. The components in magnetic surfaces respectively perpendicular, $\xi_\perp$, and parallel, $\xi_\parallel$, to the magnetic field lines are defined as

$$\xi_\perp = (\xi_\varphi B_z - \xi_z B_\varphi)/B, \qquad \xi_\parallel = \boldsymbol{\xi} \cdot \mathbf{B}/B. \tag{5}$$

In what follows we also use the Eulerian perturbation of total pressure, P', defined as

$$P' = p' + \mathbf{B} \cdot \mathbf{B}'/\mu_0, \tag{6}$$

with p' being the Eulerian perturbation of kinetic plasma pressure. Note that $\xi_\perp$ is the characteristic quantity for the Alfvén waves, ξ_r for the fast magneto-sonic waves and $\xi_\parallel$ for the slow waves.

All but two of the perturbed variables can be eliminated from the linear ideal MHD equations leading to a set of two first-order differential equations for the radial component of the Lagrangian displacement, ξ_r, and the total pressure perturbation P'. The remaining wave quantities can be expressed in terms of these two quantities (although there is one important exception). For understanding the mixed properties of the MHD waves the algebraic expressions for $\xi_\perp$, $\xi_\parallel$ and for the compression $\nabla \cdot \boldsymbol{\xi}$ are essential. The relevant equations are

$$D\frac{d(r\xi_r)}{dr} = C_1 r\xi_r - C_2 r P', \tag{7a}$$

$$D\frac{dP'}{dr} = C_3 \xi_r - C_1 P', \tag{7b}$$

$$\rho(\omega^2-\omega_A^2)\xi_\perp=\frac{i}{B}C_A, \tag{7c}$$

$$\rho(\omega^2-\omega_C^2)\xi_\parallel=\frac{if_B}{B}\frac{v_S^2}{v_S^2+v_A^2}C_S, \quad \rho\omega^2\xi_\parallel=\frac{if_B}{B}\delta p, \tag{7d}$$

$$\nabla\cdot\xi=\frac{-\omega^2 C_S}{\rho(v_S^2+v_A^2)(\omega^2-\omega_C^2)}, \tag{7e}$$

where δp is the Lagrangian variation of plasma pressure. It is defined as

$$\delta p=p'+\frac{dp}{dr}\xi_r. \tag{8}$$

The differential equations (7a) and (7b) for ξ_r and P' were first obtained in this form by Appert et al. (1974) (see also Sakurai et al. 1991a; Goossens et al. 1995; Tirry and Goossens 1996; Erdélyi and Fedun 2010). The Cartesian version of (7) with a horizontal magnetic field, the equilibrium quantities varying in the vertical direction, and no gravity are given by, e.g. Csík et al. (1997), while the same equations with gravity in the vertical direction included are given by Tirry and Goossens (1996), Tirry et al. (1998a) and Pintér et al. (2007). Goossens et al. (1992), Erdélyi et al. (1995), Erdélyi (1997) derived the corresponding equations for a steady state equilibrium, i.e. for an equilibrium with a background steady flow. The issues related to resonant absorption with background bulk motion is discussed in Sect. 4. The coefficient functions D, C_1, C_2, and C_3, and the coupling functions C_A and C_S depend on the equilibrium quantities and on the frequency ω. They take the form

$$D=\rho(v_S^2+v_A^2)(\omega^2-\omega_A^2)(\omega^2-\omega_C^2), \tag{9a}$$

$$C_1=\frac{2}{\mu_0 r}B_\varphi^2\omega^4-(v_S^2+v_A^2)(\omega^2-\omega_C^2)\frac{2mf_B}{\mu_0 r^2}B_\varphi, \tag{9b}$$

$$C_2=\omega^4-(v_S^2+v_A^2)(\omega^2-\omega_C^2)\left(\frac{m^2}{r^2}+k_z^2\right), \tag{9c}$$

$$C_3=D\left[\rho(\omega^2-\omega_A^2)+\frac{2B_\varphi}{\mu}\frac{d}{dr}\left(\frac{B_\varphi}{r}\right)\right] + \frac{4\omega^4 B_\varphi^4}{\mu_0^2 r^2}-4\rho(v_S^2+v_A^2)(\omega^2-\omega_C^2)\omega_A^2\frac{B_\varphi^2}{\mu_0 r^2}, \tag{9d}$$

$$C_A=g_B P'-\frac{2f_B B_\varphi B_z\xi_r}{\mu_0 r}, \qquad C_S=P'-\frac{2B_\varphi^2\xi_r}{\mu_0 r}. \tag{9e}$$

In (9) v_S is the adiabatic sound speed, v_A is the Alfvén speed, ω_A is the local Alfvén frequency, and ω_C is the local cusp frequency. They are defined as

$$v_S^2=\gamma p/\rho, \qquad v_A^2=B^2/(\mu_0\rho), \tag{10a}$$

$$\omega_A^2=f_B^2/(\mu_0\rho), \quad \omega_C^2=v_S^2\omega_A^2/(v_S^2+v_A^2), \tag{10b}$$

$$f_B = \mathbf{k}\cdot\mathbf{B}, \qquad g_B = mB_z/r - k_z B_\varphi, \quad \mathbf{k} = (0, m/r, k_z). \tag{10c}$$

In a nonuniform plasma v_S^2, v_A^2, ω_A^2, and ω_C^2 are functions of position.

Equations (7) define an eigenvalue problem with ω^2 as eigenvalue parameter when they are supplemented with boundary conditions. In a driven problem ω is prescribed. Equations (7a) and (7b) define what we might want to call fast magneto-sonic waves since the fast waves are primarily associated with motions normal to the magnetic surfaces. Equation (7c) defines Alfvén waves since Alfvén waves in their ideal version as they were discovered by Alfvén (1942) have motions in the magnetic surfaces perpendicular to the magnetic field lines. Finally, (7d) defines the slow magneto-sonic waves with motions along the magnetic field lines. The second version of this equation is given to show that the parallel motions are driven by the plasma pressure force. In a pressureless plasma there are no parallel motions since the Lorentz force has no component parallel to the magnetic field. The functions C_A and C_S are called the coupling functions for the good reason that they couple the four equations of (7). For example the equation for what we like to call the pure Alfvén waves is coupled with the two equations that we associate with fast magneto-sonic waves. The fact that the equations are coupled also means that there are no pure fast magneto-sonic waves and no pure Alfvén waves. The MHD waves have mixed properties or in somewhat ambiguous terminology the waves are coupled. The mixing of the fast, Alfvén and slow properties depends on the values of the functions C_A and C_S. In general the functions C_A and C_S are non-zero and depend on position. This means that a wave propagating through a non-uniform plasma can start off as a predominantly fast wave, subsequently can change into a wave that has both fast and Alfvén properties, and eventually can turn into a predominantly Alfvénic wave. The fact that MHD waves have mixed properties and that they have different appearances in different parts of the plasma is due to the inhomogeneity of the plasma. In an inhomogeneous plasma this behaviour is hard to avoid. The coupling function C_A and the local Alfvén frequency ω_A play an essential role in the analysis of resonant Alfvén waves.

There is one exception to this phenomenon of coupling of MHD waves or MHD waves with mixed properties. For axi-symmetric MHD waves with azimuthal wave number ($m = 0$) in a 1-dimensional cylindrical equilibrium model with a straight field $B_\varphi = 0$ there is no interaction or coupling between the sausage ($m = 0$) fast and slow waves and torsional ($m = 0$) Alfvén waves. For a straight magnetic field the ideal MHD equations for linear MHD waves (7) take the following simplified form

$$D\frac{d(r\xi_r)}{dr} = -C_2 r P', \tag{11a}$$

$$\frac{dP'}{dr} = \rho(\omega^2 - \omega_A^2)\xi_r, \tag{11b}$$

$$\rho(\omega^2 - \omega_A^2)\xi_\varphi = \frac{im}{r}P', \tag{11c}$$

$$\rho(\omega^2 - \omega_C^2)\xi_z = ik_z\frac{v_S^2}{v_S^2 + v_A^2}P', \quad \rho\omega^2\xi_z = ik_z\delta p', \tag{11d}$$

$$\nabla\cdot\boldsymbol{\xi} = \frac{-\omega^2 P'}{\rho(v_S^2 + v_A^2)(\omega^2 - \omega_C^2)}. \tag{11e}$$

For a straight field the φ and the z-direction are the directions in the magnetic surfaces respectively perpendicular and parallel to the magnetic field lines. As before the r-direction is normal to the magnetic surfaces. Hence, for a straight field, $\xi_\varphi = \xi_\perp$ is the characteristic quantity for the Alfvén waves and $\xi_z = \xi_\parallel$ for the slow waves. As before ξ_r characterizes the fast magneto-sonic waves. The coupling function C_A is now

$$C_A = \frac{m B_z \xi_r}{r} P'. \tag{12}$$

For an equilibrium with a straight magnetic field, the Eulerian perturbation of total pressure P' is the quantity that couples the Alfvén waves to the magneto-sonic waves except for $m = 0$. For $m = 0$ the coupling function $C_A = 0$. Hence equation (11c) becomes decoupled from the remaining equations for $m = 0$,

$$\rho(\omega^2 - \omega_A^2)\xi_\varphi = 0. \tag{13}$$

This means that we have pure Alfvén waves for $m = 0$ in a non-uniform cylindrical plasma with a straight field. The axi-symmetric MHD waves are decoupled in torsional Alfvén waves and sausage magneto-sonic waves. For an axi-symmetric non-uniform 1-dimensional cylindrical plasma this is the only case where pure Alfvén waves show up in the analysis. Each magnetic surface oscillates with its own local Alfvén frequency. The Cartesian version of these torsional Alfvén waves were studied by Heyvaerts and Priest (1983) in the context of coronal heating by phase mixing of shear Alfvén waves.

It is instructive to consider the case of a uniform cylindrical plasma of infinite extent in order to assess the effect of non-uniformity. In the case of a uniform plasma ω_A^2 is a constant and we have a solution for any m

$$\omega^2 = \omega_A^2 \tag{14}$$

with

$$\xi_\varphi \neq 0, \qquad \xi_r \neq 0, \qquad \xi_z = 0, \qquad P' = 0, \qquad \nabla \cdot \boldsymbol{\xi} = 0. \tag{15}$$

The only constraint that the solutions (14)–(15) have to satisfy is $\nabla \cdot \boldsymbol{\xi} = 0$. This can be done in many ways. The only restoring force is the magnetic tension force

$$\mathbf{T} = -\rho\omega_A^2(\xi_r \mathbf{l}_r + \xi_\varphi \mathbf{l}_\varphi) = -\rho\omega_A^2 \boldsymbol{\xi}, \tag{16}$$

where $\mathbf{l}_r$ and $\mathbf{l}_\varphi$ are the unit vectors in the r and φ-direction. In a cylindrical plasma in an infinite and uniform environment there are pure Alfvén waves for any value of the azimuthal wave number m. It is because of non-uniformity that there are no pure Alfvén waves for $m \neq 0$ which have $P' = 0$ and $\nabla \cdot \boldsymbol{\xi} = 0$ everywhere.

Spruit (1982) and Goossens et al. (2009) compared the magnetic pressure force and the magnetic tension force to decide whether the MHD waves are predominantly fast or Alfvén. For a straight field the equation of motion can be written as

$$-\rho\omega^2 \boldsymbol{\xi}_h = \nabla_h P' - \mathbf{T},$$
$$\mathbf{T} = -\rho\omega_A^2 \boldsymbol{\xi}_h,$$
$$\rho\omega^2 \xi_z = i k_z \delta p',$$

where $\boldsymbol{\xi}_h = \xi_r \mathbf{1}_r + \xi_\varphi \mathbf{1}_\varphi$ is the displacement vector in horizontal planes and ∇_h is the gradient operator in horizontal planes perpendicular to the constant vertical magnetic field.

The component of the displacement parallel to the magnetic field ξ_z is solely driven by plasma pressure and unaffected by magnetic pressure force and the magnetic tension force. Goossens et al. (2009) denote the ratio of any of the two relevant components of the pressure force to the corresponding component of magnetic tension force by $\Gamma(\omega^2)$

$$\Gamma(\omega^2) = \frac{\omega^2 - \omega_A^2}{\omega_A^2}. \tag{17}$$

In a non-uniform plasma Γ depends on position, so that the nature of the MHD wave changes according to the properties of the plasma it travels through. In a uniform plasma Γ is a constant and the nature of the wave does not change as the MHD wave always sees the same environment. Note that in ideal MHD $\Gamma(\omega^2) = 0$ at positions where $\omega^2 = \omega_A^2$.

3 Driven Resonant Waves

The aim of this section is to point out that resonant MHD waves are a natural phenomenon in non-uniform plasmas. In view of the previous discussion on mixed properties the reader is warned that the waves under study, although they are called resonant Alfvén waves, are not pure Alfvén waves in the whole of the plasma. Actually they are not pure Alfvén waves anywhere but they are extremely good approximations to the idealized concept of a pure Alfvén wave on the so-called resonant surface.

3.1 Resonant Alfvén Waves in Ideal MHD

The aim of this section is to discuss resonant MHD waves in ideal MHD in a static plasma equilibrium. The effect of plasma bulk motion, i.e. background equilibrium flows, on resonant MHD waves are discussed in Sect. 4. The main result of this subsection is that resonant Alfvén waves have singular spatial solutions in ideal MHD. Equations (8) have regular singular points at the zeroes of the coefficient function D. As a consequence we have mobile regular singular points at the positions r where

$$\omega^2 = \omega_A^2(r), \quad \omega^2 = \omega_C^2(r). \tag{18}$$

The first equality in (18) defines the Alfvén resonance point, while the second equality defines the slow resonance point. Since $\omega_A^2(r)$ and $\omega_C^2(r)$ are functions of position, these two equations define two continuous ranges in the spectrum which are classically referred to as the Alfvén continuum and the slow continuum. The local Alfvén frequency $\omega_A(r)$ maps out the Alfvén continuous part of the linear spectrum

$$[\min \omega_A(r), \max \omega_A(r)], \tag{19}$$

and the local cusp frequency $\omega_C^2(r)$ maps out the slow continuum. In a non-uniform plasma with an Alfvén velocity $v_A(r)$ and an Alfvén frequency $\omega_A(r)$ that depend on position the Alfvén continuous part of the spectrum is a natural and unavoidable phenomenon. So are the resonant Alfvén waves that are associated with this Alfvén continuous part. Actually there are infinitely many Alfvén continuous parts since an Alfvén continuous part exists for every value of the azimuthal wave number $m \geq 0$. For a straight field the Alfvén continuous parts are degenerate with respect to m; for a twisted field they depend on m.

Let us now discuss the spatial solutions of the MHD wave with a frequency in the Alfvén continuum. This review follows the analysis by Sakurai et al. (1991a), Goossens et al. (1995), Goossens and Ruderman (1995), Tirry and Goossens (1996) and Erdélyi (1997). The starting point is to focus on a frequency in the Alfvén continuum and to determine the spatial behaviour of the corresponding perturbation close to the resonant point where the condition $\omega^2 = \omega_A^2(r_A)$ is satisfied. The analysis that follows below does not say anything about the spatial solutions of the MHD wave far away from the resonant point. Since we know that MHD waves have mixed properties, the MHD wave might resemble a fast MHD wave far away from the resonant point. For example in the application by Sakurai et al. (1991b), the MHD wave far away from the resonant point was a fast magneto-sonic wave. It is convenient to introduce the new radial variable s defined as

$$s = r - r_A. \tag{20}$$

Sakurai et al. (1991a) and Goossens et al. (1995) use series expansions of the coefficient functions around $s = 0$ to obtain simplified versions of the relevant differential equations. These simplified versions are valid in the interval $[-s_A, s_A]$ around the point of resonance where the linear Taylor polynomial is a valid approximation of $\omega^2 - \omega_A^2(r)$; hence s_A has to satisfy

$$s_A \ll \left| \frac{(\omega_A^2)'}{(\omega_A^2)''} \right|. \tag{21}$$

The simplified versions of (8) close to the Alfvén resonance point are:

$$s\Delta_A \frac{d\xi_r}{ds} = \frac{g_B}{\rho B^2} C_A(s), \tag{22a}$$

$$s\Delta_A \frac{dP'}{ds} = \frac{2 f_B B_\varphi B_z}{\mu_0 r_A \rho B^2} C_A(s), \tag{22b}$$

$$s\Delta_A \xi_\perp = i\frac{C_A}{\rho B}, \tag{22c}$$

where C_A, C_S, f_B and g_B are defined by (9e) and (10c). In (22) all equilibrium quantities are evaluated at $s = 0$ ($r = r_A$), and

$$\Delta_A = \frac{d}{dr}(\omega^2 - \omega_A^2)\Big|_{r=r_A}. \tag{23}$$

The right-hand members of (22a) and (22b) have the coupling function $C_A(s)$, which is a linear combination of ξ_r and P', as a common factor. This is a key point in the analysis of resonant Alfvén waves. The differential equation for the coupling function $C_A(s)$ is (see Sakurai et al. 1991a):

$$s\frac{dC_A(s)}{ds} = 0. \tag{24}$$

It follows from this equation that $dC_A/ds = \text{const} \times \delta(s)$, where $\delta(s)$ is the Dirac delta-function. Goedbloed (1983) has shown that, in a planar equilibrium with a straight magnetic field, the large solutions of ξ_r and P' (i.e. solutions containing logarithmic terms) have to be continuous. This implies that, in this case, $dC_A/ds = 0$, so that $C_A = \text{const}$ (note that, in the case considered by Goedbloed (1983), $C_A = m B_z P'/r_A$). To our knowledge there is no similar proof for a cylindrical equilibrium model with an asymmetric twisted magnetic field, so that we make the conjecture that $dC_A/ds = 0$ also in this case. Then it follows that

$$C_A(s) \equiv g_B P' - \frac{2 f_B B_\varphi B_z \xi_r}{\mu_0 r_A} = \text{const.} \tag{25}$$

Condition (25) is the fundamental conservation law at the Alfvén resonance point first obtained by Sakurai et al. (1991a). The fundamental conservation law (25) tells us that the coupling function $C_A(s)$ that determines the degree of coupling between fast waves and Alfvén waves is to a first approximation constant in the vicinity of the resonant point. Since $C_A(s)$ is constant, the solution to (22) is

$$\xi_r(s) = \frac{g_B}{\rho B^2 \Delta_A} C_A \ln|s| + \begin{cases} \xi_-, & s < 0, \\ \xi_+, & s > 0, \end{cases} \tag{26a}$$

$$P'(s) = \frac{2 f_B B_\varphi B_z}{\mu_0 r_A \rho B^2 \Delta_A} C_A \ln|s| + \begin{cases} P'_-, & s < 0, \\ P'_+, & s > 0. \end{cases} \tag{26b}$$

The jumps in ξ_r and P' are due to dissipative effects and will be dealt with in the following subsection. The component in the magnetic surfaces and perpendicular to the magnetic field lines, $\xi_\perp$, has a $1/s$-singularity and a $\delta(s)$ contribution which dominate the $\ln|s|$ singularity and the jump found for ξ_r and P'. The singularities in the solutions are due to the fact that the ideal MHD equations do not have any dissipation and assume zero Larmor radius. The dominant singularities in the solution reside in the components in the magnetic surfaces and perpendicular to the magnetic field lines (see also Goedbloed and Poedts 2004). The dominant dynamics of the wave is contained in $\xi_\perp$ and the wave is polarized in the magnetic surfaces perpendicular to the magnetic field lines. The nonuniform plasma supports an Alfvén wave that is confined to the magnetic surface where the dispersion relation for Alfvén waves in a uniform plasma is locally satisfied. The confinement of the Alfvén wave is not absolute. The Alfvén wave is linked to the outside world as it is coupled to a wave with components normal to the magnetic surfaces. From a dynamic point of view the wave can be regarded as an almost pure Alfvén wave confined to its resonant magnetic surface and polarized perpendicular to the magnetic field lines. From the point of view of the energetics ξ_r, the component normal to the magnetic surfaces, is essential since it is this quantity which provides the unidirectional transfer of energy to the resonant surface.

Let us now return to the conservation law at the Alfvén resonance point. In an equilibrium with a straight magnetic field ($B_\varphi = 0$), the conservation law (25) reduces to

$$[P'] = 0, \tag{27}$$

where the square brackets denote the jump of a quantity across the resonant surface,

$$[f] = \lim_{s \to +0} \{f(s) - f(-s)\}. \tag{28}$$

The solution (26) for ξ_r remains unchanged, while P' can be considered to be constant across the point of resonance. Also the coupling function C_A is identically zero for $m = 0$ so that pure torsional Alfvén waves exist with $\xi_r = 0$, $P' = 0$, $\xi_z = 0$ in agreement with (26).

In an equilibrium with a twisted magnetic field the conservation law (25) implies that the jumps in P' and ξ_r are related by

$$[P'] = \frac{2 f_B B_\varphi B_z}{g_B \mu_0 r_A} [\xi_r]. \tag{29}$$

The jump for P' is proportional to B_φ. In an equilibrium model with a twisted magnetic field, both total pressure P' and ξ_r undergo a jump even for $m = 0$. This ties in with the fact that, for a twisted magnetic field, the MHD waves always have mixed properties. For an equilibrium with a curved magnetic field Goossens et al. (1995) pointed out that the conservation law (25) expresses the balance between the total pressure gradient and the inward tension force generated by the displacement, $\xi_\perp$, of the equilibrium magnetic field. In an equilibrium with a curved magnetic field coupling of Alfvén waves and fast waves is due to both the Eulerian perturbation of total pressure P' and the tension force.

3.2 Resonant Slow Waves in Ideal MHD

In this subsection we address the second equality in (18). This equality defines the slow resonance point. In a non-uniform plasma the local cusp frequency $\omega_C(r)$ maps out the slow continuous part of the linear spectrum,

$$[\min \omega_C(r),\ \max \omega_C(r)]. \tag{30}$$

The analysis for determining the spatial solutions of an MHD wave with its frequency in the slow continuum is similar to that given for resonant Alfvén waves. Again we follow Sakurai et al. (1991a). The new radial variable s is now defined as

$$s = r - r_C. \tag{31}$$

with r_C the position of the slow resonance point where the condition $\omega^2 = \omega_C^2(r_C)$ is satisfied. The simplified versions of (7) close to the slow resonance point are:

$$s\Delta_C \frac{d\xi_r}{ds} = \frac{\mu_0 \omega_C^4}{B^2\omega_A^2} C_S(s), \tag{32a}$$

$$s\Delta_C \frac{dP'}{ds} = \frac{2\omega_C^4}{r_C B^2\omega_A^2} C_S(s), \tag{32b}$$

$$s\Delta_C \xi_\parallel = \frac{i\ f_B}{\rho B \Delta_C} \frac{v_S^2}{v_S^2 + v_A^2} C_S(s). \tag{32c}$$

In (32) all equilibrium quantities are evaluated at $s = 0$ ($r = r_C$), and

$$\Delta_C = \frac{d}{dr}(\omega^2 - \omega_C^2)\Big|_{r=r_C}. \tag{33}$$

A similar analysis as for resonant Alfvén waves leads to the fundamental conservation law for slow continuum modes that the coupling function $C_S(s)$ is a constant:

$$C_S(s) \equiv P' - \frac{2B_\varphi^2 \xi_r}{\mu_0 r} = \text{const.} \tag{34}$$

Since $C_S(s)$ is constant, the solution to (32) is:

$$\xi_r(s) = \frac{\mu_0 \omega_C^4}{B^2\omega_C^2} C_S \ln|s| + \begin{cases} \xi_-, & s < 0, \\ \xi_+, & s > 0, \end{cases} \tag{35a}$$

$$P'(s) = \frac{2\omega_C^4}{r_C B^2 \omega_A^2 \Delta_C} C_S \ln|s| + \begin{cases} P'_-, & s < 0, \\ P'_+, & s > 0. \end{cases} \tag{35b}$$

The jumps in ξ_r and P' are due to dissipative effects and will be dealt with in the following subsection. The component in the magnetic surfaces and parallel to the magnetic field lines, $\xi_\parallel$, has a $1/s$ singularity and a $\delta(s)$ contribution which dominate the $\ln|s|$ singularity and the jump found for ξ_r and P'. The singularities in the solutions are due to the fact that the ideal MHD equations do not have any dissipation. The dominant singularities in the solution reside in the components in the magnetic surfaces and parallel to the magnetic field lines. The dominant dynamics of the wave is contained in $\xi_\parallel$ and the wave is polarized in the magnetic surfaces parallel to the magnetic field lines. A nonuniform plasma supports a slow wave that is confined to the magnetic surface where the dispersion relation for the accumulation point of frequencies of slow waves in a uniform plasma is locally satisfied. The confinement of the slow wave is not absolute. The slow wave is linked to the outside world as it is coupled to a wave with components normal to the magnetic surfaces. From a dynamic point of view the wave can be regarded as an almost pure slow wave confined to its resonant magnetic surface and polarized parallel to the magnetic field lines. From the point of view of the energetics ξ_r, the component normal to the magnetic surfaces, is essential since it is this quantity which provides the unidirectional transfer of energy to the resonant surface.

3.3 Resonant Alfvén Waves in Resistive MHD

The aim of the present subsection is to review how the singular solutions for the resonant Alfvén waves found in ideal MHD are modified by dissipation. For this purpose it suffices to consider non-zero electrical resistivity since this removes the singularity present in the ideal equations. There is no interest in the equations that govern arbitrary linear displacements of a cylindrical plasma in resistive MHD. The focus is on the subclass of linear displacements that correspond to resonant Alfvén waves in ideal MHD. This restriction makes a significant simplification of the equations of resistive MHD possible. The effects of dissipation are generally small and only important in the vicinity of the ideal resonance called dissipative layer. In this dissipative layer the derivatives of the perturbed quantities with respect to r are much larger than those with respect to z and φ. In addition the derivatives of equilibrium quantities can be neglected in comparison with the derivatives of the perturbed quantities with respect to r. Therefore we retain in the dissipative terms only the r-derivatives of the perturbed quantities while we neglect those of the equilibrium quantities.

The dissipative counterparts of (7) were obtained by Sakurai et al. (1991a) (see also Goossens et al. 1995) for Alfvén resonance, while they were obtained by Erdélyi (1997) and Goossens and Ruderman (1995) for slow resonance, and by Erdélyi et al. (1995) and Erdélyi (1997) for both resonances in steady states. For the Alfvén resonance they read

$$D_\eta \frac{d(r\xi_r)}{dr} = C_1 r\xi_r - C_2 r P', \tag{36a}$$

$$D_\eta \frac{dP'}{dr} = C_3\xi_r - C_1 P', \tag{36b}$$

$$\left(\omega_\eta^2 - \omega_A^2\right)\xi_\perp = \frac{i}{\rho B} C_A, \tag{36c}$$

where D_η and ω_η^2 are differential operators defined as

$$D_\eta = \rho(v_S^2 + v_A^2)(\omega_\eta^2 - \omega_A^2)(\omega^2 - \omega_C^2), \quad \omega_\eta^2 = \omega^2\left(1 - \frac{i\eta}{\omega}\frac{d^2}{dr^2}\right). \tag{37}$$

Equations (36) are the equations that govern the resonant linear displacements of a cylindrical plasma in resistive MHD. Equations (36a) and (36b) for ξ_r and P' are a set of two differential equations of third order. They are formally the same as (7a) and (7b), but the coefficient function D is replaced by the differential operator D_η. The singularities are removed from the equations, but the order of the set of differential equations is raised from 2 (in ideal MHD) to 6 (in non-ideal MHD), and, in addition, the coefficient of the derivative of highest order is proportional to η. Equation (36c) is the resistive generalization of (7c). They are formally the same, but the ideal quantity ω^2 is now replaced by the second order differential operator ω_η^2. As a consequence the resistive equation for $\xi_\perp$ is a differential equation of second order for which $s = 0$ is not a singular point.

We now focus on how solutions to the set of dissipative MHD (36) are obtained in the vicinity of the critical point r_A defined by the condition $\omega_A^2(r_A) = \omega^2$. Series expansions of the coefficient functions around $s = 0$ are used to obtain simplified versions of the dissipative MHD differential equations. Just as in ideal MHD these simplified versions are valid in the interval $[-s_A, s_A]$ around the point of resonance where the linear Taylor polynomial is a valid approximation of $\omega^2 - \omega_A^2(r)$; hence s_A has to satisfy again the inequality (17). All the remaining equilibrium quantities are replaced by their values at $s = 0$. The simplified versions of (36) are

$$D_{\eta,s}^2 \frac{d\xi_r}{ds} = \frac{g_B}{\rho B^2} C_A(s), \tag{38a}$$

$$D_{\eta,s}^2 \frac{dP'}{ds} = \frac{2 f_B B_\varphi B_z}{\mu_0 r_A \rho B^2} C_A(s), \tag{38b}$$

$$D_{\eta,s}^2 \xi_\perp = i\frac{C_A(s)}{\rho B}, \tag{38c}$$

where $D_{\eta,s}^2$ is the second order differential operator defined by

$$D_{\eta,s}^2 = s\Delta_A - i\omega\eta\frac{d^2}{ds^2}. \tag{39}$$

Equations (38) are the resistive generalizations of the ideal equations (22). The ideal factor $s\Delta_A$ is now replaced by the second order differential operator $D_{\eta,s}^2 = s\Delta_A - i\omega\eta\, d^2/ds^2$. The ideal singularity at $s = 0$ is obviously absent from the resistive equations (38). As in (22) all equilibrium quantities are evaluated at $s = 0$ ($r = r_A$). The resistive counterpart of (24) is

$$D_{\eta,s}^2 \frac{dC_A(s)}{ds} = 0. \tag{40}$$

Here also the ideal singularity is absent.

Dissipation is important when the terms $s\Delta_A$ and $\omega\eta d^2/ds^2$ on the left hand sides of (38) are comparable. This results in a dissipative layer with a thickness measured by the quantity δ_A given by

$$\delta_A = \left(\frac{\omega\eta}{|\Delta_A|}\right)^{1/3}. \tag{41}$$

The thickness of the dissipative layer therefore scales as $(\eta/|d\omega_A/dr|)^{1/3}$, a result already obtained by Kappraff and Tataronis (1977) and Hollweg and Yang (1988), and numerically verified by Poedts et al. (1990a). In view of the very large values of the magnetic Reynolds number in the solar corona we have that

$$\frac{s_A}{\delta_A} \gg 1. \tag{42}$$

Inequality (42) is important as it implies that the interval of validity of the simplified versions of the dissipative MHD equations embraces the dissipative layer and in addition contains two overlap regions to the left and the right of the dissipative layer where ideal MHD is valid. A graphical schematic overview of the dissipative layer and the two overlap regions is shown in Fig. 1 of Stenuit et al. (1998). Goossens et al. (1995) introduced the scaled variable τ,

$$\tau = \frac{s}{\delta_A}, \tag{43}$$

which is of order of unity in the dissipative layer, but, in view of inequality (42), $s \to \pm s_A$ corresponds to $\tau \to \pm\infty$. With this new variable (38) and (40) can be rewritten as

$$D_\tau^2 \frac{d\xi_r}{d\tau} = i\frac{g_B}{\rho B^2 |\Delta_A|} C_A, \tag{44a}$$

$$D_\tau^2 \frac{dP'}{d\tau} = i\frac{2 f_B B_\varphi B_z}{\rho B^2 \mu_0 r_A |\Delta_A|} C_A, \tag{44b}$$

$$D_\tau^2 \frac{dC_A}{d\tau} = 0, \tag{44c}$$

$$D_\tau^2 \xi_\perp = \frac{-C_A}{\delta_A |\Delta_A| \rho B}, \tag{44d}$$

where the differential operator D_τ^2 is given by

$$D_\tau^2 = \frac{d^2}{d\tau^2} + i\,\mathrm{sign}(\Delta_A)\tau. \tag{45}$$

Equations (44a) and (44b) were first derived by Sakurai et al. (1991a). They did not obtain the differential equations for C_A and $\xi_\perp$ in dissipative MHD. Sakurai et al. (1991a) were primarily interested in the jump conditions for ξ_r and P'. The underlying motivation was that, once the jump conditions are known, it suffices to solve the ideal MHD equations. The dissipative MHD equations can be circumvented since the solutions to the ideal MHD equations to the left and the right of the dissipative layer can be connected by the use of the jump conditions. So, unless there is an interest in the solutions in the dissipative layer itself, there is no need to solve the dissipative MHD equations. So Sakurai et al. (1991a) focused on the dissipative equations for ξ_r and P' in an attempt to find the jumps in these quantities across the dissipative layer. They assumed that the ideal conservation law (25) remains valid in dissipative MHD. Goossens et al. (1995) showed that this is indeed the case for Alfvén and Erdélyi (1997) for slow resonance. C_A being constant implies that the right-hand sides of the differential equations for ξ_r and P' are constants. Sakurai et al. (1991a) then obtained solutions for $d\xi_r/d\tau$ and $dP'/d\tau$ in terms of integrals of Hankel functions of order 1/3 of a complex argument. Integration of these expressions combined with the required asymptotic expansions enabled them to obtain the jumps in ξ_r, P' and also in $\xi_\parallel$. A related approach was used by Hollweg (1987a, 1987b) and Hollweg and Yang (1988) in a discussion of the viscous dissipative layer for planar geometry. In essence, their approach was to take P' to be constant across the whole inhomogeneous layer at the outset. Although Hollweg (1987a, 1987b) and Hollweg and Yang (1988) did not assume that there is no jump of the total pressure perturbation across the dissipative layer, the conjecture that they made implicitly implies this. This was equivalent to neglecting the inertia in the inhomogeneous layer and, consequently, in the dissipative layer. Since $C_A = g_B P'$ for planar geometry, they thus started with the constancy of C_A, and then derived an equation equivalent to (44d), which was solved in terms of Airy functions with imaginary argument.

Goossens et al. (1995) used a more compact and straightforward method for obtaining the solutions to (44). First they showed that C_A is a constant in dissipative MHD for $|s| \ll s_A$ which proves the conjecture used to derive (25). This proves the assumption by Sakurai et al. (1991a) and guarantees that the jump conditions found by Sakurai et al. (1991a) are correct. Then they obtained compact analytical solutions for ξ_r, P', and $\xi_\perp$ in the dissipative and the overlap regions which allow a very simple mathematical and physical interpretation. Goossens et al. (1995) started from the following two differential equations of second order:

$$D_\tau^2 \Psi(\tau) = 0, \qquad D_\tau^2 F(\tau) = -1 \tag{46}$$

and showed that the bounded solutions for $\Psi(\tau)$ and $F(\tau)$ are

$$\Psi(\tau) = 0, \tag{47}$$

and

$$F(\tau) = \int_0^\infty \exp(iu\tau\,\mathrm{sign}(\Delta_A) - u^3/3)\,du, \tag{48}$$

respectively. Equation (47) implies that

$$\frac{dC_A(\tau)}{d\tau} = 0,$$

so that

$$C_A(\tau) = \text{const.} \tag{49}$$

The ideal conservation law (25) obtained by Sakurai et al. (1991a) continues to hold in dissipative MHD. The coupling function C_A that determines the degree of coupling between Alfvén waves and fast magneto-sonic waves is constant across the dissipative layer. Solution (48) implies that

$$\frac{d\xi_r}{d\tau} = -i\frac{g_B}{\rho B^2|\Delta_A|} C_A F(\tau), \tag{50a}$$

$$\frac{dP'}{d\tau} = -\frac{2i f_B B_\varphi B_z}{\mu_0 r_A \rho B^2 |\Delta_A|} C_A F(\tau), \tag{50b}$$

and

$$\xi_\perp = \frac{C_A F(\tau)}{\rho B \delta_A |\Delta_A|}. \tag{51}$$

Integration of (50) gives the solutions in dissipative MHD for ξ_r and P' that remain finite as $|\tau| \to \infty$,

$$\xi_r(\tau) = -\frac{g_B C_A}{\rho B^2 \Delta_A} G(\tau) + C_\xi, \tag{52a}$$

$$P'(\tau) = -\frac{2 f_B B_\varphi B_z C_A}{\rho B^2 \mu_0 r_A \Delta_A} G(\tau) + C_P, \tag{52b}$$

where C_ξ and C_P are constants of integration and

$$G(\tau) = \int_0^\infty \frac{e^{-u^3/3}}{u} \{\exp(iu\tau\,\mathrm{sign}(\Delta_A)) - 1\}\, du. \tag{53}$$

It is worth noting that, to our knowledge, the function $G(\tau)$ (53) was first introduced by Boris (1968). Goossens et al. (1995) followed the analysis by Boris (1968).

From the definition of C_A (see (9e)), it follows that the constants C_ξ and C_P cannot be independently chosen, but are related by $C_A = g_B C_P - 2 f_B B_\varphi B_z C_\xi / \mu_0 r_A$.

The two functions $F(\tau)$ (48) and $G(\tau)$ (53) describe the dynamics of the resonant Alfvén wave in the dissipative layer and in the two overlap regions. They are in a sense universal as they determine the resonant Alfvén waves in both cylindrical and planar geometries for both static and stationary equilibrium states. See, e.g., Tirry and Goossens (1996), Tirry et al. (1998a) and Pintér et al. (2007) for planar equilibrium states; Taroyan and Erdélyi (2002, 2003a, 2003b) and Erdélyi and Taroyan (2003) for planar steady case; Erdélyi et al. (1995) and Erdélyi (1997) for cylindrical static and steady equilibrium states; and Tirry and Goossens (1995) for 2-D equilibrium. Wright and Allan (1995) derived the counterparts of the universal functions $F(\tau)$ and $G(\tau)$ for a Cartesian geometry when the energy is dissipated by finite Pedersen conductivity within ionospheric boundaries.

The real and imaginary parts of the functions $F(\tau)$ and $G(\tau)$ are plotted in Fig. 1. The remnants of the ideal delta function and the ideal $1/s$ function can be clearly recognized in the real and imaginary parts of $F(\tau)$. The same applies for the remnants of the ideal logarithmic function and the Heaviside function in respectively the real and imaginary parts of $G(\tau)$. Goossens et al. (1995) show that straightforward Maclaurin expansions give absolute convergent power series for all τ for $G(\tau)$ and $F(\tau)$. These power series can be used to obtain solutions for ξ_r, P', and $\xi_\perp$. In particular, for $|\tau| \lesssim 1$, they show that, in resistive MHD, all the physical quantities take finite values in the dissipative layer and at the ideal resonance position where the ideal MHD solutions diverge. It is easy to see from (48) and (53) that $\Re(F(\tau))$ and $\Re(G(\tau))$ are even functions of τ, while $\Im(F(\tau))$ and $\Im(G(\tau))$ are odd functions of τ, where $\Re$ and $\Im$ indicate the real and imaginary parts of a quantity. A draw-back of these power series given is that they are not well suited to find out how the resistive MHD solutions have to be connected to the ideal MHD solutions. The asymptotic behaviour of the resistive MHD solutions for $\tau \to \pm\infty$ was determined by Goossens et al. (1995). In the overlap regions the asymptotic versions of the dissipative solutions and the ideal solutions (26) represent the same solutions. The asymptotic versions recover the logarithmic behaviour of $\Re(\xi_r)$ and $\Re(P')$ already found in ideal MHD in (26), but show that this logarithmic behaviour is only valid away from the ideal resonance position for large $|\tau|$. Comparison of the asymptotic versions of the dissipative MHD solutions with the ideal MHD solutions enabled Goossens et al. (1995) to obtain expressions for $\xi_\pm$ and $P'_\pm$ in (26). With these expressions the jumps in ξ_r, P' and $\xi_\parallel$ can be computed. Alternatively, Appendix B of Goossens et al. (1995) tells us that $[G] = i\pi$. For readers who are more

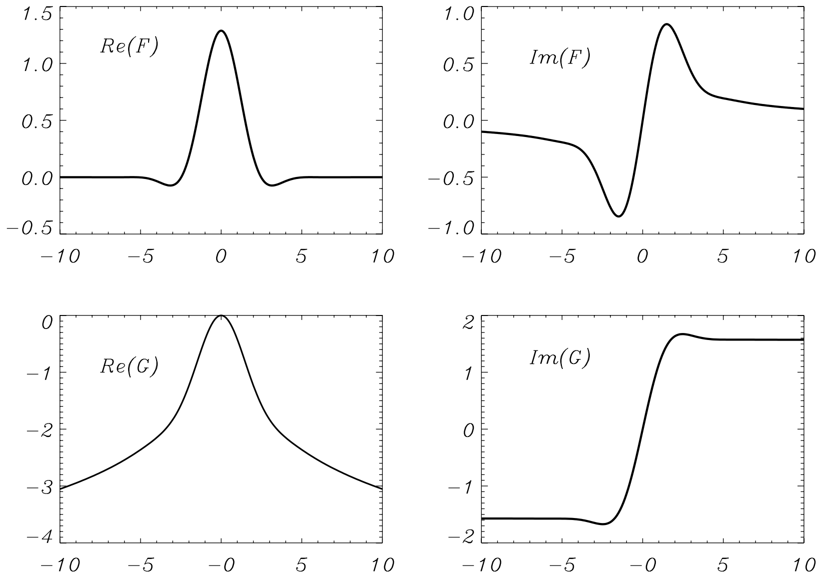

Fig. 1 The real and imaginary parts of functions $F(\tau)$ and $G(\tau)$. We can clearly see that $\Re(F)$ and $\Re(G)$ are even functions, while $\Im(F)$ and $\Im(G)$ are odd functions

graphically inclined they can observe on Fig. 1 that $[G] = i\pi$ is indeed correct. Hence the jumps in ξ_r and P' are

$$[\xi_r] = -i\pi \frac{g_B C_A}{\rho B^2 |\Delta_A|}, \tag{54a}$$

$$[P'] = -i\pi \frac{2 f_B B_\varphi B_z C_A}{\rho B^2 \mu_0 r_A |\Delta_A|}, \tag{54b}$$

$$[\xi_\parallel] = \pi \frac{2k_z f_B B_\varphi}{\rho B \mu_0 r_A |\Delta_A| \omega_A^2} \frac{v_S^2}{v_A^2} C_A. \tag{54c}$$

These jumps and the conservation law were first derived by Sakurai et al. (1991a) for the driven problem and by Tirry and Goossens (1996) for the eigenvalue problem (see Sect. 4 on eigenmodes). The jumps in P' and $\xi_\parallel$ are both proportional to B_φ. In an equilibrium with a straight magnetic field lines ($B_\varphi = 0$), both P' and $\xi_\parallel$ are constant across the dissipative layer. Also when $B_\varphi = 0$, ξ_r does not jump across the dissipative layer for waves with azimuthal wave number $m = 0$. This is in accordance with the fact that, for a straight field, axi-symmetric MHD waves with $m = 0$ are uncoupled and that the torsional Alfvén waves have $\xi_r = 0$. However, in a twisted field ξ_r does jump for waves with azimuthal wave number $m = 0$, and the jump in ξ_r is proportional to both B_φ and the longitudinal wave number k_z. An important property of resonant Alfvén wave heating is that the jumps are independent of η. This implies that the amount of absorbed wave energy and the total amount of resistive heating in the dissipative layer are also independent of η. For a straight field ($B_\varphi = 0$) the jumps are

$$[P'] = 0, \quad [\xi_r] = -i\pi \;\mathrm{sign}(\omega) \frac{m^2/r^2}{\rho |\Delta_A|} P', \quad [FluxE] = -\frac{\pi |\omega| m^2}{2\rho |\Delta_A|} |P'|^2, \tag{55}$$

where $[FluxE]$ is the jump of the energy flux when crossing the dissipative layer. In a static equilibrium this jump is always negative, meaning that the dissipative layer is a sink for the energy of the wave, so that the wave gets damped. In equilibrium models with flow the dissipative layer can be a source and quasi-modes can become over-stable as shown by, e.g., Hollweg et al. (1990), Yang and Hollweg (1991), Erdélyi and Goossens (1996), Ruderman and Wright (1998), Andries et al. (2000), Andries and Goossens (2001a, 2001b) (see also a review paper by Taroyan and Ruderman 2010). In an equilibrium with a straight magnetic field the Eulerian perturbation of total pressure does not undergo a jump when crossing the dissipative layer. $[FluxE]$ is proportional to $|P'|^2$ so that large absolute values of P' imply strong absorption. Since P' is the function that couples Alfvén waves to fast waves so larger absolute values of P' mean strong coupling.

In addition, for an equilibrium with a straight magnetic field, ξ_r and $FluxE$ do not jump, $[\xi_r] = 0$ and $[FluxE] = 0$ for waves with $m = 0$. Hence waves with $m = 0$ are not resonantly absorbed in an equilibrium with a straight field. This comes as no surprise since torsional Alfvén waves and sausage fast waves are not coupled in an equilibrium with a straight field. Here is a good point to go back to the 1980s. The result $[P'] = 0$ (55) means that the assumption used by Hollweg and Yang (1988) is correct. The approximate constancy of P' was the key ingredient in the physical discussions of resonance absorption given by Hollweg (1987a, 1987b, 1988), and Hollweg and Yang (1988). Equation (27) puts the approach by Hollweg and Yang on a rigorous mathematical footing, and extends it to cylindrical geometry.

Goossens et al. (1995) determined the asymptotic behaviour of $F(\tau)$ for $\tau \to \pm\infty$ and used it to obtain the asymptotic expansion for $\xi_\perp$,

$$\xi_\perp \simeq \frac{iC_A}{\tau\rho B\delta_A\Delta_A}. \tag{56}$$

This asymptotic version recovers the $1/\tau$ behaviour of $\Im(\xi_\perp)$, already found in ideal MHD in (25), but it shows that this behaviour is only valid away from the ideal resonance position for $\tau \to \pm\infty$. In order to understand fully the relation between the ideal and resistive solution for $\xi_\perp$ it is instructive to determine what has happened to the ideal $\delta(s)$ contribution to $\xi_\perp$. The limit of the resistive solution of $\xi_\perp$ for $\delta_A \to 0$, as a function of s is

$$\lim_{\delta_A\to 0}\xi_\perp = \frac{C_A}{\rho B}\left[\frac{\pi}{|\Delta_A|}\delta(s) + \frac{i}{\Delta_A}\mathcal{P}\left(\frac{1}{s}\right)\right], \tag{57}$$

where $\mathcal{P}$ denotes the principal Cauchy value. Hence the $\delta(s)$ contribution to $\xi_\perp$can be thought of as arising from $\Re(F(\tau))$, which is an even function of τ. The amplitude of $\Re(\xi_\perp)$ at $s = 0$ is proportional to $1/\delta_A$, while $\Re(\xi_\perp)$ becomes small when $|s| \gg \delta_A$. The area under $\Re(\xi_\perp(s))$ is thus independent of δ_A, leading to the δ-function as $\delta_A \to 0$.

In ideal MHD the dominant dynamics of resonant Alfvén waves resides in the perpendicular component of the displacement $\xi_\perp$. The logarithmic singularity and the jump contribution to the radial component of the displacement ξ_r are overruled by the s^{-1} singularity and the δ-function contribution to $\xi_\perp$. In resistive MHD all these singularities disappear and all physical variables take finite values. Goossens et al. (1995) showed that, in the dissipative layer,

$$\frac{|\xi_\perp|}{|\xi_r|} \sim R_m^{1/3}, \tag{58}$$

where $R_m = \omega L^2/\eta$, is the magnetic Reynolds number. Since R_m is very large, of the order of $10^{10} - 10^{12}$ in the solar atmosphere, (58) implies that the dominant dynamics continues to reside in the perpendicular components as was found by Poedts (1989a, 1989b, 1990a).

These analytical results provide us with the spatial solutions in the dissipative layer where dissipative MHD is required and in the two overlap regions where ideal MHD is valid. They enable us to understand the basic physics of resonant Alfvén waves and help us with the interpretation of the results of large-scale numerical simulations. In addition, the jump conditions and the conservation law provide us with a strong computational tool. They make it possible to compute the absorption of driven Alfvén waves and the damping rates of eigenmodes without having to solve the dissipative MHD equations. The solutions to the ideal MHD equations are connected over the dissipative layer by the use of the jump conditions. This method was used by Sakurai et al. (1991b) for studying the absorption of acoustic oscillations in sunspots. It was generalized to stationary equilibrium states by Goossens et al. (1992). Goossens and Hollweg (1993) used this scheme to obtain conditions for maximal and total absorption and to explain the variation of the spatial solutions with frequency. Keppens et al. (1994) used it to study the absorption in fibril models of sunspots. Stenuit et al. (1995) verified the accuracy of the so-called SGHR method by comparing results obtained with this method with known results in the literature obtained by integration of the full set of linearized non-ideal MHD equations. The agreement was remarkably good with the largest relative difference being smaller than 1.2%. A schematic overview of the various regions involved in this method is shown in Fig. 1 in Stenuit et al. (1998). Here SGHR stands for Sakurai, Goossens, Hollweg and Ruderman. They are the four authors who derived the jump conditions and connection formulae which make it possible to set up a scheme to obtain solutions for resonant MHD waves without having to solve the full system of dissipative MHD equations.

3.4 Resonant Slow Waves in Resistive MHD

The aim of the present section is to briefly recall how the singular solutions for the resonant slow waves found in ideal MHD are modified by dissipation. The analysis is similar to that given for resonant Alfvén waves. Dissipation is important in a dissipative layer with a thickness that is now measured by the quantity δ_C given by

$$\delta_C = \left(\frac{\omega \eta \omega_C^2}{|\Delta_C| \omega_A^2} \right)^{1/3}. \tag{59}$$

Here also it is assumed that

$$\frac{s_C}{\delta_C} \gg 1, \tag{60}$$

so that the interval $[-s_C,\ s_C]$ where the simplified versions of the dissipative MHD equations are valid embraces the dissipative layer and, in addition, contains two overlap regions to the left and the right of the dissipative layer where ideal MHD is valid. The scaled variable τ is now defined as

$$\tau = \frac{s}{\delta_C}, \tag{61}$$

which is of order 1 in the dissipative layer but, in view of inequality (60), $s \to \pm s_A$ corresponds to $\tau \to \pm\infty$. The outcome of the analysis is that (i)

$$C_S(\tau) = \text{const}, \tag{62}$$

so that the ideal conservation law (34) continues to hold in dissipative MHD, and that (ii) $\xi_\parallel(\tau)$ can be written with the use of the function $F(\tau)$, while (iii) $\xi_r(\tau)$ and $P'(\tau)$ can be written with the use of the function $G(\tau)$.

The jumps for the resonant slow waves are

$$[\xi_r] = -i\pi \frac{\mu_0 \omega_C^4}{B^2 |\Delta_C| \omega_A^2} C_S, \tag{63a}$$

$$[P'] = -i\pi \frac{2\omega_C^4 B_\varphi^2}{B^2 r_C \omega_A^2 |\Delta_C|} C_S, \tag{63b}$$

$$[\xi_\perp] = \pi \frac{2k_z B_\varphi \omega_C^4}{\rho r_C \omega_A^2 B |\Delta_C| (\omega_C^2 - \omega_A^2)} C_S. \tag{63c}$$

These jumps and the conservation law were first derived by Sakurai et al. (1991a). The jumps in P' and $\xi_\perp$ are both proportional to B_φ. In an equilibrium with a straight magnetic field lines ($B_\varphi = 0$), both P' and $\xi_\perp$ are constant across the dissipative layer. However, when $B_\varphi = 0$, there is a non-zero jump in ξ_r:

$$[\xi_r] = \frac{-i\pi}{\rho |\Delta_C|} \left(\frac{v_S^2}{v_S^2 + v_A^2} \right)^2 k_z^2 P'. \tag{64}$$

The jump in ξ_r is independent of the azimuthal wave number m and, for given P', proportional to k_z^2. Note that $[\xi_r]$ attains its maximal value for an incompressible plasma since then the factor containing v_S^2 is maximal and equal to 1. Equation (64) implies that axisymmetric ($m = 0$) waves are resonantly absorbed at the slow resonance point in an equilibrium with a straight field. This is in contrast with the corresponding result for Alfvén waves.

4 Resonant MHD Waves in Steady Equilibrium

In this section we review studies on resonant MHD waves in equilibrium states with flows. Equilibrium bulk motions not only influence the efficiency of resonant coupling of local magnetic structures (e.g. in flux tubes ubiquitous in the solar atmosphere or flux sheets, a popular model in solar prominence and arcade studies) and large-scale oscillations driving magnetic structures (e.g. in the magnetosheath or in magnetotail) and/or alter wave heating, but may also result in a resonant wave to become unstable. These latter, so-called resonant flow instabilities (RFI), that are physically distinct from the non-resonant Kelvin-Helmholz instabilities, can occur for velocity shears significantly below the KH threshold (see Andries et al. 2000; Andries and Goossens 2001a, 2001b and Taroyan and Ruderman 2010). Resonant MHD waves in steady state may also contribute to the explanation of the well-known frequency shifts of global solar p/f eigenmode oscillations (see Pintér and Erdélyi 2010). Aspects of these studies were earlier addressed in less comprehensive review papers by, e.g., Erdélyi (2001, 2002, 2006a, 2006b).

A key observation of the highly inhomogeneous solar atmosphere is the presence of *steady flows*. Bulk motions are observed along or nearly along the magnetic field lines which outline the magnetic structures (e.g. Doschek et al. 1976; Brekke et al. 1997; Doyle et al. 1997, 2002; Warren et al. 1997; Bellot Rubio et al. 2003; Chae et al. 2008). The question arises naturally: would an equilibrium flow influence the efficiency of the resonant coupling (i.e. absorption rate)?

First we briefly overview the mathematical consequences of a steady state, i.e. what are the new governing equations and how the connection formulae are modified (Sect. 4.1). In

Sect. 4.2 we review studies of how background bulk motions influence resonant heating in coronal structures (i.e. loops, etc.).

4.1 Theoretical Aspects

In this section we consider resonant slow and Alfvén waves in $1D$ non-uniform magnetic flux tubes in *dissipative* MHD with a *field-aligned steady* bulk motion. Analytical solutions for the Lagrangian displacement and the Eulerian perturbation of the total pressure for this stationary equilibrium state will be obtained. From these analytical solutions we obtain the fundamental conservation law and the jump conditions for resonant MHD waves in dissipative steady MHD and point out the main differences between these steady connection formulae and their static counterparts. The fundamental conservation law and the jump conditions depend on the equilibrium flow in a more complicated way than just a Doppler shift. The effects of an equilibrium flow cannot be predicted easily in general terms with the exception that the polarization of the driven resonant waves remain, i.e. the slow resonant waves remain polarized mainly parallel to the magnetic field lines, while the polarization of the driven Alfvén waves is still in magnetic surfaces and perpendicular to the magnetic field lines. Most importantly, the validity of the ideal conservation law and jump conditions obtained by Sakurai et al. (1991a) for static equilibria and Goossens et al. (1992) for steady equilibria in ideal MHD is justified in steady dissipative MHD.

The steady equilibrium state of the flux tube is again idealized as a cylindrically symmetric plasma column as outlined in Sect. 2, where the equilibrium flow, $\mathbf{v}(r)$, has only φ and z components, i.e. the background steady state is characterized by a field-aligned steady motion. The radial force balance equation is given by (2).

The equations for the linear perturbations about a steady equilibrium are now the full (3a)–(3d). In a steady state the Eulerian velocity perturbation $\mathbf{v}'$ and the Lagrangian displacement ξ are related by (see e.g. Goossens et al. 1992)

$$\mathbf{v}' = \frac{\partial \boldsymbol{\xi}}{\partial t} + \nabla \times (\boldsymbol{\xi} \times \mathbf{v}) - \boldsymbol{\xi} \nabla \cdot \mathbf{v} + \mathbf{v} \nabla \cdot \boldsymbol{\xi} \equiv \frac{\partial \boldsymbol{\xi}}{\partial t} + (\mathbf{v} \cdot \nabla)\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla)\mathbf{v}. \tag{65}$$

For the asymptotic state defined by (1) and (4) the components of the Eulerian velocity perturbation are related to the components of the Lagrangian displacement as follows:

$$v_r' = -i\Omega\xi_r, \tag{66a}$$

$$v_\varphi' = -i\Omega\xi_\varphi - r\xi_r \frac{d}{dr}\left(\frac{v_\varphi}{r}\right), \tag{66b}$$

$$v_z' = -i\Omega\xi_z - \xi_r \frac{dv_z}{dr}. \tag{66c}$$

Here we introduced the Doppler-shifted frequency

$$\Omega = \omega - \omega_f, \tag{67}$$

where ω_f is the flow frequency defined as

$$\omega_f = \frac{m}{r} v_\varphi + k_z v_z. \tag{68}$$

Remarkably, the ideal ($\eta = 0$) MHD equations for linear motions in a steady state, (3), can be reduced to two linear first-order differential equations that look *formally* identical to their counterparts (7a)–(7b) governing the linear motions superimposed on a static equilibrium state. However, now the coefficients D, C_1, C_2, and C_3 depend on the equilibrium flow as well. For a *steady* equilibrium state

$$D = \rho(v_S^2 + v_A^2)(\Omega^2 - \omega_A^2)(\Omega^2 - \omega_C^2), \tag{69a}$$

$$C_1 = Q\Omega^2 - 2m(v_S^2 + v_A^2)(\Omega^2 - \omega_C^2)T/r^2, \tag{69b}$$

$$C_2 = \Omega^4 - (v_S^2 + v_A^2)(\Omega^2 - \omega_C^2)\left(\frac{m^2}{r^2} + k_z^2\right), \tag{69c}$$

$$\begin{aligned} C_3 = D\left\{\rho(\Omega^2 - \omega_A^2) + r\frac{d}{dr}\left[\frac{1}{\mu_0}\left(\frac{B_\varphi}{r}\right)^2 - \rho\left(\frac{v_\varphi}{r}\right)^2\right]\right\} \\ + Q^2 - 4(v_S^2 + v_A^2)(\Omega^2 - \omega_C^2)\frac{T^2}{r^2}, \end{aligned} \tag{69d}$$

$$T = \rho\Omega v_\varphi + \frac{f_B B_\varphi}{\mu_0}, \tag{69e}$$

$$Q = -(\Omega^2 - \omega_A^2)\frac{\rho v_\varphi^2}{r} + \frac{2\Omega^2 B_\varphi^2}{\mu_0 r} + \frac{2\Omega f_B B_\varphi v_\varphi}{\mu_0 r}. \tag{69f}$$

The governing equations for steady state were first obtained by Bondeson et al. (1987) and later derived with coefficients D, C_1, C_2, and C_3 in this form by Goossens et al. (1992), Erdélyi et al. (1995), and Erdélyi (1996).

Since the Alfvén frequency ω_A, the cusp frequency ω_C, and the Doppler-shifted frequency $\Omega = \omega - \omega_f$ all depend on the radial coordinate r, (7a)–(7b) through the coefficients D, C_1, C_2, and C_3 defined by (69) have mobile regular singularities at $r = r_A$ and $r = r_C$ defined by

$$\Omega^2(r_A) = \omega_A^2(r_A), \qquad \Omega^2(r_C) = \omega_C^2(r_C) \tag{70}$$

for stationary equilibria. These equations determine the Alfvén and slow resonance points, respectively, in a steady equilibrium.

Let us now express the steady counterparts of the parallel, $\xi_\parallel$, and perpendicular, $\xi_\perp$, components of the Lagrangian displacement vector $\boldsymbol{\xi}$ defined by (5). For a steady equilibrium the component of the Lagrangian displacement parallel and perpendicular to the equilibrium magnetic field lines in the magnetic surfaces can be expressed in terms of ξ_r and P' as

$$\begin{aligned} (\Omega^2 - \omega_C^2)\xi_\parallel = \frac{i f_B}{\rho B}\frac{v_S^2}{v_S^2 + v_A^2}\left(P' - \frac{Q\xi_r}{\Omega^2}\right) \\ - i(\Omega^2 - \omega_C^2)\frac{2\Omega B_\varphi v_\varphi + f_B v_\varphi^2}{B\Omega^2 r}, \end{aligned} \tag{71a}$$

$$(\Omega^2 - \omega_A^2)\xi_\perp = \frac{i}{\rho B}\left(g_B P' - \frac{2B_z T\xi_r}{r}\right). \tag{71b}$$

Now, following the mathematical method outlined in Sect. 3.1 for Alfvén waves and by Erdélyi (1997) for slow waves we can determine the spatial solutions close to the Alfvén and slow singular point where $\Omega^2 = \omega_A^2(r_A)$ and $\Omega^2 = \omega_C^2(r_C)$, respectively, are satisfied in ideal steady MHD.

4.1.1 Alfvén Resonance in Ideal Steady State

The simplified versions of (7a)–(7b) close to the Alfvén resonant point are

$$s\Delta_A \frac{d\xi_r}{ds} = \frac{g_B}{\rho B^2} C_A(s), \tag{72a}$$

$$s\Delta_A \frac{dP'}{ds} = \frac{2 f_B T}{\mu_0 r_A \rho B^2} C_A(s), \tag{72b}$$

where the Alfvén coupling function $C_A(s)$ takes now the form

$$C_A(s) = g_B P' - \frac{2B_z T\xi_r}{r_A}, \tag{73}$$

and all equilibrium quantities in (72a) and (72b) are evaluated at the Alfvén resonant point $s = 0$ $(r = r_A)$. Note that now

$$\Delta_A = \frac{d}{dr}(\Omega^2 - \omega_A^2)\Big|_{r=r_A}, \tag{74}$$

i.e., care has to be exercised when evaluating Δ_A as now it also depends on the equilibrium flow. The spatial solutions for ξ_r, P' and $\xi_\perp$ close to the Alfvén resonant point (i.e. to r_A) in a steady equilibrium take now the form

$$\xi_r(s) = \frac{g_B}{\rho B^2 \Delta_A} C_A \ln|s| + \begin{cases} \xi_-, & s < 0, \\ \xi_+, & s > 0, \end{cases} \tag{75a}$$

$$P'(s) = \frac{2B_z T}{r_A \rho B^2 \Delta_A} C_A \ln|s| + \begin{cases} P'_-, & s < 0 \\ P'_+, & s > 0, \end{cases} \tag{75b}$$

$$s\xi_\perp = i\frac{C_A}{\rho B \Delta_A}, \tag{75c}$$

where the Alfvén coupling function for steady state becomes

$$C_A = g_B P' - \frac{2B_z T\xi_r}{r_A} \equiv \text{const.} \tag{76}$$

Condition (76) is the fundamental conservation law at the Alfvén resonant point in a steady equilibrium. Condition (76) was obtained by Goossens et al. (1992) in ideal MHD and its universal validity is shown to remain in dissipative MHD by Erdélyi et al. (1995). The jumps in ξ_r and P' are due to dissipative (resistive and/or viscous) effects and will be specified further down. These results indicate that $\xi_\perp$ dominates the solutions even in steady state, so the solutions remain polarized perpendicular to the magnetic field lines. Although the equilibrium flow modifies the spatial dependence, it does not affect the polarization properties of the spatial solutions, i.e. the $1/s$ singularity and the $\delta(s)$ contribution found already for $\xi_\perp$ in static MHD will continue to dominate the $\ln|s|$ singularity of ξ_r and P'.

4.1.2 Slow Resonance in Ideal Steady State

Let us know summarize in a concise manner similar to Sect. 4.1.2 the main results for slow resonant waves in ideal steady MHD. The spatial solutions for ξ_r and P' close to the slow resonant point (i.e. to r_C) in a stationary equilibrium take the form

$$\xi_r(s) = \frac{\mu_0 \omega_C^4}{B^2 \omega_A^2 \Delta_C} C_S \ln|s| + \begin{cases} \xi_-, & s < 0, \\ \xi_+, & s > 0, \end{cases} \tag{77a}$$

$$P'(s) = \frac{\mu_0 Q_C}{B^2 \omega_A^2 \Delta_C} C_S \ln|s| + \begin{cases} P'_-, & s < 0, \\ P'_+, & s > 0. \end{cases} \tag{77b}$$

The parallel component of the displacement is given by

$$s\xi_\parallel = \frac{i f_B v_S^2}{\rho B \Delta_C \omega_C^2 (v_S^2 + v_A^2)} C_S + i s \xi_r \frac{2\Omega B_\varphi v_\varphi + f_B v_\varphi^2}{B \omega_C^2 r_C}. \tag{78}$$

Now the coupling function for the steady slow resonance is given by

$$C_S = \omega_C^2 P' - Q_C \xi_r \equiv \text{const} \tag{79}$$

with

$$Q_C = \frac{2\omega_C B_\varphi}{\mu_0 r}(\omega_C B_\varphi + f_B v_\varphi) + \frac{\rho v_A^2 \omega_C^2 v_\varphi^2}{r^2 v_S^2}, \tag{80}$$

and

$$\Delta_C = \frac{d}{dr}(\Omega^2 - \omega_C^2)\Big|_{r=r_C}. \tag{81}$$

The above solutions given by (77) were obtained by using the radial variable s defined by (31), and all equilibrium quantities have to be evaluated at the slow resonant point r_C. Condition (79) is the fundamental conservation law at the slow resonance point in ideal steady MHD. Condition (79) is obtained by Goossens et al. (1992) in ideal MHD and its universal validity is shown to remain in dissipative MHD by Erdélyi (1997). The jumps in ξ_r and P' are due to dissipative (resistive) effects and will be specified later. These results also imply that $\xi_\parallel$ remains to dominate the solutions in steady state, so the solutions are polarized parallel to the magnetic field lines even if background flows are present in the equilibrium. Again, although the steady state changes the actual functional dependence of the spatial solutions, the equilibrium flow does not affect the polarization properties of these spatial solutions.

4.1.3 Resonant MHD Waves in Dissipative Inhomogeneous Steady Plasmas

In this section we review how the singular solutions for resonant slow and Alfvén waves found in ideal static (see Sect. 3.3) and steady (see Sect. 4.1) MHD are modified by taking into account both the effect of dissipation and background bulk motions. The finite electrical resistivity η (or viscosity if the application requires, see Erdélyi and Goossens 1995) removes the singularity in the ideal steady equations. The perturbed linear resistive MHD equations (see (32–39) in Erdélyi et al. 1995, for their rather elaborate form) in steady MHD reduce by eliminating all but two of the perturbed variables (namely ξ_r and P') to the system of (36a) and (36b) but with C_1, C_2 and C_3 defined by (69b)–(69d). The differential operator D_η in these equations is equal to $D_{\eta,A}$ given by

$$D_{\eta,A} = \rho(v_S^2 + v_A^2)(\Omega^2 - \omega_C^2)(\Omega_\eta^2 - \omega_A^2) \tag{82}$$

for resonant Alfvén waves, and to $D_{\eta,C}$ given by

$$D_{\eta,C} = \rho(v_S^2 + v_A^2)(\Omega_\eta^2 - \omega_C^2)(\Omega^2 - \omega_A^2) \tag{83}$$

for resonant slow waves. The operator Ω_η^2 is defined by

$$\Omega_\eta^2 = \Omega\left(\Omega - i\eta\frac{d^2}{dr^2}\right). \tag{84}$$

Again, notice some remarkable properties here. The dissipative MHD equations for linear resonant motions in an equilibrium state with a flow can be reduced to two linear third-order differential equations that look *formally* identical (except for the operator $D_{\eta,C/A}$) to their counterparts (7a)–(7b) governing the resonant waves in static ideal equilibrium, or to (36a)–(36b) governing resistive resonant MHD waves. The latter feature saves us to go through the very lengthy mathematics of obtaining dissipative solutions in stationary MHD as we can simplify the analysis following the methods outlined in Sect. 3.3. First we present the solutions for resistive slow waves in steady MHD, followed by its counterpart for resistive Alfvén waves in a magnetized plasma with background steady state.

The governing equation in dissipative (resistive) MHD for the parallel component of the Lagrangian displacement characterizing slow resonant waves is for a stationary equilibrium:

$$(\Omega_\eta^2 - \omega_C^2)\xi_\parallel = \frac{if_B}{\rho B}\frac{v_S^2}{v_S^2 + v_A^2}\left(P' - \frac{Q_C\xi_r}{\Omega^2}\right) - i(\Omega^2 - \omega_C^2)\frac{2\Omega B_\varphi v_\varphi + f_B v_\varphi^2}{B\Omega^2 r}. \tag{85}$$

Equation (85) is the resistive counterpart of (71a) and was first obtained by Erdélyi (1997). The detailed calculation to obtain analytic solutions to the governing equations for ξ_r, P' and $\xi_\parallel$ are given in §3.2 of Erdélyi (1997). Here we only quote the main results.

The simplified equation for the parallel component of the displacement at cusp resonances for steady dissipative MHD is

$$\left(s\Delta_C - i\eta\Omega\frac{\omega_C^2}{\omega_A^2}\frac{d^2}{ds^2}\right)\xi_\parallel = i\frac{f_B v_S^2 C_S}{\rho B(v_S^2 + v_A^2)\Omega^2} - is\xi_r\frac{2\Omega B_\varphi v_\varphi + f_B v_\varphi^2}{B\Omega^2 r_C}, \tag{86}$$

where all the equilibrium quantities are evaluated at the resonant point, $s = 0$. This is the resistive generalization of (55) in Goossens et al. (1992). The second term on the right hand side is proportional to $s\ln|s|$ and can be neglected in comparison with the first one. This governing equation for slow resonant waves in steady state has no singularity at $s = 0$ in contrast to its ideal counterpart. The final solution is:

$$\xi_r = -\frac{\mu_0\omega_C^2}{|\Delta_C|\omega_A^2 B^2 C_S}G(\tau) + C_{\xi,C}, \tag{87a}$$

$$P' = -\frac{\mu_0 Q_C}{|\Delta_C|\omega_A^2 B^2}C_S G(\tau) + C_{P,C}, \tag{87b}$$

$$\xi_{\parallel} = \frac{f_B v_S^2 C_S}{\delta_C |\Delta_C| \rho B (v_S^2 + v_A^2)\Omega^2} F(\tau), \tag{87c}$$

$$C_S \equiv \text{const}, \tag{87d}$$

where $C_{\xi,C}$ and $C_{P,C}$ are constants of integration and

$$G(\tau) = \int_0^\infty \frac{e^{-u^3/3}}{u} \{\exp(iu\tau\,\text{sign}(\Omega\Delta_C)) - 1\}\, du, \tag{88}$$

$$F(\tau) = \int_0^\infty \exp(iu\tau\,\text{sign}(\Omega\Delta_C) - u^3/3)\, du. \tag{89}$$

The fact that $C_S \equiv$ const shows that the ideal conservation law for cusp resonances remains valid in dissipative steady MHD.

Using the asymptotic expansion of $G(\tau)$ for large values of τ the jump conditions for ξ_r and P' are as follows:

$$[\xi_r] = -i\pi \frac{\mu_0 \omega_C^2 C_S}{B^2 \omega_A^2 |\Delta_C|} \text{sign}(\Omega), \tag{90a}$$

$$[P'] = -i\pi \frac{\mu_0 Q C_S}{B^2 \omega_A^2 |\Delta_C|} \text{sign}(\Omega). \tag{90b}$$

These jump conditions and the conservation law $C_S =$ const for slow resonances were first derived by Goossens et al. (1992) in *ideal* MHD and confirmed by Erdélyi (1997) to remain valid in dissipative (resistive) MHD. The effects of an equilibrium shear flow are hidden in the constants of the conservation law; the equilibrium flow also modifies Δ_C which appears in the expressions for jumps and for the width of the dissipative layer. Note also, as in the case of Alfvén resonances, that the jumps are independent of the dissipation coefficient η, which is another remarkable property of resonant MHD waves and is analogous to the jump conditions across shock in hydrodynamics where the jumps are also independent of the micro-physics. This universal character implies that the amount of absorbed wave energy and the total amount of dissipative (resistive or viscous) heating in the dissipative layer are also independent of η.

Finally, from the asymptotic behaviour of $F(\tau)$ for $\tau \to \pm\infty$ we now easily get the following asymptotic expansion for $\xi_{\parallel}$,

$$\xi_{\parallel} \simeq \frac{i f_B v_S^2 C_S}{\tau \delta_C |\Delta_C| \rho B (v_S^2 + v_A^2)\Omega^2}. \tag{91}$$

This asymptotic expansion recovers the $1/\tau$ behaviour of $\Im(\xi_{\parallel})$ far away from the ideal resonance position, i.e. for $\tau \to \pm\infty$. The equilibrium flow does not change the asymptotic behaviour. Note that the analysis breaks down at the critical point of flow resonance, i.e. where $\omega = \omega_f$.

Last but not least in the summary of the theoretical aspects, let us comment on resonant Alfvén waves in dissipative steady MHD. From a mathematical perspective the analysis is similar to the one of slow waves given above. Again, the detailed governing equations are rather elaborate and can be found in Erdélyi et al. (1995). Since the Alfvén waves are characterized by $\xi_\perp$, let us focus on the governing equation determining this component of the Lagrangian displacement vector. The equation for the perpendicular component of the Lagrangian displacement in the magnetic surface in dissipative steady MHD is

$$\left(\Omega_\eta^2 - \omega_A^2\right)\xi_\perp = \frac{i}{\rho B}\left(g_B P' - \frac{2B_z T}{r}\xi_r\right), \tag{92}$$

where $D_\eta \equiv D_{\eta,A}$ is given by (82). Equation (92) is the resistive and steady state generalization of (36c) and the resistive extension of (71b) or (49) in Goossens et al. (1992). Remarkably, (92) is formally similar to (71b), but Ω^2 is now replaced by Ω_η^2 that is a second-order differential operator, and $\omega_A = \Omega$ is not singularity anymore.

Using the techniques outlined in Goossens et al. (1995) and Erdélyi et al. (1995) we can solve the governing equations for ξ_r, P', $\xi_\perp$, and for the coupling function C_A in dissipative (resistive) steady MHD to obtain the following solution:

$$\xi_r = -\frac{g_B C_A}{\rho B^2 \Delta_A} G(\tau) + C_{\xi,A}, \tag{93a}$$

$$P' = -\frac{2B_z T C_A}{\rho B^2 r \Delta_A} G(\tau) + C_{P,A}, \tag{93b}$$

$$\xi_\perp = \frac{C_A \operatorname{sign}(\Omega)}{\delta_A |\Delta_A| \rho B} F(\tau), \tag{93c}$$

$$C_A \equiv \text{const}, \tag{93d}$$

where $C_{\xi,A}$ and $C_{P,A}$ are constants of integration and $G(\tau)$ and $F(\tau)$ are formally the same as (88)–(89) but must be evaluated for Alfvén resonance. This result confirms that the ideal conservation law for Afvén resonance found by Goossens et al. (1992) remains fully valid in dissipative MHD.

Again, similarly to the case of slow resonance, the asymptotic expansion of $G(\tau)$ for large values of τ the jump conditions for ξ_r and P' can be derived for the steady dissipative Alfvén resonance yielding:

$$[\xi_r] = -i\pi \frac{g_B C_A}{\rho B^2 |\Delta_A|} \operatorname{sign}(\Omega), \tag{94a}$$

$$[P'] = -i\pi \frac{2B_z T C_A}{\rho B^2 r_A |\Delta_A|} \operatorname{sign}(\Omega). \tag{94b}$$

These jump conditions and the conservation law $C_A = \text{const}$ for Alfvén resonances were first derived by Goossens et al. (1992) in *ideal* MHD and confirmed by Erdélyi et al. (1995) to remain valid in dissipative (resistive) MHD. The jumps and the conservation law derived in the framework of resistive dissipative MHD for steady equilibrium are in their structure similar to their counterparts obtained for static equilibrium. Again, the effects of an equilibrium shear flow are rather hidden in the constants of the conservation law. The equilibrium flow also modifies Δ_A, a very important point that must be emphasized, which appears in the expressions for jumps and for the width of the dissipative layer. The jumps remain still independent of the dissipation coefficient, η, similar to the case of slow resonance. This seems to be a very robust and universal feature of resonant MHD waves in general.

Finally, from the asymptotic behaviour of $F(\tau)$ for $\tau \rightarrow \pm\infty$ one obtains for $\xi_\perp$ in steady state that

$$\xi_\perp \simeq \frac{iC_A \mathrm{sign}(\Omega)}{\tau \Omega B \delta_A |\Delta_A|}. \tag{95}$$

This asymptotic behaviour far away from the ideal Alfvén resonance position in stationary MHD is in agreement with its counterpart in static MHD found by Goossens et al. (1995) as it confirms the τ^{-1} behaviour of $\Im(\xi_\perp)$ for $\tau \rightarrow \pm\infty$. The equilibrium flow does not change the asymptotic behaviour. Note that again the analysis breaks down at the critical point of flow resonance, i.e. where $\omega = \omega_f$.

4.2 Heating by Resonant Absorption in Steady State

In this section we are concerned with a *driven* problem in *steady* state. If the inhomogeneous plasma is driven externally at a frequency ω that falls in the band of the continuous eigenfrequencies of the system modified by the equilibrium flow, the amplitude of the oscillations excited in the system peaks at the resonance point where the frequency of the local field line matches the Doppler-shifted frequency ω. The resonance, just like in a static equilibrium, causes energy to build up at the resonant magnetic surface at the expense of the global motion. Again, in ideal MHD the resonantly accumulated energy will be infinite at the resonant position(s) (see e.g. Erdélyi 1996; Erdélyi and Goossens 1996).

In order to compute the absorption of wave energy and the heating of a plasma we have to include dissipative effects in the stationary MHD equations (Erdélyi et al. 1995). The inclusion of non-ideal effects removes the Doppler-shifted singularity of the governing equations, and both the energy density and the spatial gradients have large but finite values. In this case the energy transferred to the resonant magnetic surface can be converted into heat. However, under certain circumstances as the driving waves that are passing through the resonant layer in a steady state may carry in fact more energy *away* from the resonant location than that of the incident waves, i.e. resonant waves can *gain* additional energy, their amplitude may increase as a result of this dynamic resonant interaction even leading to instabilities (see Erdélyi and Goossens 1996; Tirry et al. 1998b). This effect is called overreflection. Until the mid-90s resonant absorption of MHD waves was studied for static equilibrium states almost exclusively, as is outlined in previous Sections. However, observations reveal both upwards and downwards mass-flows in magnetic flux tubes along their longitudinal axes in the solar atmosphere. Typical speeds are 5–50 km/s (see e.g. Doyle et al. 1997).

Steady equilibrium flows (velocity fields) change the properties of MHD waves. The most obvious effect of an equilibrium flow is the Doppler-shift of the Alfvén and slow continua. For coronal heating by resonant absorption this Doppler-shift is important because the frequency range where the mechanism operates is changed as shown by Erdélyi (1996) and Erdélyi and Goossens (1996).

In addition to the Doppler-shift, an equilibrium flow can have more subtle effects on the absorption of waves which are hidden in the equations. One example of such effects might be the modification of connection formulae, as we have seen in Sect. 4.1 that are crucial for determining the absorbed energy (see e.g. Goossens et al. 1992; Erdélyi and Goossens 1995). Another example is the change of thickness of the dissipative layers (see e.g. Erdélyi 1996; Erdélyi and Goossens 1996).

Erdélyi (1998) studied the effect of a steady velocity field on the rate of resonant absorption of Alfvén waves in coronal loops. His numerical results indicate that an equilibrium

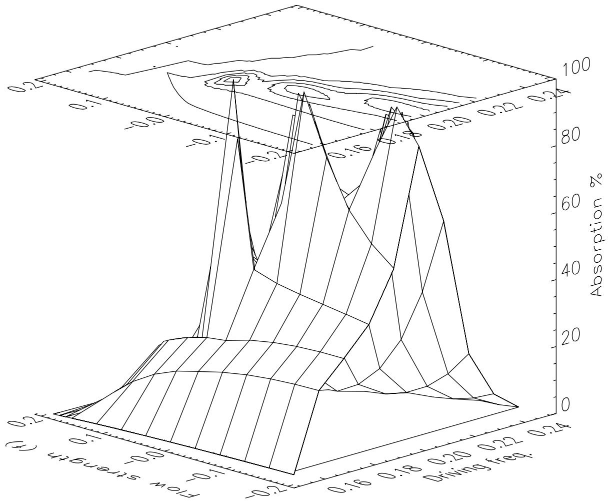

Fig. 2 Absorption coefficient α as a function of the equilibrium flow strength parameter f, and driving frequency ω. For more details on the coronal loop model see Erdélyi (1998)

mass flow can *significantly* influence the absorption of Alfvén waves (see Fig. 2). It is therefore important to take into account the presence of an equilibrium flow when determining the power loss of MHD waves, due to their interaction with coronal loops.

Alfvén waves can carry energy only along the magnetic field lines and slow waves are able to carry only 1–2% of energy under coronal (i.e. low plasma-β) conditions. However fast magneto-acoustic waves might also have an important contribution in explaining the coronal temperatures as it has been shown, e.g., by Čadež et al. (1997) and Csík et al. (1998). Fast magneto-acoustic waves are magnetic waves which can propagate carrying energy *across* the magnetic field lines. They are compressive and therefore subject to dissipation by, e.g. viscosity, heat conduction, Landau and transit-time damping, etc.

Linear theory shows that, in the vicinity of the resonant position, the amplitudes of the variables can be very large even when they are small far away from this position. This observation implies that linear theory can break down in this region. Ruderman (1997a, 1997b) developed a nonlinear theory of resonant slow waves in isotropic plasmas which has been extended to anisotropic plasmas by Ballai (1998a, 1998b). They have derived the nonlinear governing equations in a strongly anisotropic plasma and have found a generalized form of the connection formulae (see Ruderman 2000). The effect of a steady (shear) flow on the resonant behaviour of nonlinear slow waves was discussed first by Ballai and Erdélyi (1998). An application to the magnetic canopy was given in Erdélyi and Ballai (1999) where the approximation of weak nonlinearity and long wavelength was used for the analytical progress. The qualitative result of these papers is the decrease of absorption caused by nonlinearity can be overruled by even a small amount field-aligned bulk motion. For the review of the nonlinear theory of slow resonant waves see Ballai and Ruderman (2010).

Finally, we recall a detailed study of coronal heating by nonlinear resonant FMA waves carried out by Erdélyi et al. (2001). Again, the coefficient of wave energy resonant absorption is derived using (i) weak nonlinearity and (ii) long-wavelength approximation. Among their conclusions we point out that (i) an equilibrium flow in the slow dissipative layer can *either* increase or decrease the coefficient of the wave energy absorption. Thus, a field-aligned flow has an important effect on the resonant interaction of fast waves and nonlinear slow resonances and energy transfer, (ii) negative absorption rate (i.e. RFI or over-reflection) has been found for a wide range of parameters. A more accurate quantitative analysis would require observations and diagnostics of large-scale flow fine structures in the low and middle solar atmosphere.

5 Quasi-Modes

5.1 Motivation

This section focuses on the third objective of this review. This objective is to explain that any eigenmode of ideal MHD, whether it is a surface Alfvén wave or a body fast magneto-acoustic wave, that has a frequency with its real part in the Alfvén continuum, is transformed into a damped quasi-mode. The existence of damped quasi-modes is due to the intrinsic coupling of MHD waves and with the fact that the Alfvén resonance is a sink for the wave energy. Damped quasi-modes are not a rarity, whenever an eigenmode has its frequency in the Alfvén continuum a quasi-mode shows up. The attentive reader might have noticed that, in our discussion of resonant Alfvén waves in dissipative MHD in Sect. 3.3, the frequency ω was treated as a real quantity. This means that the analysis presented in Sect. 3.3 applies to driven waves. The present section is concerned with eigenmodes. The linear spectrum of MHD waves of a non-uniform static plasma equilibrium consists of discrete (fast and slow magneto-sonic and Alfvén) eigenmodes and continuum Alfvén and slow eigenmodes.

In Sect. 2 it was explained that there is always coupling between magneto-sonic waves and Alfvén waves except for a straight field and $m = 0$. This means that, for an equilibrium with a straight magnetic field, the real eigenvalues of discrete fast sausage eigenmodes can lie in the continuum of the torsional Alfvén continuum eigenmodes, but there is no coupling and hence no damping. On the other hand, for $m \neq 0$, the discrete fast eigenmodes with an eigenfrequency in the Alfvén continuum couple to a local Alfvén continuum eigenmode. The jump of energy when crossing the dissipative layer $[FluxE]$ (55) is always negative in a static equilibrium and the dissipative layer is a sink for the energy of the wave. Hence the wave gets damped and becomes a damped quasi-mode. Quasi-modes or collective modes showed up in ideal MHD during the construction of the solution of the initial value problem by using the Laplace transform technique in the complex frequency plane (see e.g. Sedlaček 1971; Grossmann and Tataronis 1973; Goedbloed 1983; Goedbloed and Poedts 2004). The term quasi-modes or collective modes refers to the fact that the modes possess complex frequencies even in ideal MHD. Studies of quasi-modes or collective modes have been motivated by schemes to heat plasmas by means of Alfvén wave absorption. Grossmann and Tataronis (1973), Hasegawa and Chen (1974), Ionson (1978) and Wentzel (1979b) proposed to excite the surface Alfvén wave and this wave should then be absorbed at the spatial Alfvén resonance and its energy dissipated to the plasma. Chen and Hasegawa (1974a) used a slab geometry and found that absorption rate is strongly enhanced when the non-uniformity of the equilibrium is sharp and the driving frequency is close to real part of the frequency of the collective surface mode. Hasegawa and Uberoi (1982) (see references therein) calculated the damping rate of the damped surface quasi-mode of a slab model with a linear variation in density. A similar result for a linear variation of the square of the local Alfvén frequency was obtained by Goedbloed (1983).

The studies in the 1970s and the early 1980s on plasma heating by resonant absorption of Alfvén waves was mainly concerned with the surface Alfvén wave often in planar geometry. In the 1980s and 1990s it was shown, first in fusion plasma physics (see e.g. Appert et al. 1981; Balet et al. 1982) and, subsequently, in solar physics (see e.g. Poedts et al. 1989a, 1989b, 1990a, 1990b, 1990c; Steinolfson and Davila 1993) that a similar phenomenon takes place in an equilibrium with an arbitrary non-uniformity and for frequencies not necessarily associated with the Alfvén surface wave but also with fast body waves. In the fusion plasma physics the surface Alfvén wave introduced by Grossmann and Tataronis (1973) and Hasegawa and Chen (1974) was also referred to as the first (lowest frequency) radial eigenmode of the fast magneto-acoustic wave (see e.g. Balet et al. 1982). Balet et al. (1982) identified quasi-modes in an original way. They imposed an initial displacement on the plasma column and then let it oscillate freely. Fourier analysis of the displacement at different radii leads to a peak in the Fourier amplitude at the same frequency at the different radii. Hence, Balet et al. (1982) identified this global motion as a collective mode of the plasma response. Poedts and Kerner (1991) studied the quasi-mode for a plasma-vacuum-wall system and used a numerical eigenvalue code to show that the ideal quasi-mode corresponds to a normal mode of resistive MHD of which the damping becomes independent of resistivity for sufficiently small resistivity. Ofman et al. (1995) studied resonant absorption for a Cartesian planar slab of low beta plasma. They identified the frequencies of the quasi-modes as the frequencies of maximal absorption and determined the dependence of the quasi-mode frequencies on the wave numbers. Wright and Rickard (1995) computed the time-dependent behaviour of a Cartesian nonuniform MHD cavity driven by a random boundary motion which has a broadband frequency spectrum. These authors identified two criteria for efficient excitation of the Alfvén resonance: first, the fast or global eigenfrequencies of the cavity must lie in the spectrum of the driving motions and second, the fast or global eigenfrequencies of the cavity must lie in the Alfvén continuum. Wright and Rickard (1995) showed that there are indeed fast eigenmodes which have the real part of their frequencies in the Alfvén continuum. They did not compute complex frequencies of the eigenmodes. They showed by time dependent computations in resistive MHD that efficient absorption occurs at the frequencies of the global fast waves.

5.2 Quasi-Modes in Ideal and Eigenmodes in Resistive MHD

To explain the notion of quasi-modes we consider a simple example. The equilibrium magnetic field is in the z-direction of Cartesian coordinate system x, y, z. The plasma density and the magnetic field magnitude are ρ_- and B_- for $x < 0$, and ρ_+ and B_+ for $x > 0$. Assume that the plasma is incompressible. Then the surface wave can propagate on the surface of magnetic interface situated at $x = 0$, and the phase speed of this wave is given by

$$\frac{\omega_{\mathrm{surf}}^2}{k^2} = v_{\mathrm{surf}}^2 \equiv \frac{\rho_- v_{A-}^2 + \rho_+ v_{A+}^2}{\rho_- + \rho_+}, \quad v_{A\pm}^2 = \frac{B_\pm}{\mu_0 \rho_\pm}, \tag{96}$$

where ω and k are the frequency and wave number of the surface wave, and we assume that the wave propagates in the z-direction. The surface wave is an eigenmode of ideal MHD, and its frequency given by equation (96) is an eigenfrequency.

Let us now change the equilibrium and add a transition layer defined by $|x| < a$, where the density and magnetic field vary continuously from ρ_- to ρ_+ and from B_- to B_+ respectively in such a way that $v_A(x)$ is a monotonic function. We obtain the equilibrium configuration which is often called "finite-thickness magnetic interface." If $ak \ll 1$, then the existence of the transitional layer does not affect very much ω, at least its real part. Let us, for the sake of definiteness, assume that $v_{A-} < v_{A+}$, so that $v_A(x)$ is a monotonically growing function. Then it is easy to see that $v_{A-} < v_{\mathrm{surf}} < v_{A+}$. This result implies that there is a point $x_A \in (-a, a)$ where $\omega = v_A(x_A)k \equiv \omega_A(x_A)$, i.e. there is Alfvén resonant position in $(-a, a)$. Actually, the restriction to incompressible motions means that the Alfvén and slow resonance coincides as pointed out by Goossens et al. (1992) and Ruderman (2009), so it would be more appropriate to call x_A the mixed resonant position. However, traditionally it bears the name Alfvén resonant position. As we have already seen, the solution at the Alfvén resonance has a non-integrable singularity, so that now there is no eigenmode in the form of the surface wave with its frequency equal to ω.

However the frequency $\omega_{\text{surf}} = kv_{\text{surf}}$ still plays an important role in the description of waves on a finite-thickness magnetic interface, at least when $ak \ll 1$. This frequency appears when we calculate the asymptotic state of the oscillation of an arbitrarily perturbed interface for large time. To do this we first solve the initial value problem using the Laplace transform. Initially the Laplace transform is defined only in the upper part of the complex ω-plane. It turns out that the Laplace-transformed solution has logarithmic branch-points at $\pm\omega_{A-} = \pm kv_{A-}$, $\pm\omega_{A+} = \pm kv_{A+}$, and $\pm\omega_A(x) = \pm kv_A(x)$. To obtain a single-valued branch of the Laplace-transformed solution we make cuts in the ω-plane, and then construct the Riemann surface of the Laplace-transformed solution that consists of infinite number of sheets attached to each other at the branch cuts. This procedure is described in a seminal paper by Sedlaček (1971). Note that Sedlaček (1971) considered the electrostatic oscillations of a cold inhomogeneous plasma. However, mathematically, this problem is the same as that describing propagation of surface waves on a thick interface.

The Bromwich integration contour used to calculate the inverse Laplace transform is a straight line above the real axis and parallel to this axis on the principal Riemann sheet. To calculate the asymptotic behaviour of the solution for large time we have to close this contour. This closure goes to non-principal Riemann sheets through the cuts. The asymptotic solution for large time consists of two terms. The first term describes oscillations of individual magnetic field lines with local Alfvén frequencies. It decays as t^{-1} due to phase mixing. The second term arises from two simple poles of the Laplace-transformed solution at $\omega = \omega_r + i\omega_i$ and $\omega = -\omega_r + i\omega_i$ situated on non-principal Riemann sheets. It describes collective oscillation of the plasma with the frequency ω_r in the form of a surface wave. These oscillations damp with the decrement $\gamma_{\text{d}} = -\omega_i$. When $ka \ll 1$, $\omega_r \approx kv_{\text{surf}}$ and $\gamma_{\text{d}}/\omega_r \sim ka$, so that the collective oscillation is only weakly damped. This second term in the asymptotic solution is called quasi-mode. Since the quasi-mode decays as $\exp(-\gamma_{\text{d}}t)$, it cannot be an eigenmode of the ideal MHD because all such eigenmodes have either real or purely imaginary frequencies (e.g. Goedbloed and Poedts 2004).

Since the first term in the asymptotic solution decays as t^{-1} while the second term decays exponentially, for very large time the first term always dominates over the second one. However, when $ka \ll 1$, the second term dominates over the first one for large but "not very large" time (we do not discuss here the exact meaning of this restriction). We say that, in this case, the second term gives the so-called intermediate asymptotic. Hence, for $ka \ll 1$, the intermediate asymptote is the surface wave with its properties very similar to those of a surface wave on a true interface.

If we add dissipation then a quasi-mode is converted in a normal mode of dissipative MHD. The characteristic property of this normal mode is that its decrement tends to the decrement of the corresponding quasi-mode of ideal MHD when dissipative coefficients tend to zero. The dissipative solution tends to the ideal solution uniformly with respect to the spatial variable on any fixed interval not containing the resonant position when the dissipative coefficients tend to zero.

Eigenmodes of dissipative MHD were first studied for one of the simplest magnetic plasma configuration, which is a finite-thickness magnetic interface. Mok and Einaudi (1985) and Einaudi and Mok (1985) studied the resistive eigenmodes of a finite-thickness magnetic interface in an incompressible plasma under the conditions that $ka \ll 1$ and

$a^2k \ll \delta_A \ll a$ (δ_A is the thickness of the dissipative layer in the driven problem defined by (41), where we have to substitute ω_r for ω). Note that the second condition, in particular, implies that $\delta_A \gg a(\gamma_d/\omega_r)$, where $\gamma_d = -\omega_i \sim ak\omega_r$ is the decrement of the quasi-mode and the corresponding dissipative eigenmode. It was found that the plasma motion in the dissipative layer is the same as in the driven problem. In particular, the plasma displacement in the direction perpendicular to the interface is described by the G function, and that in the equilibrium magnetic field direction by the F function.

Ruderman et al. (1995) studied the same problem as Mok and Einaudi (1985), but they relaxed the assumption $a^2k \ll \delta_A \ll a$ and allowed δ_A to be arbitrarily small. As a result they found that, when $\delta_A \lesssim a^2k$, the character of the motion in the dissipative layer is quite different from that in the case $\delta_A \gg a^2k$. In the general case, the plasma displacement in the direction perpendicular to the interface is described by the G_Λ function, and that in the equilibrium magnetic field direction by the F_Λ function, where

$$F_\Lambda(\tau) = \int_0^\infty \exp(iu\tau\,\mathrm{sign}(\Delta_A) - u^3/3 + \Lambda u)\,du, \tag{97}$$

$$G_\Lambda(\tau) = \int_0^\infty \frac{e^{-u^3/3}}{u}\{\exp(iu\tau\,\mathrm{sign}(\Delta_A) + \Lambda u) - 1\}\,du, \tag{98}$$

and the parameter Λ is given by

$$\Lambda = \frac{2\omega_A\gamma_d}{\delta_A|\Delta_A|} \sim \frac{\gamma_d/\omega_r}{\delta_A/a}, \tag{99}$$

i.e. Λ is of the order of the ratio of two small quantities. It is easy to see that $F_\Lambda(\tau) = F(\tau)$ and $G_\Lambda(\tau) = G(\tau)$ when $\Lambda = 0$. Moreover, qualitatively $F_\Lambda(\tau)$ and $G_\Lambda(\tau)$ are quite similar to $F(\tau)$ and $G(\tau)$ respectively when $\Lambda \lesssim 1$. However, when $\Lambda \gg 1$ the behaviour of $F_\Lambda(\tau)$ and $G_\Lambda(\tau)$ is qualitatively different from that of $F(\tau)$ and $G(\tau)$. The graphs of the real and imaginary parts of $F_\Lambda(\tau)$ and $G_\Lambda(\tau)$ are shown in Fig. 3 for $\Lambda = 5$. It follows from (97) and (98), and also can be seen in Fig. 3, that $\Re(F_\Lambda)$ and $\Re(G_\Lambda)$ are even functions, while $\Im(F_\Lambda)$ and $\Im(G_\Lambda)$ are odd functions.

An important property of function $G_\Lambda(\tau)$ is that

$$\lim_{\tau\to\pm\infty} G_\Lambda = \pm\frac{1}{2}\pi i. \tag{100}$$

so that $[G_\Lambda(\tau)] = \pi i$. We can see in Fig. 3 that, for large Λ, $F_\Lambda(\tau)$ and $G_\Lambda(\tau)$ have the form of wave packets. This is confirmed by the asymptotic analysis. In particular, using the steepest descent method, we can obtain that, for large Λ,

$$\begin{aligned} F_\Lambda(\tau) \sim \frac{\pi}{\sqrt[4]{\Lambda^2+\tau^2}}\left(\sqrt{\frac{1}{2}\left(\sqrt[4]{\Lambda^2+\tau^2}+\Upsilon_+\right)} - i\sqrt{\frac{1}{2}\left(\sqrt[4]{\Lambda^2+\tau^2}-\Upsilon_+\right)}\right) \\ \times \exp\left(\frac{2}{3}[(\Lambda\Upsilon_+ - \tau\Upsilon_-) + i(\tau\Upsilon_+ - \Lambda\Upsilon_-)]\right), \end{aligned} \tag{101}$$

where

$$\Upsilon_+ = \sqrt{\frac{\sqrt{\Lambda^2+\tau^2}+\Lambda}{2}}, \qquad \Upsilon_- = \sqrt{\frac{\sqrt{\Lambda^2+\tau^2}-\Lambda}{2}}. \tag{102}$$

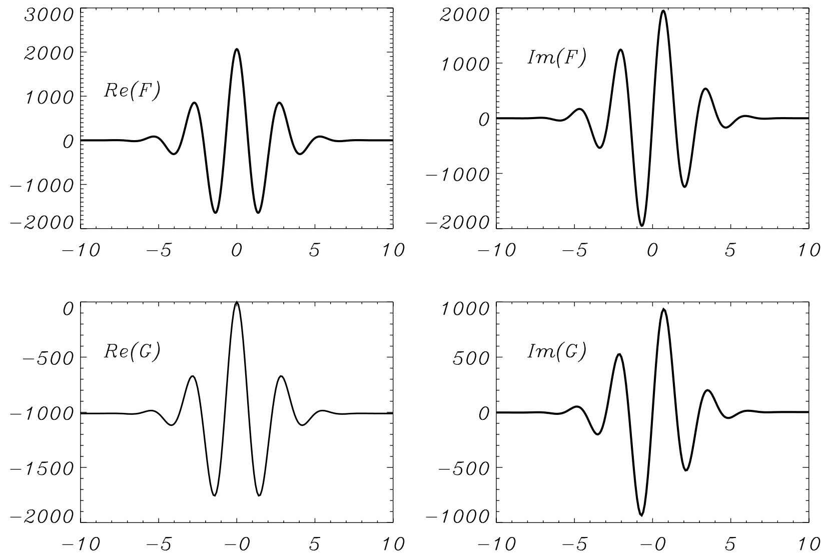

Fig. 3 The real and imaginary parts of functions $F_\Lambda(\tau)$ and $G_\Lambda(\tau)$. We can see that $\Re(F_\Lambda)$ and $\Re(G_\Lambda)$ are even functions, while $\Im(F_\Lambda)$ and $\Im(G_\Lambda)$ are odd functions

We can define the thickness of the dissipative layer as the distance where $|F_\Lambda(\tau)| = e^{-1}|F_\Lambda(0)|$. Using (101) it is easy to obtain that, for large Λ, this distance is approximately equal to $\tau_d = \Lambda\sqrt{3}$. Then, in the dimensional variables, the thickness of the dissipative layer is of the order of

$$\tau_d \delta_A \sim \Lambda \delta_A = a(\gamma_\mathrm{d}/\omega_r). \tag{103}$$

We see that, for large Λ, the thickness of dissipative layer is independent of the Reynolds number and determined by the decrement.

It is also interesting to determine the characteristic scale of variation of perturbations in the dissipative layer. We can define this scale as the distance from $\tau = 0$ to the first zero of $\Re(F_\Lambda)$. Simple calculations show that this distance is of the order of $1/\sqrt{\Lambda}$. Hence, the dimensional characteristic scale of variation of perturbations in the dissipative layer is of the order of

$$\frac{\delta_A}{\sqrt{\Lambda}} \sim \sqrt{\frac{\eta}{\gamma_\mathrm{d}}} \sim a(ak)^{-1} R_m^{-1/2}, \tag{104}$$

where $R_m = a/\eta\omega_r$, and we have used the estimates $\Delta_A \sim \omega_r^2/a$ and $\gamma_\mathrm{d} \sim ak\omega_r$. Hence, while the thickness of the dissipative layer becomes independent of R_m when $R_m \gg 1$, the characteristic scale of the plasma parameters variation in the dissipative layer is decreasing as $R_m^{-1/2}$.

The analysis by Ruderman et al. (1995) was extended by Tirry and Goossens (1996) to cylindrical geometry. Their analysis can be also considered as an extension of the analysis of resonant Alfvén waves by Sakurai et al. (1991a) and Goossens et al. (1995) for the eigenmode problem. Tirry and Goossens (1996) also simplified the analysis by Ruderman et al. (1995). While Ruderman et al. (1995) used the method of multiple scales and introduced "slow" times, Tirry and Goossens (1996) simply took into account the imaginary part of ω when writing the approximate expression for $\omega^2 - \omega_A^2$ in the vicinity of the resonant position. In what follows we briefly outline the method used by Tirry and Goossens (1996), and present the results that they obtained.

Let us write the frequency of a dissipative eigenmode in the form

$$\omega = \omega_r + i\omega_i, \tag{105}$$

ω_r and ω_i being real quantities. Strictly speaking damping (in a static equilibrium) should be weak meaning that $|\omega_i| \ll |\omega_r|$. Then the modification of Sect. 3.2 is straightforward. In (38) and (40) the differential operator $D^2_{\eta,s}$ defined in (39) should be replaced with

$$D^2_{\eta,s} = 2i\omega_A\omega_i + s\Delta_A - i\omega_r\eta\frac{d^2}{ds^2}. \tag{106}$$

In (44) the differential operator D^2_τ defined in (45) should be replaced with

$$D^2_{\Lambda,\tau} = \frac{d^2}{d\tau^2} + i\,\mathrm{sign}(\Delta_A)\,\tau - \Lambda, \tag{107}$$

where Λ is defined by (99). The solution to the modified version of (44) were obtained by Tirry and Goossens (1996) with the use of a Fourier transform technique. It reads

$$C_A(\tau) = \mathrm{const}, \tag{108a}$$

$$\xi_r(\tau) = -\frac{g_B C_A}{\rho B^2 \Delta_A} G_\Lambda(\tau) + C_\xi, \tag{108b}$$

$$P'(\tau) = -\frac{2 f_B B_\varphi B_z C_A}{\rho B^2 \mu_0 r \Delta_A} G_\Lambda(\tau) + C_P, \tag{108c}$$

$$\xi_\perp(\tau) = \frac{C_A}{\delta_A |\Delta_A| \rho B} F_\Lambda(\tau), \tag{108d}$$

where the function $F_\Lambda(\tau)$ and $G_\Lambda(\tau)$ are defined by (97) and (98), and C_ξ and C_P are constants that have to be determined from the matching with the ideal solution in the overlap regions.

As we have already seen the jump of function $G_\Lambda(\tau)$ is the same as that of function $G(\tau)$ (see (100)). This implies that the jump conditions (54) for driven waves continue to hold for eigenmodes. In retrospect this is an important observation since the jump conditions obtained for driven Alfvén waves were used for computing eigenmodes by e.g. Goossens et al. (1992) and Goossens and Hollweg (1993). The resonant interaction, damping or amplification, is translated in a complex frequency. In a static equilibrium the jump in energy is always negative, meaning that the dissipative layer is a sink for the energy of the wave so that the wave gets damped. In equilibrium models with flow the quasi-modes can become over-stable as shown by, e.g., Hollweg et al. (1990), Ruderman and Wright (1998), Andries et al. (2000), and Andries and Goossens (2001a, 2001b).

To the best of our knowledge damped global eigenmodes that are coupled to resonant Alfvén waves in a non-uniform equilibrium state can be computed by three methods. The first method was introduced by Balet et al. (1982). It uses Fourier analysis at different positions of the free oscillation that is excited by imposing an initial displacement for determining the frequency and damping rate of the quasi-mode. The second method is to use a numerical code that integrates the resistive MHD equations in the whole volume of the equilibrium state to determine a selected mode or part of the resistive spectrum of the system. The third method was used by, e.g., Mok and Einaudi (1985), Ruderman et al. (1995) and Tirry and Goossens (1996). It circumvents the numerical integration of the non-ideal MHD equations and only requires numerical integration (or closed analytical solutions) of the linear ideal MHD equations. The method relies on the fact that dissipation is important only in a narrow layer around the resonant point where the real part of the frequency of dissipative eigenmode equals the local Alfvén frequency. The jump conditions are used to connect the solutions to the ideal MHD equations to the left and right of the dissipative layer. This method is the eigenvalue counterpart of the SGHR method for driven resonant Alfvén waves. It was used for computing eigenmodes of various non-uniform plasma configurations by e.g. Tirry et al. (1998a, 1998b), Stenuit et al. (1998, 1999), Andries et al. (2000), Andries and Goossens (2001a), and Van Doorsselaere et al. (2004).

A drastic variant of the method that strongly simplifies solving the non-ideal MHD equations uses the so-called thin boundary (TB) approximation. In this lazy version it is assumed that the non-uniform layer is so thin that we can neglect the total pressure variation across this layer. Then the system of linear ideal MHD equations is reduced to only one equation for the plasma displacement in the direction perpendicular to the non-uniform layer. This equation can be easily solved analytically. The TB approximation has gained popularity in the context of damped transverse oscillations of solar coronal loops as it allows to obtain simple analytical expressions for the damping rate. It was used by e.g. Hollweg and Yang (1988), Goossens et al. (1992, 2002a, 2009), Goossens and Hollweg (1993), Ruderman and Roberts (2002), Ruderman (2003) and Dymova and Ruderman (2006a). A discussion of the TB approximation can be found in Goossens (2008) and Goossens et al. (2009). In this respect it is instructive to refer to Van Doorsselaere et al. (2004) and Terradas et al. (2006a) who computed damped eigenmodes of 1-D cylindrical models of coronal loops with a numerical eigenvalue code. Comparison of results obtained by using the TB approximation with those obtained by the numerical eigenvalue code shows that the thin boundary approximation is surprisingly accurate far beyond its theoretical domain of applicability.

Let us recall that, for an equilibrium with a straight magnetic field, discrete fast sausage eigenmodes can have real eigenvalues that lie in the continuum of the torsional Alfvén continuum eigenmodes, but there is no coupling and hence no damping. On the other hand, for $m \neq 0$ the discrete fast eigenmodes with an eigenfrequency in the Alfvén continuum couple to a local Alfvén continuum eigenmode. Tirry and Goossens (1996) gave a nice illustration of this transformation of undamped sausage fast magneto-sonic eigenmodes with frequencies in the Alfvén continuum into kink fast magneto-sonic eigenmodes damped by resonant absorption. These authors considered a pressureless cylindrical plasma equilibrium with a constant straight magnetic field B_z and a density profile that varies with distance r to the axis of the cylinder as $\rho(r) = \rho(0)[1 + 0.9\exp(-(r/R)^4)]$. This density profile was also used by Ofman et al. (1995) and Wright and Rickard (1995) albeit in Cartesian coordinates. Tirry and Goossens (1996) computed the first three sausage ($m = 0$) fast magneto-sonic eigenmodes of this equilibrium model for 31 equidistant values of $k_z R$ from 0 to 3. For each k_z the corresponding Alfvén continuum was computed. For $k_z R \geq 1$ the fundamental fast sausage mode has its frequency in the Alfvén continuum. This happens for the first overtone for $k_z R \geq 2$ and for the second overtone for $k_z R \geq 3$. Hence there are many sausage modes with frequencies in the Alfvén continuum. In order to see what happens with these modes when going from a sausage mode $m = 0$ to a kink mode $m = 1$ Tirry and Goossens (1996) carried out a numerical experiment in which they varied the value of m in a continuous manner from 0 to 1. The outcome of this mathematical exercise is that the frequency becomes complex with a non-zero negative imaginary part and a real part that undergoes a shift. The excursion of the frequency in the complex plane has been computed for different values of the magnetic Reynolds number $R_m = 10^7, 10^8, 10^9, 10^{10}$. The result is a numerical confirmation of the damping rate being independent of dissipation for sufficiently small dissipation as could have been predicted on the basis of the expression for $[FluxE]$. The same conclusion was reached by Poedts and Kerner (1991) in a study of the kink mode in a fusion related setup. Quasi-modes are the natural oscillation modes of a system. They combine properties of a localized resonant Alfvén wave and of a global fast eigenmode. They are damped due to resonant coupling and the damping is independent of dissipation for small dissipation.

Let us recall that the use of the jump conditions in combination with the TB approximation has gained popularity in the context of damped transverse oscillations of loops. It allows to obtain simple analytical expressions for the damping rate that can be used for the interpretation of observational data and for seismological investigation. In addition it turns out that this method is surprisingly accurate far beyond its theoretical domain of applicability. Let us as an example consider the damping of kink MHD waves in a pressureless plasma with a straight magnetic field (see e.g. Goossens et al. 2009). For a uniform plasma with constant but different densities ρ_i inside and ρ_e outside the loop, we can follow the analysis by Spruit (1982) and Edwin and Roberts (1983). Edwin and Roberts (1983) gave a general formulation of the dispersion equation for MHD waves on a straight homogeneous magnetic cylinder which can be written as follows:

$$F_\star \frac{J'_m(x_0)K_m(y_0)}{J_m(x_0)K'_m(y_0)} = 1 \qquad (109)$$

with the quantity $F_\star$ given by

$$F_\star = \frac{k_i}{k_e}\frac{\rho_e(\omega^2-\omega_{Ae}^2)}{\rho_i(\omega^2-\omega_{Ai}^2)}, \qquad (110)$$

where $x_0 = k_i R$, $y_0 = k_e R$, and

$$k_i^2 = \frac{(\omega^2-k^2 v_{Si}^2)(\omega^2-\omega_{Ai}^2)}{(v_{Si}^2+v_{Ai}^2)(\omega^2-\omega_{Ci}^2)}, \qquad k_e^2 = \frac{(k^2 v_{Si}^2-\omega^2)(\omega_{Ai}^2-\omega^2)}{(v_{Si}^2+v_{Ai}^2)(\omega_{Ci}^2-\omega^2)}. \qquad (111)$$

J_m and K_m are the Bessel function and modified Bessel function of the second kind. In (109) $m = 0$ corresponds to the sausage mode, $m = 1$ to kink mode, and $m > 1$ to fluting modes. The dispersion equation (109) is written for body waves ($k_i^2 > 0$). To obtain the dispersion equation for surface waves we have to substitute $I_m(|x_0|)$ and $I'_m(|x_0|)$ for $J_m(x_0)$ and $J'_m(x_0)$.

The dispersion relation (109) can be solved numerically. However, it is instructive and also accurate to consider the so-called thin tube (TT) approximation. The Bessel functions $J_m(x)$ and $K_m(y)$ in (109) are replaced with their first order asymptotic expansions. The dispersion relation (109) is reduced to

$$1 + F_\star \frac{k_e}{k_i} = 0. \qquad (112)$$

In the cold plasma approximation the expressions (111) for k_i^2 and k_e^2 reduce to

$$k_i^2 = \frac{\omega^2-\omega_{Ai}^2}{v_{Ai}^2}, \qquad k_e^2 = \frac{\omega_{Ae}^2-\omega^2}{v_{Ae}^2}. \qquad (113)$$

Then the solution to (112) is

$$\omega^2 = \frac{\rho_i \omega_{Ai}^2 + \rho_e \omega_{Ae}^2}{\rho_i + \rho_e} = \omega_k^2, \tag{114}$$

and for the radial wave numbers k_i and k_e we obtain

$$k_i^2 = k_e^2 = k_z^2 \frac{\rho_i - \rho_e}{\rho_i + \rho_e}. \tag{115}$$

The right hand side of (114) is almost invariably called the square of the kink frequency and denoted as ω_k^2. In the thin tube approximation the frequency is independent of the azimuthal wave number $m \geq 1$ as already noted by Goossens et al. (1992). Hence all flute modes with $m \geq 2$ have the same frequency as the kink mode with $m = 1$. The radial wave numbers k_i and k_e depend in a simple way on the density contrast.

Let us now replace the discontinuous variation of density from its internal value ρ_i to ρ_e by a continuous variation in a non-uniform layer $[R - l/2, R + l/2]$. A fully non-uniform equilibrium state corresponds to $l = 2R$. The continuous variation of ω_A has the important effect that the kink MHD wave, which has its frequency in the Alfvén continuum, interacts with local Alfvén continuum waves and gets damped. This resonant damping is translated in a complex frequency. In the TB approximation we need to add an additional term to the dispersion relation which takes into account the jump in the radial component across the resonant layer where the real part of the kink eigenmode is equal to the local Alfvén frequency $\omega = \omega_A(r_A)$, with r_A being the resonant position. In the thin boundary approximation $r_A = R$ when the Alfvén speed profile is symmetric with respect to $r = R$. The jump in ξ_r is given by (55). The modified version of the ideal dispersion relation (109) is

$$F_\star \frac{J_m'(x_0) K_m(y_0)}{J_m(x_0) K_m'(y_0)} - i G_\star \frac{K_m(y_0)}{K_m'(y_0)} = 1,$$

where $F_\star$ is given by (110) and $G_\star$ is defined as

$$G_\star = \pi \frac{m^2/r_A^2}{\rho|\Delta_A|} \frac{\rho_e(\omega^2 - \omega_{Ae}^2)}{k_e}. \tag{116}$$

$G_\star$ contains the effect of the resonance. When we combine the thin tube (TT) approximation with the thin boundary (TB) approximation, we can simplify the dispersion relation to

$$1 + F_\star \frac{k_e}{k_i} - i G_\star \frac{k_e R}{m} = 0. \tag{117}$$

The zero order solution to (117), i.e. the solution when the effect of the resonance is not taken into account is of course (114). In order to take the effect of the resonance into account we use a complex frequency ω defined by (105) and approximate ω^2 with $\omega_k^2 - 2i\omega_k\gamma_{\mathrm{d}}$ (recall that $\gamma_{\mathrm{d}} = -\omega_i$). The solution for the decrement γ_{d} is

$$\frac{\gamma_{\mathrm{d}}}{\omega_k} = \frac{\pi/2}{\omega_k^2} \frac{m}{R} \frac{\rho_i^2 \rho_e^2}{(\rho_i + \rho_e)^3} \frac{(\omega_{Ai}^2 - \omega_{Ae}^2)^2}{\rho(r_A)|\Delta_A(r_A)|}. \tag{118}$$

Equation (118) agrees with equation (77) of Goossens et al. (1992) when that equation is corrected for a typo as the factor $(\omega_{Ai}^2 - \omega_{Ae}^2)$ should be squared. This is surprising since that result was obtained by Goossens et al. (1992) for surface waves in incompressible plasmas.

In the same section of that paper it was noted that there is no distinction between compressible and incompressible plasmas for surface kink and fluting waves in thin tubes.

Equation (118) shows that the decrement depends linearly on m. Since we are mainly interested in $m = 1$ we shall specialize to that value in the remainder of this subsection. If the variation of ω_A^2 is solely due to the variation of density ρ as is the case here since we have considered a constant vertical magnetic field, (118) can be rewritten as

$$\frac{\gamma_{\mathrm{d}}}{\omega_k} = \frac{\pi}{8}\frac{m}{R}\frac{(\rho_i - \rho_e)^2}{\rho_i + \rho_e}\frac{1}{|\frac{d\rho}{dr}|_{r_A}}. \tag{119}$$

For a linear profile of density

$$\left|\frac{d\rho}{dr}\right|_{r_A} = \frac{\rho_i - \rho_e}{l},$$

so that

$$\frac{\gamma_{\mathrm{d}}}{\omega_k} = \frac{\pi}{8}\frac{l}{R}\frac{\rho_i - \rho_e}{\rho_i + \rho_e}, \qquad \frac{\tau_D}{T} = \frac{4}{\pi^2}\frac{1}{l/R}\frac{\rho_i + \rho_e}{\rho_i - \rho_e}. \tag{120}$$

In (120) τ_D is the damping time and T the period. Note that the expression for $\gamma_{\mathrm{d}}/\omega_k$ given by (120) agrees with (79b) of Goossens et al. (1992).

For a sinusoidal profile of density

$$\left|\frac{d\rho}{dr}\right|_{r_A} = \frac{\pi}{2}\frac{\rho_i - \rho_e}{l}$$

so that

$$\frac{\gamma_{\mathrm{d}}}{\omega_k} = \frac{1}{4}\frac{l}{R}\frac{\rho_i - \rho_e}{\rho_i + \rho_e}, \qquad \frac{\tau_D}{T} = \frac{2}{\pi}\frac{1}{l/R}\frac{\rho_i + \rho_e}{\rho_i - \rho_e}. \tag{121}$$

Here the results agree with those obtained by Ruderman and Roberts (2002). The works by Ruderman and Roberts (2002) and of Goossens et al. (1992) are general and not restricted to a specific profile of density. Ruderman and Roberts (2002) illustrated their work by taking a sinusoidal profile for density. Goossens et al. (1992) gave results for a linear profile of density and a linear profile of Alfvén velocity. These results on the damping rates of kink oscillations in coronal loops illustrate the remarkable versatility of the jump conditions in combination with the TB approximation and the TT approximation. The accuracy of the scheme is illustrated in a recent analytical seismological study by Goossens et al. (2008) which complemented a fully numerical seismology investigation by Arregui et al. (2007a). This might be a good place to recall that Goossens et al. (1992) obtained a general expression for the damping rate of MHD wave modes in the presence of an equilibrium flow, see their equation (76). This expression has been used in a recent study on the damping of kink MHD waves in the presence of a equilibrium flow by Vasheghani Farahani et al. (2009) and Terradas et al. (2010a).

The role of quasi-modes for magnetic solar and space plasmas has been discussed in the first Sect. 5.1. Beautiful examples of mode coupling and quasi-modes can be found in the work on foot point driving by e.g. De Groof and Goossens (2000, 2002), Goossens and De Groof (2001) and De Groof et al. (2002). The quasi-modes and resonant damping are robust phenomena as shown by Terradas et al. (2008) for complicated plasma structures. Quasi-modes and resonant damping have become very popular in the context of transverse MHD waves in coronal loops (see e.g. Ruderman and Roberts 2002; Goossens et al. 2002a, 2008, 2009; Goossens 2008).

The theory on quasi-modes reviewed in this section is general in the sense that the only condition imposed is that the discrete eigenmode must have a frequency with its real part in the Alfvén continuum. The discrete eigenmode can be a surface Alfvén eigenmode or a fast body eigenmode. The surface Alfvén wave is important for at least two reasons. The first is historical. It is the easiest case where an eigenmode with its frequency in the Alfvén continuum can be identified. In ideal MHD the square of the frequency of the surface Alfvén wave is the weighted mean of the squares of the Alfvén frequencies of the two adjacent uniform plasmas. When the true discontinuity in the local Alfvén frequency is replaced with a continuous variation, the frequency of the surface Alfvén eigenmode must be in the Alfvén continuum and the Alfvén resonance cannot be avoided. From this viewpoint it is understandable that the first schemes to heat plasmas by means of Alfvén wave absorption proposed to excite the surface Alfvén wave (see e.g. Grossmann and Tataronis 1973; Hasegawa and Chen 1974; Ionson 1978; Wentzel 1979b). The second reason is that kink waves in a 1-dimensional cylindrical plasma have become popular as a first approximation for explaining the observed transverse oscillations of coronal loops. In particular the fundamental radial mode is important in that respect. The frequency of this fundamental radial mode is in the part of the spectrum where the wave is propagating in the interior of the flux tube and evanescent in the exterior (see e.g. Edwin and Roberts 1983; Goossens et al. 2009). Hence form that point of view the wave could be called a body wave. However, inspection of the eigenfunctions (see e.g. Wentzel 1979a; Goossens et al. 2009 (Figs. 1–2)) reveals that the fundamental radial mode is a surface Alfvén wave both in the uniform and non-uniform case. This phenomenon is also known in the fusion plasma physics literature (see e.g. Appert et al. 1984; Cramer 2001) where it is related to the presence of the Alfvén resonance. Note that the properties of the non-axisymmetric MHD waves deduced by Spruit (1982) for uniform thin flux tubes were recovered and extended by Goossens et al. (2009). These properties lead Spruit to call the non-axisymmetric MHD waves in thin flux tubes transversal. For $m = 1$ the non-axisymmetric MHD wave is the kink transversal MHD wave.

5.3 Initial Value Problem

In this subsection we consider the initial value problem. We divide our discussion in two parts. In the first part we consider the transition to the steady state of oscillation when a resonant wave starts to be driven with a constant frequency at the initial moment of time. In the second part we consider the resonant damping of a wave that was excited by the initial forcing. We will see that these two problems are quite different.

5.3.1 Transition to the Steady State of Oscillation

We consider the transition to the steady state of periodic oscillation when a resonant MHD wave started to be driven by an external periodic driver at the initial moment of time. To our knowledge this problem was first addressed by Kappraff and Tataronis (1977). These authors considered the excitation of resonant MHD waves in a plasma slab bounded by two vacuum regions. The waves are excited by the electrical current in a coil situated in the vacuum regions. The only dissipative process that they took into account was resistivity. The wave started to be driven by the external current at the initial moment of time. Kappraff and Tataronis (1977) solved the initial value problem and showed that initially the perturbation amplitude near the resonant position grows, but then it saturates and the system attains the steady state of driven oscillation. The characteristic time of transition to the steady state of oscillation is proportional to $R_m^{1/3}$, and the amplitude of the steady state oscillation in the resonant layer is also proportional to $R_m^{1/3}$.

Ruderman and Wright (2000) studied the transition to the steady state of driven oscillation of a magnetic cavity. The equilibrium configuration that they used was as follows. In Cartesian coordinates x, y, z the magnetic field is in the z-direction and has a constant magnitude. The plasma is confined between two boundaries at $z = 0$ and $z = L$, and the magnetic field lines are frozen in the dense plasma beyond these boundaries. The plasma density continuously varies in the x-direction in such a way that it is constant in $x < 0$ and $x > a$, while it monotonically decreases in the interval $[0, a]$. As a result the Alfvén speed, $v_A(x)$, is constant in $x < 0$ and $x > a$, while it monotonically increases in the interval $[0, a]$. The transitional layer is assumed to be thin, $a \ll L$. The plasma is driven by a harmonic motion at $z = 0$ polarized in the x-direction. The driver amplitude varies harmonically in the y-direction with the characteristic scale of variation much larger than a, i.e. its amplitude is proportional to $\sin(k_y y)$ with $ak_y \ll 1$.

Ruderman and Wright (2000) studied the plasma motion in the cavity using the linearized MHD equations for a cold plasma. They took resistivity into account, but assumed that the magnetic Reynolds number is large, $R_m \gg 1$. The cavity possesses a resistive eigenmode (quasi-mode in the ideal plasma approximation). The real parts of the frequency of the fundamental eigenmode and the overtones are approximately equal to the surface wave frequency on the magnetic interface that we obtain in the limit $a \to 0$. They are given by (Ruderman 1991)

$$\omega_{gn}^2 = \frac{1}{2}\left\{(v_{A1}^2 + v_{A2}^2)\kappa_n^2 - [(v_{A1}^2 - v_{A2}^2)^2\kappa_n^4 + 4v_{A1}^2 v_{A2}^2 k_y^4]^{1/2}\right\}. \tag{122}$$

For a general discussion of surface waves on magnetic interfaces see, e.g., Roberts (1981). In (122) $n = 1$ corresponds to the fundamental mode in the z-direction, and $n = 2, 3, \ldots$ to the overtones. The quantities v_{A1} and v_{A2} are the values of v_A in $x < 0$ and $x > a$ respectively, and κ_n is given by

$$\kappa_n^2 = k_y^2 + n^2 k_z^2,$$

where $k_z = \pi/L$. It is straightforward to show that $v_{A1} n k_z < \omega_{gn} < v_{A2} n k_z$. This inequality implies that there is an infinite series of resonant positions x_{gn}, $n = 1, 2, \ldots$, defined by $v_A(x_{gn}) n k_z = \omega_{gn}$. The first resonant position corresponds to the fundamental mode in the z-direction, and the others to the overtones. The resonances cause damping of the fundamental mode and overtones with the decrements γ_n of the order of $(a/L)\omega_{gn}$. Hence, the damping is weak when the transitional layer is thin, $\gamma_n/\omega_{gn} \ll 1$.

Ruderman and Wright (2000) assumed that the driving frequency Ω is within the fundamental continuum, $v_{A1} k_z < \Omega < v_{A2} k_z$. This implies that there is one additional resonant position x_A defined by $v_A(x_A) k_z = \Omega$.

It is important to distinct between the resonant and non-resonant driving. We call the driving non-resonant when $|\Omega - \omega_{g1}| \gg \gamma_1$. When $t \to \infty$ the cavity attains the steady state of oscillation with the frequency Ω. The oscillation amplitude is of the order of the driver amplitude. The cavity as a whole oscillates in the x-direction. This global motion is described by the equation of the damped oscillator, so that it attains the steady of oscillation after a time of the order of γ_1^{-1}. The local motions are the motions in the vicinity of the resonant positions. They are mainly polarized in the y-direction. They attain the steady state of oscillations after the time t_{tr} given by

$$t_{tr} = \max\left(\gamma_1^{-1}, R_m^{1/3}\Omega^{-1}\right). \tag{123}$$

The oscillation amplitude in the dissipative layer embracing x_A is of the order of $R_m^{1/3}$ times the driver amplitude. The oscillation amplitude in the dissipative layer embracing x_{q1} is of the same order when $a/L \lesssim R_m^{-1/3}$ (so that $\gamma_1/\Omega \lesssim R_m^{-1/3}$), but it is only of the order of L/a times the driver amplitude when $a/L \gg R_m^{-1/3}$ (so that $\gamma_1/\Omega \gg R_m^{-1/3}$).

In the case of the resonant driving, defined by the condition $|\Omega - \omega_{g1}| \lesssim \gamma_1$, the transitional times for the global and local motions are the same. However now the amplitude of the global motion is much larger than in the case of non-resonant driving. It is now of the order of L/a times the driver amplitude. The amplitudes of local motions in the resonant layers embracing x_A and x_{g1} are of the order of $(L/a)R_m^{1/3}$. Note that these two layers overlap when $a/L \lesssim R_m^{-1/3}$. The main results of this analysis are summarized in the table taken from Ruderman and Wright (2000).

		Non-resonant driving		Quasi-resonant driving	
		$\gamma \lesssim \Omega R^{-1/3}$	$\gamma \gg \Omega R^{-1/3}$	$\gamma \lesssim \Omega R^{-1/3}$	$\gamma \gg \Omega R^{-1/3}$
Global or coherent motion	Amplitude	$\sim f_0$	$\sim f_0$	$\sim Lf_0/a$	$\sim Lf_0/a$
	Transitional time to the steady state	$\sim \gamma^{-1}$	$\sim \gamma^{-1}$	$\sim \gamma^{-1}$	$\sim \gamma^{-1}$
Dissipative layer embracing x_A	Amplitude of v	$\sim f_0 R^{1/3}$	$\sim f_0 R^{1/3}$	$\sim f_0(L/a)R^{1/3}$	$\sim f_0(L/a)R^{1/3}$
	Transitional time to the steady state	$\sim \gamma^{-1}$	$\sim \Omega^{-1}R^{1/3}$	$\sim \gamma^{-1}$	$\sim \Omega^{-1}R^{1/3}$
Dissipative layer embracing x_{gn}	Amplitude of v	$\sim f_0 R^{1/3}$	$\sim Lf_0/a$	$\sim f_0 R^{1/3}$ $(n > 1)$	$\sim Lf_0/a$ $(n > 1)$
	Transitional time to the steady state	$\sim \gamma^{-1}$	$\sim \Omega^{-1}R^{1/3}$	$\sim \gamma^{-1}$	$\sim \Omega^{-1}R^{1/3}$

In this table f_0 is the driver amplitude, $\gamma = \gamma_n$, $R = R_m$ and v is the velocity component in the y-direction.

It follows from this table that, in the case where $\gamma \gg \Omega R^{-1/3}$, $t_{tr} \sim R_m^{1/3}$ no matter if the driving is resonant or non-resonant. However it was obtained in the numerical simulations by Poedts and Kerner (1992) and Ofman et al. (1994) that $t_{tr} \sim R_m^{\alpha}$ with $\alpha \approx 0.22$. At present it is not clear what causes this discrepancy between the analytical theory and numerical modelling. One possible conjecture is that both Poedts and Kerner (1992) and Ofman et al. (1994) took not large enough values of R_m to obtained the asymptotic formula for t_{tr} in the case of resonant driving. However we should note that Poedts and Kerner (1992) took R_m in the interval from 10^6 to 10^8, which gives $R_m^{1/3} = 100$–464. These values of $R_m^{1/3}$ look sufficiently large to produce the asymptotic scaling for t_{tr}.

5.3.2 *Impulsive Excitation*

The problem of impulsive excitation of eigenmodes in dissipative MHD and quasi-mode in ideal MHD has been studied by many authors (e.g. Ionson 1978; Rae and Roberts 1982; Lee and Roberts 1986; Hollweg 1987a, 1988; Steinolfson and Davila 1993). At present this problem has a very important application to solar physics because it is related to the description of damped transverse oscillations of coronal loops. Motivated by observations of these oscillations Ruderman and Roberts (2002) solved the initial value problem describing the excitation and subsequent damping of kink oscillations of arbitrarily perturbed coronal loop. In what follows we briefly describe their analysis and the main results.

Ruderman and Roberts (2002) have used the cold plasma approximation relevant for the solar corona. The equilibrium state is as follows. In cylindrical coordinates r, φ, z the magnetic field is in the z-direction and has constant magnitude. The plasma density is equal to ρ_e in $r > a$ and ρ_i in $r < a - \ell$, where ρ_e and ρ_i are constants. The density monotonically decreases from ρ_i to ρ_e in the annulus $a - \ell < r < a$. The plasma is confined between the planes $z = 0$ and $z = L$, and the magnetic field lines are frozen in the dense plasma beyond these planes.

The system is arbitrarily perturbed in the initial moment of time. The evolution of the perturbed system is described by the system of linearized viscous MHD equations. Viscosity is assumed to be weak, so it is only taken into account in the dissipative layer embracing the resonant position. Since the compressional viscosity does not remove the Alfvén singularity and practically does not affect the motion in the Alfvén dissipative layer (e.g. Ofman et al. 1994; Erdélyi and Goossens 1995), only shear viscosity is taken into account.

Ruderman and Roberts (2002) have used the thin tube thin boundary (TTTB) layer approximation and assumed that $a \ll L$ and $\ell \ll a$. The second assumption enabled them to neglect the magnetic pressure variation in the annulus. The solution of the initial value problem was obtained with the use of the Laplace transform. It is represented by the integral along the Bromwich contour from the solution to the Laplace-transformed equations. This solution is a multi-valued function of ω. To obtain a single-valued branch we need to make cuts in the complex ω-plane. Here we use the opportunity to correct one error made by Ruderman and Roberts (2002). They claimed that the solution to the Laplace-transformed equations has four logarithmic branch-points at $\omega = \pm\omega_{Ai}$ and $\omega = \pm\omega_{Ae}$, where $\omega_{Ai} = k_z v_{Ai}$, $\omega_{Ae} = k_z v_{Ae}$, v_{Ai} and v_{Ae} are the values of the Alfvén velocity in $r < a - \ell$ and $r > a$ respectively, and $k_z = \pi/L$. However thorough examination shows that there are logarithmic branch points only at $\omega = \pm\omega_{Ae}$, while $\omega = \pm\omega_{Ai}$ are regular points. Correspondingly the branch cuts in Fig. 3 in Ruderman and Roberts (2002) have to start not from $\omega = \pm\omega_{Ai}$ but from $\omega = \pm\omega_{Ae}$. Fortunately this error did not affect the results obtained in the paper.

To study the asymptotic behaviour of the solution to the initial value problem we have to close the Bromwich integration contour. Then the asymptotic behaviour is determined by the simple poles of the Laplace transform at $\omega = \omega_{\rm d}$ and $\omega = -\omega_{\rm d}^*$, where the asterisk indicates the complex conjugate quantity, and $\omega_{\rm d} = \omega_k - i\gamma_{\rm d}$ with ω_k and $\gamma_{\rm d}$ being given by (114) and (119) respectively. These simple poles correspond to the viscous eigenmode of the magnetic tube configuration considered here. The main result of the analysis is that the oscillation with the frequency ω_k emerges after the time $t_{\rm tr}$ satisfying $t_{\rm tr}\omega_k \gg 1$. Since the oscillation period is $2\pi/\omega_k$, this condition is satisfied for $t_{\rm tr}$ of the order of just a couple of oscillation periods. After the harmonic oscillations emerged from an arbitrary initial perturbation, it starts to decay with the decrement $\gamma_{\rm d}$. It is very important that $\gamma_{\rm d}$ is independent of dissipation coefficients and is completely defined by the ratio ℓ/a and the density variation in the annulus. Thus, if we accept that the damping of coronal loop kink oscillations is caused by resonant absorption, then, contrary to the statement made by Nakariakov et al. (1999), the observations of damped transverse oscillations of coronal magnetic loops cannot be used to determine the dissipative coefficients in the corona.

When $\gamma_{\rm d} \lesssim R_e^{-1/3}\omega_k$, where R_e is the Reynolds number calculated using the shear viscosity, the motion in the dissipative layer embracing the resonant position at $r = r_A$, where

r_A is determined by $v_A(r_A) = \omega_k$, is quasi-stationary. The thickness of the dissipative layer is of the order of δ_A given by (41), and the maximum amplitude of this motion is of the order of $AR_e^{1/3}$, where A is the amplitude of the global eigenmode. However, in the solar corona $\gamma_{\rm d} \gg R_e^{-1/3}\omega_k$. In that case the thickness of the dissipative layer is of the order of ℓ^2/a, and the maximum amplitude of motion in the dissipative layer is of the order of aA/ℓ. The characteristic time of damping of the local motion is of the order of $R_e^{1/3}\omega_k^{-1}$.

When $\gamma_{\rm d} \gg R_e^{-1/3}\omega_k$, the damping of the global coherent motion is not related with the energy dissipation. Rather it is the energy conversion from the global motion to the local incoherent motion in the vicinity of the resonant position at r_A. Although traditionally this process is considered as the energy transfer from the global mode to the local Alfvénic oscillations, in fact this is simply the evolution of one wave mode. This evolution occurs in such a way that, initially, the energy is distributed over the whole volume of the oscillating magnetic tube. But then, when the time progresses, the energy concentrates more and more near r_A. The energy dissipation occurs only when the local motion damps.

For a typical damped transverse oscillation of a coronal magnetic loop the oscillation period is about a few minutes, and the damping time is about 10 min. The typical value of the Reynolds number in the solar corona is $R_e \sim 10^{12}$. Hence, the typical time of energy dissipation is of the order of 100 hours. We should emphasize that this estimate obtained using the classical formula for the viscosity coefficients given by Braginskii (1965). Quite possible that, due to the presence of large shear velocities, the motion near r_A becomes unstable. This instability can lead to turbulence of the motion that can strongly increase the viscosity coefficients and, consequently, reduce the energy dissipation time. However this possible reduction in the energy dissipation time does not affect at all the theoretically predicted damping time of the coronal loop kink oscillations due to resonant absorption.

6 Summary

This paper has tried to give an overview of our knowledge of resonant MHD waves in the solar atmosphere. Our understanding of MHD waves in the Sun's atmosphere has evolved from a mainly theoretical approach in the 1960s and 1970s with limited observational support, however due to improvements in instrumentation now they are an observationally well documented phenomenon. Observations of MHD waves are becoming increasingly more accurate as advancements in technology make it possible to collect more detailed data with higher spatial and temporal resolution. It is now clear that MHD waves are ubiquitous in the solar atmosphere. The plasma of the solar atmosphere is characterized by structuring and non-uniformity and it is this non-uniformity that produces resonant MHD waves.

The emphasis of the present paper has been on the mathematical analysis and basic properties of resonant MHD waves. The main results of the mathematical analysis are: (i) MHD waves in non-uniform plasmas have mixed properties and behave differently in different parts of the plasma according to the local plasma conditions. An MHD wave that behaves predominantly as a fast wave can be transformed into a wave that is predominantly an Alfvén wave during its journey in a spatially varying plasma background; (ii) Resonant MHD waves are a natural phenomenon in non-uniform plasmas. They are difficult to avoid in non-uniform plasmas. (iii) Damping of resonant MHD waves is an efficient damping mechanism.

Resonant MHD waves are here to stay and have still undiscovered fascinating behaviour in store for us. As already said in the previous paragraph our emphasis has been on mathematical analysis and we have not been able, partly because of matters of space, to explore all possible relations of the present theory to observations. The observations that we have linked to resonant MHD waves in the present paper have been limited to transverse standing MHD waves in coronal loops. Resonant MHD waves can explain the observed strong damping and the theory of resonantly damped transverse MHD waves allows us to do seismology in the time domain using observed and computed values of the period and damping time. This is definitely a success story of resonant MHD waves and we are happy about that. However, standing transverse MHD waves in coronal loops are rare phenomena as they are caused a nearby flare or other larger scale energetic events like coronal mass ejections (CMEs). A far larger reservoir of possibilities has been opened recently. Observations by e.g. Tomczyk et al. (2007) and Tomczyk and McIntosh (2009) have shown that propagating MHD waves are everywhere in the corona. Verth et al. (2010) have used theoretical results by Terradas et al. (2010b) to show that the observed damping of these propagating coronal MHD waves can be explained by resonant damping. Hence resonant MHD waves are important both in their standing and propagating version. This widens the role of resonant waves in the solar atmosphere considerably. In addition resonant MHD waves are very robust. They do not require a fully ionized plasma. Soler et al. (2009b) have shown that resonant MHD waves survive in partially ionized plasmas and this finding opens an avenue of applications on standing and propagating MHD waves, e.g., in chromospheric and prominence plasmas. Such damping of MHD waves in chromospheric spicules has now been observed by He et al. (2009a, 2009b). In short, resonant MHD waves will continue to help us understand the fascinating physics of MHD waves observed in the solar atmosphere.

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