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Convection in the Presence of an Inclined Axis of Rotation with Applications to the Sun

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Abstract Thermal convection in a horizontal fluid layer heated from below and rotating about an arbitrary axis is studied analytically with the attention focused on mean flows and drifts generated by the convection velocity field. Mean flows occur in both horizontal directions when the angle between the rotation vector and the vertical is finite but less than 90°. In the case of a hexagonal convection pattern, a wavelike drift is found in the presence of a horizontal component of rotation. Applications to solar convection are discussed. Considering the simplicity of the model the agreement with observations is surprisingly good.

Keywords Supergranulation · Rotation

1. Introduction

Analytical models for buoyancy-driven convection in natural systems have long been used for understanding basic dynamical features of those systems. Through the explicit parameter dependencies and symmetry properties of the analytical solutions, important insights may be gained that are often not easily available from results of numerical computations. Although nonlinear features of convection can usually be studied analytically only through perturbation expansions in powers of the amplitude, such expansions suffice to describe, at least qualitatively, most nonlinear properties even when the resulting equations are truncated at the cubic order of the amplitude. Although a quantitatively correct description of convection in natural systems cannot be achieved in this way, it is surprising how well the so-called weakly-nonlinear analysis fits the observations when the action of turbulence caused by the neglect of motions of smaller scales is taken into account through the use of eddy diffusivities. Such successes are the result of the property that the basic modes of convection and their interaction treated in the weakly nonlinear analysis are still the dominant ones in the large-scale convection flows of the natural systems.

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In this spirit, the analysis of this paper is devoted to the effects of rotation on convection in a fluid layer heated from below when the angular velocity vector ($\mathbf{\Omega}$) of rotation possesses a horizontal as well as a vertical component. This program of research has been carried out in part already in the early 1980s when Hathaway (1982) and Hathaway and Somerville (1983) numerically investigated convection in a fluid layer heated from below and rotating about an inclined axis of rotation. Since they used a symmetric layer with no-slip conditions at top and bottom boundaries, the mean flow vanished when averaged over the height of the layer, but the interior shear clearly reflects the Reynolds stress generated by the convection flow. A similar, but analytical, analysis has been presented by Busse (1982), who used stress-free conditions at top and bottom boundaries. Here we shall revisit the weakly-nonlinear analysis in the case when the Coriolis force is treated as a perturbation and more general boundary conditions are considered. Particular attention will be devoted to mean flows and wavelike drifts generated by convection since these can be related most readily to observations on the Sun. We shall apply the results to convection on the supergranular scale because this scale is known to be large enough for flows to be influenced by the rotation of the Sun.

In the following, the mathematical problem and the method of solution will be formulated in Section 2. The various orders of the expansion in powers of the amplitude (ϵ) and of the rotation parameters will be described in Sections 3 to 5. In Section 6 solar applications will be discussed.

2. Mathematical Formulation of the Problem

We are considering a horizontal fluid layer of height d that is heated homogeneously from below and cooled from above such that the average temperatures T_2 and T_1 are attained at the lower and the upper boundaries of the layer, respectively. The layer is assumed to be rotating with a constant angular velocity vector ($\mathbf{\Omega}$) that is inclined with respect to the vertical as indicated in Figure 1. Accordingly, we assume $\mathbf{\Omega} = (0, \Omega_y, \Omega_z)$, where a Cartesian coordinate system has been introduced with the z-coordinate in the vertical direction. This formulation of the problem applies locally to convection in a thin, rotating, spherical shell at a colatitude ϕ given by $\tan\phi = \Omega_y/\Omega_z$. In contrast to most other analyses, symmetry of the convection layer with respect to its midplane will not be assumed. Different boundary conditions at top and bottom will be used, but for most of the analysis they do not have to be specified explicitly. We also allow for deviations from the Boussinesq approximation

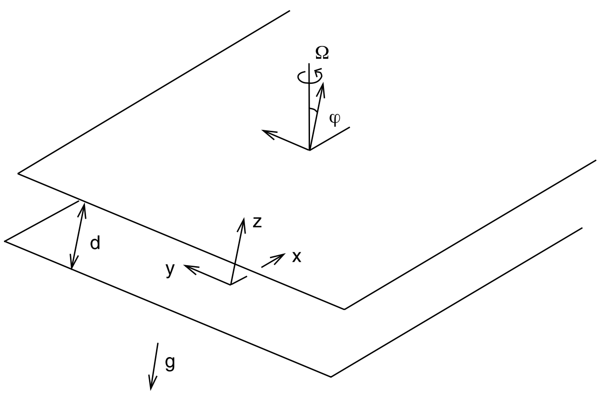


Figure 1 Sketch of the geometrical configuration of the problem.

without denoting them explicitly. They can be taken into account as additional perturbations of the solutions as in the paper of Busse (1967).

For the dimensionless description of the problem we use d, d^2/κ, and $(T_2 - T_1)/R$ as scales for length, time, and temperature, respectively, where κ is the thermal diffusivity of the fluid and R is the Rayleigh number to be defined in the following. The dimensionless equations for the velocity vector ($\mathbf{v}$) and the deviation (Θ) of the temperature field from the static state of pure conduction can be written in the form

$$\nabla^2\mathbf{v} + \mathbf{k}\Theta - \tau_z\mathbf{k}\times\mathbf{v} - \tau_y\mathbf{j}\times\mathbf{v} - \nabla\pi = P^{-1}(\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}), \tag{1a}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{1b}$$

$$\nabla^2\Theta + R\mathbf{k}\cdot\mathbf{v} = \mathbf{v}\cdot\nabla\Theta + \partial_t\Theta, \tag{1c}$$

where π is the deviation of the pressure from its static distribution and where Rayleigh number R, Prandtl number P, and Coriolis parameters τ_y and τ_z are defined by

$$R = \alpha g(T_2 - T_1)d^3/\kappa\nu, \qquad P = \nu/\kappa, \qquad \tau_y = 2\Omega_y d^2/\nu, \qquad \tau_z = 2\Omega_z d^2/\nu. \tag{2}$$

Here α is the coefficient of thermal expansion and ν is the kinematic viscosity of the fluid. The unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ correspond to the x-, y-, and z-directions. It is convenient to introduce the general representation for a solenoidal vector field:

$$\mathbf{v} = \mathbf{U} + \nabla\times(\nabla\Phi\times\mathbf{k}) + \nabla\Psi\times\mathbf{k} \equiv \mathbf{U} + \boldsymbol{\delta}\Phi + \boldsymbol{\eta}\Psi, \tag{3}$$

where the poloidal and toroidal functions Φ and Ψ can be chosen such that their averages over planes $z =$ constant vanish, $\overline{\Phi} = \overline{\Psi} = 0$, and where $\mathbf{U}$ represents the horizontal average of $\mathbf{v}$, $\mathbf{U} = \overline{\mathbf{v}}$. By taking the z-components of the curl and of the curl curl of (1a) we obtain equations for Φ and Ψ. In addition we need equations for Θ and $\mathbf{U}$:

$$\left(P^{-1}\partial_t - \nabla^2\right)\Delta_2\Psi - (\tau_y\partial_y + \tau_z\partial_z)\Delta_2\Phi = -P^{-1}\boldsymbol{\eta}\cdot(\mathbf{v}\cdot\nabla\mathbf{v}), \tag{4a}$$

$$\left(P^{-1}\partial_t - \nabla^2\right)\nabla^2\Delta_2\Phi + \Delta_2\Theta + (\tau_y\partial_y + \tau_z\partial_z)\Delta_2\Psi = -P^{-1}\boldsymbol{\delta}\cdot(\mathbf{v}\cdot\nabla\mathbf{v}), \tag{4b}$$

$$\left(\partial_t - \nabla^2\right)\Theta + R\Delta_2\Phi = -(\boldsymbol{\delta}\Phi + \boldsymbol{\eta}\Psi + \mathbf{U})\cdot\nabla\Theta, \tag{4c}$$

$$\left(\partial^2_{zz} - P^{-1}\partial_t\right)\mathbf{U} - \tau_z\mathbf{k}\times\mathbf{U} = -\partial_z\overline{\Delta_2\Phi(\nabla_2\partial_z\Phi + \boldsymbol{\eta}\Psi)}/P, \tag{4d}$$

where ∇_2 and Δ_2 are defined by $\nabla_2 = \nabla - \mathbf{k}\mathbf{k}\cdot\nabla$ and $\Delta_2 = \nabla_2^2$. We shall solve these equations in terms of power series in the small parameters ϵ, τ_y and τ_z,

$$\Phi = \epsilon\left(\Phi_1^{(0)} + \tau_y^2\hat{\Phi}_1^{(2)} + \tau_y\tau_z\check{\Phi}_1^{(2)} + \tau_z^2\tilde{\Phi}_1^{(2)} + \cdots\right) + \epsilon^2\left(\Phi_2^{(0)} + \tau_y\hat{\Phi}_2^{(1)} + \cdots\right) + \cdots, \tag{5a}$$

$$\Psi = \epsilon\left(\tau_y\hat{\Psi}_1^{(1)} + \tau_z\tilde{\Psi}_1^{(1)} + \cdots\right) + \epsilon^2\left(\Psi_2^{(0)} + \cdots\right) + \cdots, \tag{5b}$$

$$\Theta = \epsilon\left(\Theta_1^{(0)} + \tau_y^2\hat{\Theta}_1^{(2)} + \tau_y\tau_z\check{\Theta}_1^{(2)} + \tau_z^2\tilde{\Theta}_1^{(2)} + \cdots\right) + \epsilon^2\left(\Theta_2^{(0)} + \tau_y\hat{\Theta}_2^{(1)} + \cdots\right) + \cdots, \tag{5c}$$

$$R = R_0^{(0)} + \epsilon^2 R_2^{(0)} + \cdots. \tag{5d}$$

In writing these representations we have already taken into account the property that $\Psi_1^{(0)}$ vanishes according to the linear order of Equation (4a) in the absence of rotation. As a consequence the effects of rotation enter only quadratically into the expressions for Θ and Φ.

The amplitude ϵ is defined by $\epsilon = \langle \Theta \Theta_1^{(0)} \rangle$, where the angle brackets indicate the average over the entire fluid layer. This condition yields the normalization condition

$$\left\langle \Theta_m^{(n)} \Theta_1^{(0)} \right\rangle = \delta_{1m} \delta_{0n}. \tag{6}$$

3. Solutions in the Linear Order of the Problem

Since the lowest order of Equation (4) describes convection in a horizontally isotropic layer the solution can be written in the form

$$\Theta_1^{(0)} = g(z) \sum_{l=-N}^{N} A_l \exp\{\mathrm{i}\mathbf{q}_l \cdot (\mathbf{r} - \mathbf{c}t)\}, \qquad \Phi_1^{(0)} = f(z) \sum_{l=-N}^{N} A_l \exp\{\mathrm{i}\mathbf{q}_l \cdot (\mathbf{r} - \mathbf{c}t)\}, \tag{7}$$

where the vectors $\mathbf{q}_l$ obey the restriction $\mathbf{k} \cdot \mathbf{q}_l = 0$, $|\mathbf{q}_l| = q_c = \text{const.}$, $\mathbf{q}_{-l} = -\mathbf{q}_l$ for $l = 1, \ldots, N$. To have a real expression (11) $A_l = A_{-l}^+$ must be required, where A_l^+ denotes the complex conjugate of A_l. The normalization condition (9) is satisfied by the requirement

$$\sum_{l=-N}^{N} |A_l|^2 = 1, \qquad \left\langle g(z)^2 \right\rangle = 1. \tag{8}$$

We have included in expressions (7) the possibility of a small drift with the phase-velocity vector $\mathbf{c}$. Such a phase velocity can enter the problem only as a nonlinear property since in the linear order of Equation (4) only steady solutions exist in the absence of rotation. In a rotating layer, wavelike solution may enter, but only when the rotation exceeds a finite threshold value that is excluded in our formulation. Solutions (7) of special interest are those with $N = 1$ and $N = 3$ describing convection rolls and hexagonal convection cells, respectively. For the latter

$$\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 = 0 \tag{9}$$

must be required. Solutions with $A_l \equiv A = 1/\sqrt{6}$, $-3 \le l \le 3$, describe hexagonal cells with rising motion in the center, whereas those with $A_l \equiv A = -1/\sqrt{6}$, $-3 \le l \le 3$, correspond to descending motion in the center.

Given expressions (7), solutions $\hat{\Psi}_1^{(1)}$ and $\tilde{\Psi}_1^{(1)}$ can be written in the form

$$\hat{\Psi}_1^{(1)} = \mathrm{i}\hat{h}(z) \sum_{l=-N}^{N} \mathbf{q}_l \cdot \mathbf{j} A_l \exp\{\mathrm{i}\mathbf{q}_l \cdot (\mathbf{r} - \mathbf{c}t)\},$$
$$\tilde{\Psi}_1^{(1)} = \tilde{h}(z) \sum_{l=-N}^{N} A_l \exp\{\mathrm{i}\mathbf{q}_l \cdot (\mathbf{r} - \mathbf{c}t)\}, \tag{10}$$

where $\hat{h}(z)$ and $\tilde{h}(z)$ are real functions. For later use we also need an expression for $\check{\Phi}_1^{(2)}$,

$$\check{\Phi}_1^{(2)} = \mathrm{i}\check{f}(z) \sum_{l=-N}^{N} \mathbf{q}_l \cdot \mathbf{j} A_l \exp\{\mathrm{i}\mathbf{q}_l \cdot (\mathbf{r} - \mathbf{c}t)\}. \tag{11}$$

In the particular case investigated by Busse (2003), a no-slip boundary at the bottom and a stress-free boundary at the top were used together with nearly insulating conditions for the temperature Θ,

$$\Phi = \partial_z \Phi = \Psi = 0, \qquad \mathbf{U} = 0, \qquad \partial_z \Theta = \beta q \Theta \quad \text{at } z = 0, \tag{12a}$$

$$\Phi = \partial^2_{zz} \Phi = \partial_z \Psi = 0, \qquad \partial_z \mathbf{U} = 0, \qquad \partial_z \Theta = -\beta q \Theta \quad \text{at } z = 1. \tag{12b}$$

Here β is a small parameter that denotes the ratio between the thermal conductivities of the boundary and the fluid, and q is the wavenumber for which we usually assume the critical value q_c. The functions $g(z)$, $f(z)$, $\hat{h}(z)$, and $\tilde{h}(z)$ assume the form

$$\begin{aligned} &g(z) \equiv 1, \qquad f(z) = z^2(z-1)\left(z - \frac{3}{2}\right)/4!, \\ &\hat{h}(z) = \left[-z^6 + \frac{15}{4}\left(z^5 - z^4\right) + \frac{9}{4}z\right]/6!, \end{aligned} \tag{13}$$

where terms of the order $\beta^{2/3}$ given in Busse (2003) have been neglected. Here we add the functions

$$\begin{aligned} &\tilde{h}(z) = z^3\left(-z^2 + \frac{25}{8}z - \frac{5}{2}\right)/5!, \\ &\check{f}(z) = z^2\left(-\frac{2}{9}z^7 + \frac{5}{4}z^6 - 2z^5 + \frac{21}{4}z^2 + \frac{55}{9}z - \frac{11}{6}\right)/8!, \end{aligned} \tag{14}$$

which had not been included in Busse (2003). It should be mentioned that because of a sign error in Equations (5a) and (5b) of that paper, τ_y should be replaced by $-\tau_y$ in all explicit results of that paper.

4. Mean Flow Driven by Convection

Equation (4d) yields in the lowest nonvanishing orders

$$\begin{aligned} \frac{d^2}{dz^2}\mathbf{U} &= \tau_z \mathbf{k} \times \mathbf{U} \\ &\quad - \epsilon^2 \partial_z \overline{\left(\Delta_2 \Phi_1^{(0)}\left(\tau_y \boldsymbol{\eta} \hat{\Psi}_1^{(1)} + \tau_y \tau_z \nabla_2 \partial_z \check{\Phi}_1^{(2)}\right) + \tau_y \tau_z \Delta_2 \check{\Phi}_1^{(2)} \nabla_2 \partial_z \Phi_1^{(0)}\right)}/P \\ &= \tau_z \mathbf{k} \times \mathbf{U} - \epsilon^2 q_c^2 P^{-1} \\ &\quad \times \sum_{l=-N}^{N} \mathbf{q}_l \cdot \mathbf{j}\,|A_l|^2 \frac{d}{dz}\big(\mathbf{q}_l \times \mathbf{k}\tau_y f(z)\hat{h}(z) \\ &\quad + \mathbf{q}_l \tau_y \tau_z\left[f(z)\check{f}'(z) - \check{f}(z) f'(z)\right]\big), \end{aligned} \tag{15}$$

where $f'(z)$ indicates the derivative of $f(z)$ and where we have used the property of the functions $\hat{\Phi}_1^{(2)}$, $\tilde{\Phi}_1^{(2)}$, and $\tilde{\Psi}_1^{(1)}$ that they can contribute under the horizontal average only in higher orders because of their spatial symmetry.

Integration of (15) yields

$$\mathbf{U} = -\epsilon^2 q_c^2 P^{-1} \left\{ \sum_{l=-N}^{N} \mathbf{q}_l \cdot \mathbf{j} |A_l|^2 \big(\mathbf{q}_l \times \mathbf{k} \tau_y F(z) + \mathbf{q}_l \tau_y \tau_z \big[G(z) + H(z) \big] \big) \right\} + \cdots , \quad (16)$$

where terms of higher order than those retained have not been denoted explicitly. It is important to consider both the order τ_y and the order $\tau_y \tau_z$ of the mean flow in this expression, since they are obviously directed in different horizontal directions. The functions $F(z)$, $G(z)$, and $H(z)$ are defined as the solutions of the differential equations

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}z} F(z) = f(z) \hat{h}(z), \qquad \frac{\mathrm{d}}{\mathrm{d}z} G(z) = -\int_z^1 F(\zeta)\,\mathrm{d}\zeta, \\ &\frac{\mathrm{d}}{\mathrm{d}z} H(z) = f(z) \check{f}'(z) - \check{f}(z) f'(z), \end{aligned} \quad (17)$$

where a stress-free condition for the mean flow at the upper boundary at $z = 1$ has been used. Since it is assumed that the vertical component of the velocity vanishes at that boundary, $f(1) = \check{f}(1) = 0$, $\frac{\mathrm{d}}{\mathrm{d}z}\mathbf{U}$ does indeed vanish at $z = 1$. The condition for $\mathbf{U}$ at the lower boundary will determine the constants of integration for the solutions of (17).

In the particular case defined by the boundary conditions (12), $F(z)$ and $H(z)$ are given by

$$F(z) = z^4 \left(-\frac{8}{11} z^7 + 5z^6 - 13z^5 + 15z^4 - \frac{45}{7} z^3 + 3z^2 - 9z + \frac{27}{4} \right) / (8 \times 6! \times 4!), \quad (18a)$$

$$\begin{aligned} H(z) = &- z^4 \left(\frac{80}{13} z^9 - 50z^8 + \frac{1725}{11} z^7 - 225z^6 \right. \\ &\left. + 120z^5 + \frac{505}{7} z^3 - 145z^2 + 66z \right) / (8 \times 9! \times 4!). \end{aligned} \quad (18b)$$

These functions as well as $G(z)$ have been plotted in Figure 2.

5. A Wavelike Drift

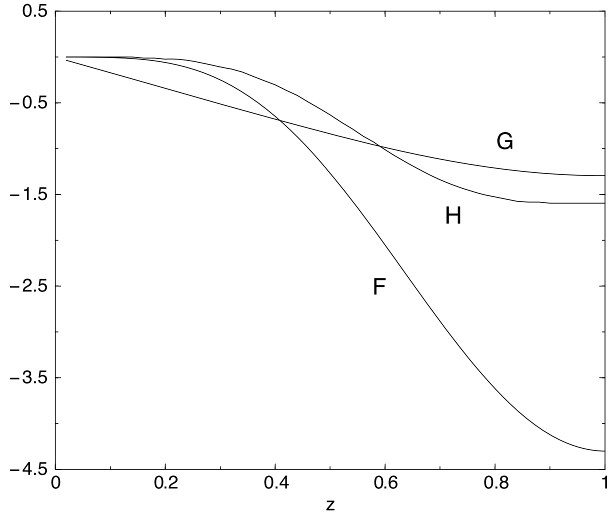
A finite phase-velocity vector $\mathbf{c}$ of the order $\epsilon^2 \tau_y$ is required when the solvability condition in the order $\epsilon^2 \tau_y$ of Equations (4b), (4c) is considered,

$$\nabla^4 \Delta_2 \hat{\Phi}_2^{(1)} - \Delta_2 \hat{\Theta}_2^{(1)} = P^{-1} \big(\boldsymbol{\delta} \cdot \big(\boldsymbol{\delta} \Phi_1^{(0)} \cdot \nabla \boldsymbol{\eta} \Psi_1^{(1)} + \boldsymbol{\eta} \Psi_1^{(1)} \cdot \nabla \boldsymbol{\delta} \Phi_1^{(0)} \big) - \mathbf{c}_1^{(1)} \cdot \nabla \Phi_1^{(0)} \big), \quad (19a)$$

$$\nabla^2 \hat{\Theta}_2^{(1)} - R_0^{(0)} \Delta_2 \hat{\Phi}_2^{(1)} = \boldsymbol{\eta} \Psi_1^{(1)} \cdot \nabla \Theta_1^{(0)} - \mathbf{c}_1^{(1)} \cdot \nabla \Theta_1^{(0)} + R_1^{(1)} \Delta_2 \Phi_1^{(0)}, \quad (19b)$$

where we have used $\mathbf{c} = \epsilon \tau_y \mathbf{c}_1^{(1)} + \cdots$. The solvability condition for this system of equations requires that the right-hand side must be orthogonal to all solutions of the homogeneous system. Accordingly, we multiply Equation (19a) by $f(z) R_0^{(0)} \exp\{-\mathrm{i}\mathbf{q}_n \cdot (\mathbf{r} - \mathbf{c}t)\}$ and Equation (19b) by $g(z) \exp\{-\mathrm{i}\mathbf{q}_n \cdot (\mathbf{r} - \mathbf{c}t)\}$, add the results, and average them over the fluid layer. Because the homogeneous system is self-adjoint, the left-hand side vanishes,

Figure 2 The functions $-F(z)(\times 10^6)$, $G(z)(\times 10^6)$ and $H(z)(\times 10^8)$.



while the right-hand side provides the desired solvability conditions for $-N \leq n \leq N$,

$$\begin{aligned}
&\mathbf{c}_1^{(1)} \cdot \mathbf{q}_n A_n \langle q_c^2 R_0^{(0)} (q_c^2 ff - ff'')/P + g^2 \rangle \\
&\quad = -\sum_{l,n} A_l A_m \mathbf{q}_l \cdot \mathbf{j} \big(\mathbf{k} \cdot \mathbf{q}_l \times \mathbf{q}_m \big[\big(\mathbf{q}_n \cdot \mathbf{q}_m \langle hf'f' \rangle + q_c^4 \langle hff \rangle \big) R_0^{(0)}/P + \langle hg^2 \rangle \big] \\
&\qquad + \mathbf{k} \cdot \mathbf{q}_l \times \mathbf{q}_n \big[\mathbf{q}_m \cdot \mathbf{q}_n \langle hf'f' \rangle + q_c^2 \langle hff'' \rangle \big] R_0^{(0)}/P \big) \delta(\mathbf{q}_l + \mathbf{q}_m - \mathbf{q}_n). \qquad (20)
\end{aligned}$$

Only the imaginary part of the solvability conditions has been written since the real part vanishes identically with $\hat{R}_1^{(1)} = 0$. Obviously the right-hand sides of Equation (20) yield a finite value only if the argument of the δ function vanishes (*i.e.*, when wave vectors $\mathbf{q}_l, \mathbf{q}_m$ exist with $\mathbf{q}_l + \mathbf{q}_m = \mathbf{q}_n$). This condition requires an angle of 120° between the vectors $\mathbf{q}_l$ and $\mathbf{q}_m$. If such vectors do not exist, then Equation (16) are satisfied with $\hat{\mathbf{c}}_1^{(1)} = 0$. But if there is one pair $\mathbf{q}_l, \mathbf{q}_m$ with $\mathbf{q}_l + \mathbf{q}_m = \mathbf{q}_n$ then all wave vectors must have this property and the corresponding coefficients A_l must be equal. This situation can only be realized for the hexagon solution with the property (9) or superpositions thereof, as indicated in Figure 3. In the case of the hexagon solution, $N = 3$, $A_n = A = \pm 1/\sqrt{6}$, (20) are satisfied with

$$\mathbf{c}_1^{(1)} = -\mathbf{i} A q_c^2 \frac{3\langle \hat{h}[(f'f' + q_c^2 ff + ff'')q_c^2 R_0^{(0)}/P + g^2] \rangle}{2\langle q_c^2 R_0^{(0)} (q_c^2 ff - ff'')/P + g^2 \rangle}. \qquad (21)$$

For positive A, corresponding to convection cells with rising motion in the center, we thus obtain a drift in the retrograde sense of rotation provided that both averages in expression (21) have the same sign, as will usually be the case. The opposite drift will occur, of course, for cells with descending motion in the center. It is remarkable that the result (21) does not depend on the orientation of the hexagonal cells in the x–y plane. This property results from the fact that $(\mathbf{q}_l - \mathbf{q}_n) \cdot \mathbf{j}/\mathbf{q}_n \cdot \mathbf{i} = 3/2$ whenever $\mathbf{q}_l + \mathbf{q}_m = \mathbf{q}_n$ holds and $\mathbf{q}_l$ is obtained from $\mathbf{q}_n$ by a clockwise rotation with an angle of 60°.

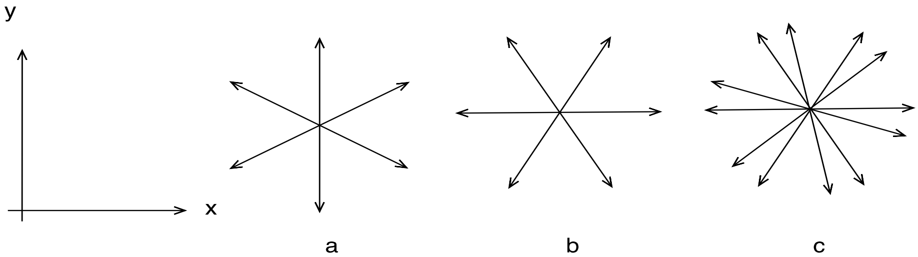


Figure 3 **q**-vector distributions of hexagonal convection (a) and (b), and a superposition thereof (c).

6. Applications to the Sun

In applying the results of the previous sections to the Sun we shall only consider ratios of velocities. There is thus no need to determine the dependence of the amplitude (ϵ) on the Rayleigh number, although such a relationship has been derived by Busse (2003). We shall also not consider the question of the stability of solutions of the form (7). We just shall assume that the deviations from the Boussinesq approximation are sufficiently strong to promote the stability of convection cells of hexagonal type (Busse, 1967). There are two properties of hexagonal convection cells that make them useful for understanding asymmetric convection structures: (*i*) the pattern asymmetry, where a central vertical velocity contrasts with a vertical velocity of opposite sign at the rimlike boundary of the hexagonal cell, and (*ii*) the velocity asymmetry, wherein a concentrated strong velocity contrasts with a more-distributed weaker velocity of opposite sign. We argue here that the latter property of hexagonal cells is appropriate for the interpretation of supergranular convection.

Supergranular cells are characterized by strong downdrafts mostly occurring near what traditionally is called the boundary of the cell, but also in the interior of the cells (Zahn, 2000). When images of flow divergence obtained from correlation-tracking maps (DeRosa, Duvall, and Toomre, 2000; DeRosa and Toomre, 2004; Meunier and Roudier, 2007) are viewed, it is found that downdrafts are concentrated in centers about as much as updrafts are. Because of their much-lower vertical velocity and lower density, the updraft spots are larger and more impressive than the smaller downdraft spots. Helioseismological measurements (Duvall *et al.*, 1997; Duvall and Gizon, 2000; Hindman *et al.*, 2004) give a similar impression. It thus appears that the pattern asymmetry with a fairly homogeneous rim of descending velocity around a central upwelling is not a good model for supergranular cells, apart from the fact that the ratio of areas of rising and descending velocities does not fit the observations. We thus favor the hexagon model based on the asymmetry in the strength of the vertical velocity. Accordingly, we describe the dynamics of supergranular convection by hexagonal cells with descending flow in the center (*i.e.*, $A = -1/\sqrt{6}$). The focus on the high velocities of the downdrafts is of particular importance for the nonlinear aspects of the theory since the downdrafts will dominate properties of supergranular convection that are nonlinear in the amplitude.

According to Equation (21), the ratio between the phase velocity in the x-direction and the z-average of the maximum vertical velocity, $|u_z| = \epsilon q_c^2 6|A|\langle f(z)\rangle$, is given by

$$\mathbf{c}\cdot\mathbf{i}/|u_z| = -\frac{\epsilon\tau_y 3q_c^2 A\langle\hat{h}(z)\rangle}{2\sqrt{6}\epsilon q_c^2\langle f(z)\rangle} = \tau\sin\phi\frac{2}{21}, \tag{22}$$

where we have introduced the colatitude ϕ with $\tau_y = \tau \sin\phi$ and $\tau_z = \tau \cos\phi$ and where we have assumed the particular model of Busse (2003) with the special boundary conditions (12). For this model the horizontal wavenumber q_c is of the order $\beta^{1/3}$ such that the terms proportional to $R_0^{(0)}$ in Equation (21) can be neglected in first approximation. When the ratio (22) is identified with the ratio between the observed phase velocity near the equator, $c \approx 40$ m/s (Gizon, Duvall, and Schou, 2003), and a typical value of the vertical velocity in descending regions of supergranules, $u_z^d \approx 150\ \mathrm{m\,s^{-1}}$ (Simon, 2001), we obtain from Equation (22) the value $\tau = 2.8$. Using $\Omega = 3 \times 10^{-6}\ \mathrm{s^{-1}}$ and $d = 10^7$ m as a typical vertical extent of supergranular cells, we find that this value of τ corresponds to an eddy viscosity $\nu_e = 2.14 \times 10^8\ \mathrm{m^2\,s^{-1}}$, which is close to values quoted in the literature; see, for instance, Simon and Weiss (1997).

With the thus-obtained value of the eddy viscosity we can make predictions of the mean flows expected to be observed on the Sun on the basis of Equation (16). The signs of the mean-flow components are independent of the sign of the hexagon solution. In the order τ_y, a flow $\mathbf{U} = U_x \mathbf{i}$ in the negative x-direction is obtained. For its dimensional counterpart U_x^d divided by u_z^d we find

$$\frac{U_x^d}{|u_z^d|} = -\frac{\epsilon \tau_y q_c^2 F(1)}{12AP\langle f(z)\rangle} = -\tau \sin\phi \frac{d|u_z^d| F(1)}{12\nu_e \langle f(z)\rangle^2} \approx -0.72 \sin\phi. \tag{23}$$

This result can be compared with the decrease by about 15 nHz of the equatorial rate of solar rotation throughout the uppermost 2% of the solar radius. The mean flow of order $\tau_y \tau_z$ is positive and in the y-direction. For its dimensional counterpart, U_y^d divided by u_z^d, we obtain

$$\frac{U_y^d}{|u_z^d|} = \frac{\epsilon \tau_y \tau_z q_c^2 (G(1) + H(1))}{12AP\langle f(z)\rangle} = -\tau^2 \sin\phi \cos\phi \frac{d|u_z^d|(G(1) + H(1))}{12\nu_e \langle f(z)\rangle^2} \approx 0.30 \sin 2\phi. \tag{24}$$

Typical values of the observed meridional circulation on the Sun are 20 $\mathrm{m\,s^{-1}}$ (Braun and Fan, 1998). The predictions (23) and (24) for the azimuthal and the meridional mean flows on the Sun thus agree qualitatively with the observations, but the amplitudes exceed the measured values by about a factor of two. This is not surprising in view of the simplicity of the model. It should also be remarked that the correlation between poloidal and toroidal components of large-scale flows in a turbulent fluid is far from the perfect correlation obtained in our model. This tends to diminish the Reynolds stresses generating the mean flows. We thus conclude that the dynamics of convection on the supergranular scale can explain the mean flows and observed azimuthal drift on the Sun, including their latitudinal dependence.

The mechanisms of mean-flow generation and pattern drift considered in this paper should be clearly distinguished from the mechanism of the generation of the solar differential rotation. That mechanism is also generally attributed to the action of convection, but on a much larger scale of the order of the depth of the convection zone such that the spherical shape of the latter plays an important role. The generation of differential rotation by the Reynolds stress of "giant cell"-type convection was first demonstrated by Busse (1970) on the basis of an analytical model, which may still be of interest in the present connection. Remarkably, it exhibits a differential rotation dependent on latitude, but not on depth, albeit for reasons that are probably different from those responsible for the nearly depth-independent solar differential rotation.

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