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NON-AXISYMMETRIC OSCILLATIONS OF THIN PROMINENCE FIBRILS

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Abstract. We study non-axisymmetric oscillations of thin prominence fibrils. A fibril is modeled by a straight thin magnetic tube with the ends frozen in dense plasmas. The density inside and outside the tube varies only along the tube and it is discontinuous at the tube boundary. Making a viable assumption that the tube radius is much smaller than its length, we show that the squares of the frequencies of non-axisymmetric tube oscillations are given by the eigenvalues of the Sturm–Liouville problem for a second-order ordinary differential equation on a finite interval with the zero boundary conditions. For an equilibrium density that is constant outside the tube and piecewise constant inside we derived a simple dispersion equation determining the frequencies of non-axisymmetric oscillations. We carry out a parametric study of this equation both analytically and numerically, restricting our analysis to the first even mode and the first odd mode. In particular, we obtained a criterion that allows to find out if each of these modes is a normal or leaky mode.

1. Introduction

Quiescent prominences are typically large sheets of cool, dense plasma supported in the corona by a large-scale magnetic field. They are embedded in the hot and rarefied coronal environment. These sheets are not uniform but exhibit a fibril structure (e.g. Dunn, 1960; Simon *et al.*, 1986; Démoulin *et al.*, 1987; Engvold *et al.*, 1987; Schmieder and Mein, 1989; Schmieder, Raadu, and Wiik, 1991; Engvold, 2001). Observations of quiescent prominences show that they are composed of small-scale threads. There is evidence of both vertical and horizontal filamentary structure within prominences. A filament is composed of small-scale loops anchored at many different footpoints that are not aligned along the filament axis. The typical length of a fibril in the prominence is known to be hundreds of times larger than its thickness. Ballester and Priest (1989) constructed a model for the fibril structure of prominences in terms of slender magnetic flux tubes containing hot plasma over most of their length and cool plasma near the summits representing the cool region of the prominence. This model of the horizontal fibril structure of prominences is taken as a basis in the present study.

Quiescent prominences are observed to display a variety of oscillations, with periods extending from a few minutes to over an hour (see, e.g., a review paper by Oliver and Ballester, 2002). The existence of small amplitude, periodic velocity oscillations in quiescent solar prominences associated with their fibril structure is a well-known phenomenon. We rely heavily on the observations revealing that individual fibrils or groups of fibrils may oscillate independently with their own periods. Similar to magnetic loop oscillations, they too may prove important as diagnostic tools of local conditions in prominences and coronal plasmas (e.g., Uchida, 1970; Roberts, Edwin, and Benz, 1984; Roberts, 1991a; Nakariakov, 2001).

To our knowledge Joarder, Nakariakov, and Roberts (1997) were the first who attempted to study oscillations of prominence fibrils using a magnetic slab as a model. Diaz *et al.* (2001) substantially improved and extended the analysis by Joarder, Nakariakov, and Roberts (1997). They developed a very general method for studying oscillations of inhomogeneous magnetic slabs used as models of prominence fibrils. Diaz, Oliver, and Ballester (2002) extended this method for cylindrical geometry which enabled them to model prominence fibrils by magnetic tubes. Their method can be used to study oscillations of magnetic tubes with the equilibrium density varying along the tube axis for an arbitrary ratio of the tube radius to its length. A disadvantage of this method is that its realization results in complicated numerical calculations.

An important property of prominence fibrils is that the size of their cross-sections is much smaller than their length. This property implies that prominence fibrils can be modeled by slender magnetic tubes with the radius much smaller than the length. Hence, the problem contains a small parameter. The main aim of this paper is to use this small parameter to develop a method for studying oscillations of slender magnetic tubes with stratified density that is much simpler than that developed by Diaz, Oliver, and Ballester (2002). The paper is organized as follows. In the next section we describe the equilibrium state and formulate the problem. In Section 3 the governing equation for pressure perturbation at the tube boundary is derived. We obtain the dispersion relations for a piecewise constant equilibrium density profile in Section 4, and carry out its parametric analysis in Section 5. Section 6 contains the summary of the results and our conclusions.

2. Basic Assumptions and Governing Equations

A solar prominence is considered to be made of piling up smaller-scale structures (fibrils) both in the horizontal and vertical directions. The fibrils are curved magnetic loops having a cool part near the apex and anchored in the photosphere. However, for the sake of simplicity, in our equilibrium model a fibril is modeled by a straight axisymmetric magnetic flux tube (the effects of gravity and curvature are considered negligible). Figure 1 shows the sketch of the equilibrium state of the system. Cylindrical coordinates (r, φ, z) are introduced. Total length of the tube is $2L$ and radius R. The tube is embedded in the hot coronal plasma. Densities inside and outside the tube, respectively, $\rho_i(z)$ and $\rho_e(z)$, are both functions of the vertical coordinate z only. The loop is anchored in the photosphere, so its footpoints are

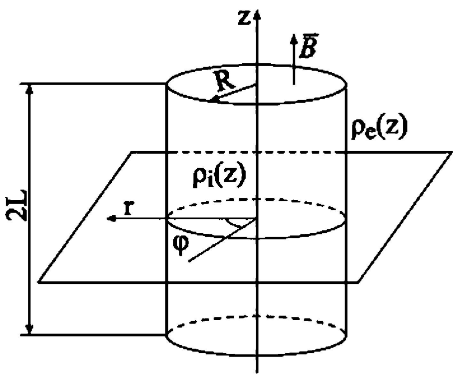

Figure 1. Sketch of the equilibrium state. The ends of the magnetic tube are assumed to be frozen in a dense photospheric plasma.

subject to the line-tying conditions. The plasma is permeated by a uniform magnetic field directed along the tube axis.

In what follows we use the cold plasma approximation ($\beta \to 0$), where the plasma β is defined by $\beta = 2\mu p/B^2$. This is by all means an adequate approximation for solar prominence fibrils, since $\beta \lesssim 0.04$ for typical prominence plasma and even lower values are obtained for the solar corona (e.g., Ofman and Mouradian, 1996; Diaz, Oliver, and Ballester, 2002). Hence only the magnetic pressure is taken into account and we can neglect the thermal pressure in the momentum equation. We consider oscillations with wave lengths of order L (long wave limit) in a cold plasma neglecting viscous effects and curvature of the fibril, but taking the density variation along the fibril into account. The following system of linearized scalar ideal MHD equations with $p = 0$ govern the plasma motion

$$\frac{\partial u}{\partial t} = \frac{1}{\rho}\left(\frac{B}{\mu}\frac{\partial b_r}{\partial z} - \frac{\partial P}{\partial r}\right), \tag{1a}$$

$$\frac{\partial v}{\partial t} = \frac{1}{\rho}\left(\frac{B}{\mu}\frac{\partial b_\varphi}{\partial z} - \frac{1}{r}\frac{\partial P}{\partial \varphi}\right), \tag{1b}$$

$$\frac{\partial b_r}{\partial t} = B\frac{\partial u}{\partial z}, \quad \frac{\partial b_\varphi}{\partial t} = B\frac{\partial v}{\partial z}, \tag{1c}$$

$$\frac{\partial P}{\partial t} = -\frac{B^2}{r\mu}\left(\frac{\partial (ru)}{\partial r} + \frac{\partial v}{\partial \varphi}\right), \tag{1d}$$

$$\frac{1}{r}\frac{\partial (rb_r)}{\partial r} + \frac{1}{r}\frac{\partial b_\varphi}{\partial \varphi} + \frac{\partial b_z}{\partial z} = 0. \tag{1e}$$

Here $\mathbf{V} = (u, v, 0)$ is the velocity, $\mathbf{B} = (b_r, b_\varphi, b_z + B)$ the magnetic field, $\rho = \rho(r, z)$ the unperturbed plasma density, $P = Bb_z/\mu$ the perturbation of the magnetic pressure and μ the magnetic permeability of empty space. After some algebra the set of Equations (1) reduces to a single equation for the magnetic pressure perturbation P,

$$\frac{\partial^2 P}{\partial t^2} - V_A^2 \nabla^2 P = 0, \tag{2}$$

where ∇^2 is the Laplace operator and $V_A(r, z) = B[\mu\rho(r, z)]^{-1/2}$ is the Alfvén speed. In what follows we also use the equation relating the radial component of the velocity and the perturbation of the magnetic pressure,

$$\frac{\partial^2 u}{\partial t^2} - V_A^2 \frac{\partial^2 u}{\partial z^2} = -\frac{1}{\rho} \frac{\partial^2 P}{\partial r \partial t}. \tag{3}$$

Note that this equation remains valid even when the plasma is not cold, i.e. $\beta \neq 0$ (see, e.g., Equation (6.15) in Roberts, 1991b)

At the tube boundary ($r = R$) the conditions of continuity of u and P have to be satisfied,

$$[u] = 0, \quad [P] = 0, \tag{4}$$

where $[f]$ indicates the jump of function f across the boundary. The frozen-in conditions at the tube ends read

$$u = v = 0 \quad \text{at } z = \pm L. \tag{5}$$

It follows from (1d) and (5) that

$$P = 0 \quad \text{at } z = \pm L. \tag{6}$$

Finally, all perturbations have either to decay as $r \to \infty$, which corresponds to normal modes of the tube oscillation, or to satisfy the radiation condition, which corresponds to leaky modes.

3. Governing Equation for Pressure Perturbation at the Tube Boundary

In this section we apply the method of asymptotic expansions to derive the governing equation for the pressure perturbation at the tube boundary. Let us introduce a dimensionless parameter $\epsilon = R/L$. For quiescent prominence fibrils the typical numerical values of geometric parameters under coronal conditions are (e.g., Ofman and Mouradian, 1996; Joarder, Nakariakov, and Roberts, 1997; Diaz *et al.*, 2001; Diaz, Oliver, and Ballester)

$$R \simeq 50 \div 250\,\text{km}, \quad L \simeq 25\,000 \div 100\,000\,\text{km}.$$

Hence the condition $R \ll L$ is valid ($R/L \simeq 0.001 \div 0.01$) and ϵ can be regarded as a small parameter in the present study.

Since the equilibrium quantities depend on r and z only, we can Fourier-analyze the perturbed quantities with respect to φ and t and take them proportional to

$\exp(-i\omega t + im\varphi)$, where m (a positive integer) is the azimuthal wave number and $\omega > 0$ is the frequency of oscillations. Then, in what follows, the perturbed quantities are functions of r and z only.

While inside the tube the characteristic scale in the radial direction is R, this scale is equal to L outside the tube. This observation and the inequality $R \ll L$ inspires us to introduce the stretching coordinate: $\zeta = \epsilon z$ in the vertical direction both inside and outside the tube and $\xi = \epsilon r$ in the radial direction outside the tube. We substitute the new variables in Equation (2) and neglect terms of order ϵ^2. Then, inside the tube, Equation (2) reduces to the equation

$$\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right) - \frac{m^2}{r}P = 0. \tag{7}$$

Outside the tube it reduces to

$$\frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial P}{\partial \xi}\right) + \frac{\partial^2 P}{\partial \zeta^2} + \left(\frac{\tilde{\omega}^2}{V_A^2} - \frac{m^2}{\xi^2}\right)P = 0, \tag{8}$$

where $\tilde{\omega} = \epsilon^{-1}\omega$. Note that the time dependence is lost in Equation (7), while it is present in Equation (8) through the term proportional to $\tilde{\omega}^2$. The reason for this is that the term containing the time derivative in Equation (2) is much smaller than the terms containing the derivatives with respect to r inside the tube, however it is of the order of all other terms in Equation (2) outside the tube.

The solution of (7) regular at $r = 0$ is

$$P = C(\zeta)\left(\frac{r}{R}\right)^m, \tag{9}$$

where $C(\zeta)$ is an arbitrary function of ζ.

To solve (8) we separate the variables. Let us take $P(\xi, \zeta) = F(\xi)G(\zeta)$. Substituting this anzatz in (8), using (6) and separating the variable, we obtain the equation for F,

$$\frac{d^2 F}{d\xi^2} + \frac{1}{\xi}\frac{dF}{d\xi} - \left(k^2 + \frac{m^2}{\xi^2}\right)F = 0, \tag{10}$$

and the Sturm–Liouville problem for $G(\zeta)$,

$$\frac{d^2 G}{d\zeta^2} + \frac{\tilde{\omega}^2}{V_A^2}G = -k^2 G, \tag{11}$$

$$G = 0 \quad \text{at } \zeta = \pm\zeta_0, \tag{12}$$

where k is the separation constant and $\zeta_0 = \epsilon L$. It is well known (e.g., Coddington and Levinson, 1955) that the eigenvalues of the Sturm–Liouville problem (11–12) constitute a monotonically increasing sequence $\{k_n^2\}$, $n = 1, 2, \ldots$, such that $k_n^2 \to \infty$ as $n \to \infty$. We use the notation G_n for the eigenfunction of the Sturm–Liouville problem (11–12) corresponding to the eigenvalue k_n^2. There are two possibilities:

either $k_1^2 > 0$ and, consequently all the eigenvalues are positive, or $k_1^2 < 0$, so that there are a few negative eigenvalues.

When $k^2 = k_n^2 > 0$, (10) is the modified Bessel equation. Its solution decaying at infinity is $F(\xi) = K_m(k_n\xi)$, where K_m is the modified Bessel function of the second kind (McDonald function). When $k^2 = k_n^2 < 0$, (10) is the Bessel equation. Its solution satisfying the radiation condition is $F(\xi) = H_m^{(1)}(\kappa_n\xi)$ (e.g., Cally, 1986, 2003), where $\kappa_n^2 = -k_n^2$ and $H_m^{(1)}$ is the Hankel function of the first kind. Finally, if $k = 0$ then the solution of (10) decaying at infinity is $F(\xi) = \xi^{-m}$. Let N be the number of negative eigenvalues, i.e. $k_n^2 < 0$ for $n = 1, \dots N$, $k_{N+1}^2 \geq 0$ and $k_n^2 > 0$ for $n \geq N + 2$. By definition we put $N = 0$ when $k_1^2 \geq 0$. The general solution of (8) is given by

$$P = \sum_{n=1}^{N} A_n G_n(\zeta) H_m^{(1)}(\kappa_n\xi) + \chi A_{N+1} G_{N+1}(\zeta)\xi^{-m} + \sum_{n=N+1+\chi}^{\infty} A_n G_n(\zeta) K_m(\xi k_n). \tag{13}$$

where A_n are arbitrary constants. In (13) $\chi = 1$ when $k_{N+1} = 0$. If $k_{N+1}^2 > 0$, then $\chi = 0$ and the second term on the right-hand side of (13) is absent. Finally, if $N = 0$ then the first sum on the right-hand side of (13) is absent.

To evaluate the magnetic pressure perturbation outside the tube, P_e, near the tube boundary, where $\xi = \epsilon r \ll 1$, we use the asymptotic expansions of the Bessel functions (Abramowitz and Stegun, 1964)

$$K_m(x) \sim \frac{(m-1)!}{2}\left(\frac{x}{2}\right)^{-m}, \quad H_m^{(1)}(x) \sim -\frac{i(m-1)!}{\pi}\left(\frac{x}{2}\right)^{-m}, \tag{14}$$

valid for $|x| \ll 1$ (recall that we consider $m > 0$). Using these asymptotic expansions we obtain from (13) that for small values of ξ the quantity P_e is given by the approximate expression

$$P_e = A(\zeta)\left(\frac{\epsilon R}{\xi}\right)^m, \tag{15}$$

where

$$A(\zeta) = (\epsilon R)^{-m}\left\{2^m(m-1)!\left[-\frac{i}{\pi}\sum_{n=1}^{N}\frac{A_n G_n(\zeta)}{\kappa_n^m} + \frac{1}{2}\sum_{n=N+1+\chi}^{\infty}\frac{A_n G_n(\zeta)}{k_n^m}\right] + \chi A_{N+1} G_{N+1}(\zeta)\right\}. \tag{16}$$

Using (9), (15) and the second boundary condition in (4), which is satisfied at $r = R$ and $\xi = \epsilon R$, we obtain that $C(\zeta) = A(\zeta)$.

Equation (3) can be rewritten as

$$V_A^2 \frac{\partial^2 u}{\partial \zeta^2} + \tilde{\omega}^2 u = -\frac{i\tilde{\omega}}{\epsilon\rho}\frac{\partial P}{\partial r}. \tag{17}$$

It follows from the first boundary condition in (4) that $\partial^2 u/\partial\zeta^2$ is continuous at the tube boundary. This implies that we can drop the indexes 'i' and 'e' at u when considering (17) at the tube boundary. Using (9) to evaluate the right-hand side of (17) inside the tube and then taking $r = R$ we obtain the following equation valid at the tube boundary,

$$\frac{B^2}{\mu\rho_i}\frac{\partial^2 u}{\partial \zeta^2} + \tilde{\omega}^2 u = -\frac{i\tilde{\omega}m}{\epsilon R\rho_i}A(\zeta). \tag{18}$$

Using (15) to evaluate the right-hand side of (17) outside the tube and then taking $\xi = \epsilon R$ we obtain a similar equation valid at the tube boundary,

$$\frac{B^2}{\mu\rho_e}\frac{\partial^2 u}{\partial \zeta^2} + \tilde{\omega}^2 u = \frac{i\tilde{\omega}m}{\epsilon R\rho_e}A(\zeta). \tag{19}$$

It is straightforward to obtain from (18) and (19) that

$$u = -\frac{2imA(\zeta)}{\epsilon R\tilde{\omega}(\rho_i - \rho_e)}, \quad \frac{\partial^2 u}{\partial \zeta^2} = \frac{im\mu\tilde{\omega}(\rho_i + \rho_e)}{\epsilon RB^2(\rho_i - \rho_e)}A(\zeta). \tag{20}$$

Differentiating the first equation in (20) twice with respect to ζ, comparing the result with the second equation and returning to the original variable z, we obtain

$$\frac{\partial^2 Q}{\partial z^2} + \frac{\omega^2}{C_k^2(z)}Q = 0, \tag{21}$$

where

$$Q = \frac{A}{\rho_i - \rho_e}, \quad C_k^2 = \frac{2B^2}{\mu(\rho_i + \rho_e)}. \tag{22}$$

When there is no density variation in the z-direction (ρ_i and ρ_e are constants), C_k is the kink speed and it is equal to the phase speed of propagation of non-axisymmetric perturbations along a thin magnetic tube. We retain the name 'kink speed' for C_k even in the case when there is density variation in the z-direction.

In accordance with (6) the variable Q has to satisfy the boundary conditions

$$Q = 0 \quad \text{at } z = \pm L. \tag{23}$$

Hence the frequency of the tube oscillation ω is an eigenvalue of the Sturm–Liouville problem (21), (23). It is the frequency of a normal mode of the tube oscillation if P decays exponentially when $r \to \infty$, and it is the frequency of a leaky mode otherwise.

It is worth to note that Equation (21) does not contain m. This means that the frequencies of all non-axisymmetric oscillations are the same in the thin tube approximation.

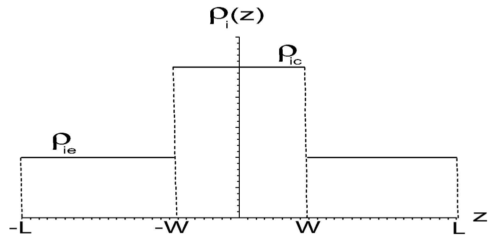

Figure 2. Density profile inside the tube.

4. Oscillations of Fibrils with Piecewise Constant Density

Following Diaz, Oliver, and Ballester (2002) we study oscillation of a tube with a piecewise constant density (see Figure 2). Outside the tube the plasma density is constant ($\rho_e = \text{const}$). Inside the tube the plasma has a constant density ρ_{ic} in the central region determined by the inequality $|z| < W$, and a constant density ρ_{ie} in the two side regions determined by the inequality $|z| > W$. In what follows we take $\rho_{ic} \gg \rho_{ie}, \rho_e$, so that the central region represents the cool dense part of the prominence. Equation (21) is now an equation with constant coefficients, so that its general solution is straightforward:

$$Q_{c,e}(z) = \alpha_{c,e} \cos \frac{\omega z}{C_{kc,e}} + \beta_{c,e} \sin \frac{\omega z}{C_{kc,e}}, \tag{24}$$

where the subscripts 'c' and 'e' indicate whether a quantity is taken in the central region or in the side regions respectively. The kink speeds C_{kc} and C_{ke} are given by (22) with $\rho_i = \rho_{ic}$ and $\rho_i = \rho_{ie}$ respectively.

The boundaries $|z| = W,\ 0 \le r \le R$ are contact discontinuities. This means that at these boundaries the following boundary conditions have to be satisfied (e.g., Cowling, 1960; Goedbloed, 1983; Landau, Lifshitz, and Pitaevskii, 1984),

$$[\mathbf{V}] = 0, \quad [\mathbf{B}] = 0, \tag{25}$$

where now the square brackets indicate the jump of a quantity either across the surface $z = -W$, or across the surface $z = W$. It follows from these boundary conditions and (1e) that P and $\partial P/\partial z$ have to be continuous at $z = \pm W$, which implies the following boundary conditions for Q:

$$[(\rho_i - \rho_e)Q] = 0, \quad \left[(\rho_i - \rho_e)\frac{\partial Q}{\partial z}\right] = 0. \tag{26}$$

Substituting (24) in (26) we obtain the system of four linear algebraic homogeneous equations for the variables α_c, α_e, β_c and β_e. This system has a non-trivial solution only if its determinant is zero. This gives the dispersion equation determining ω.

This dispersion equation can be split in two dispersion equations, one for even modes,

$$\tan\left(\Omega(1-l)\sqrt{\frac{1+e}{2}}\right) - \sqrt{\frac{1+e}{1+c}}\cot\left(\Omega l\sqrt{\frac{1+c}{2}}\right) = 0, \tag{27}$$

and the other for odd modes,

$$\tan\left(\Omega(1-l)\sqrt{\frac{1+e}{2}}\right) + \sqrt{\frac{1+e}{1+c}}\tan\left(\Omega l\sqrt{\frac{1+c}{2}}\right) = 0, \tag{28}$$

where we have introduced the dimensionless quantities

$$\Omega = \frac{\omega L}{V_{Ae}}, \quad c = \frac{\rho_{ic}}{\rho_e}, \quad e = \frac{\rho_{ie}}{\rho_e}, \quad l = \frac{W}{L}. \tag{29}$$

In the even modes Q (and, consequently, A) is an even function of z, while in the odd modes Q (and, consequently, A) is an odd function of z. In what follows we assume that $c > 1$ and $c > e$.

It is interesting to note that the dispersion Equations (27) and (28) are similar to the dispersion equations for kink oscillations in a slab model of prominences obtained by Joarder and Roberts (1992, 1993) (see also Roberts and Joarder, 1994). However, their equations are not completely identical to Equations (27) and (28), which is not surprising because Joarder and Roberts used different prominence geometry and considered a finite-β plasma.

5. Parametric Analysis of the Dispersion Relations

We start our analysis by noting that, if Ω is a root of either (27) or (28), then $-\Omega$ is a root of the same dispersion equation. This observation enables us to consider only positive roots of (27) and (28).

For what follows it is convenient to introduce the notation $a = \sqrt{(1+c)/2}$ and $b = \sqrt{(1+e)/2}$. Let $\Omega(l) > 0$ be a root of (27). Differentiating (27) with respect to l and using (27) once again to transform the result, we eventually obtain

$$[(1-l)\tan^2(b(1-l)\Omega) + l\cot^2(al\Omega) + 1]\frac{\mathrm{d}\Omega}{\mathrm{d}l} = -\Omega\frac{c-e}{c+1}\cot^2(al\Omega). \tag{30}$$

A similar procedure applied to (28) results in

$$[(1-l)\tan^2(b(1-l)\Omega) + l\tan^2(al\Omega) + 1]\frac{\mathrm{d}\Omega}{\mathrm{d}l} = -\Omega\frac{c-e}{c+1}\tan^2(al\Omega). \tag{31}$$

Equations (30) and (31) show that any positive root of either dispersion Equation (27) or (28) is a monotonically decreasing function of l. This result is not surprising at all if we consider the analogy between oscillations of a magnetic tube and a string. It is well known that the frequencies of string eigenmodes decrease when the string mass increases.

In what follows we restrict our analysis to the first even mode, and the first odd mode only. The characteristic property of the first even mode is that $A(z)$ has no zeros in the interval $[-L, L]$. In the first odd mode $A(z)$ has exactly one zero at $z = 0$ in the interval $[-L, L]$. The frequency of the first even mode is the smallest positive root of (27), and the frequency of the first odd mode is the smallest positive root of (28). When $l \ll e/c = \rho_{ie}/\rho_{ic}$ (which corresponds to $W \approx 0$), the frequencies of the first even mode and the first odd mode are given by $\Omega \approx \Omega_k$ and $\Omega \approx 2\Omega_k$ respectively, where $\Omega_k = \pi/\sqrt{2(1+e)}$. In the dimensional variables these two frequencies are approximately equal to ω_k and $2\omega_k$, where ω_k in the frequency of kink oscillations of a thin homogeneous tube with the density ρ_{ie} inside the tube and ρ_e outside. The square of this frequency is given by (e.g., Ryutov and Ryutova, 1976; Edwin and Roberts, 1983)

$$\omega_k^2 = \frac{\pi^2 B^2}{2\mu L^2(\rho_{ie} + \rho_e)}. \tag{32}$$

This result is not surprising at all because the condition $l \ll e/c$ means that the mass in the cool dense part of the tube is negligible in comparison with the mass in the hot rarefied part. We obtain a similar result if we take $W = L$, so once again we consider a homogeneous tube, but now with the internal density equal to ρ_{ic}. In that case the frequencies of the fundamental mode and the first overtone are equal to $\bar{\omega}_k$ and $2\bar{\omega}_k$, where $\bar{\omega}_k^2$ is given by Equation (32), however with ρ_{ic} substituted for ρ_{ie}.

Let us verify if there are negative eigenvalues of the Sturm–Liouville problem (11–12) for a mode with the frequency ω. It is straightforward to obtain that

$$k_1^2 = \frac{1}{\epsilon^2 L^2}\left(\frac{\pi^2}{4} - \Omega^2\right). \tag{33}$$

For the first even mode the sign of k_1^2 determines if the solution outside the tube given by (13) decays exponentially as $r \to \infty$ or it does not, and thus if the first even mode is a normal mode or a leaky mode. However this criterion is not applicable to the first odd mode. It follows from the continuity of P at the tube boundary that $A_n = 0$ for odd n in (13), so that the summation starts from $n = 2$. This means that the properties of the solution outside the tube are determined by k_2^2. A straightforward calculation yields

$$k_2^2 = \frac{\pi^2 - \Omega^2}{\epsilon^2 L^2}. \tag{34}$$

When $\rho_{ie} > \rho_e$ it is easy to see that $\Omega_k < \pi/2$, so that $k_1^2 > 0$ and $k_2^2 > 0$ when $l \ll e/c$, which implies that both the first even mode and the first odd mode are normal (non-leaky) modes for $l \ll e/c$. Since the frequencies of these modes decrease monotonically when l increases, we conclude that the both modes are normal modes for any value of l.

In the case where $\rho_{ie} < \rho_e$ the situation is different. Now $\Omega_k > \pi/2$, so that $k_1^2 < 0$ and $k_2^2 < 0$ when $l \ll e/c$, which implies that both the first even mode and the first odd mode are leaky modes for $l \ll e/c$. On the other hand, $\Omega \approx \pi/\sqrt{2(1+c)}$ for the first even mode and $\Omega \approx 2\pi/\sqrt{2(1+c)}$ for the first odd mode when $1-l \ll 1/c$. This implies that $k_1^2 > 0$ and $k_2^2 > 0$ for $1-l \ll 1/c$ and both the first even mode and the first odd mode are normal modes in this limit. On the basis of this analysis we conclude that there are two threshold values of l, l_e and l_o. The first even mode is a leaky mode when $l < l_e$ and a normal mode when $l > l_e$. Similarly, the first odd mode is a leaky mode when $l < l_o$ and a normal mode when $l > l_o$. Assuming that $l_{e,o} \ll 1$, we can easily obtain the approximate expression for l_e and l_o:

$$l_e \approx \frac{2\sqrt{2(1+e)}}{\pi(1+c)} \cot\left(\frac{\pi}{2}\sqrt{\frac{1+e}{2}}\right), \tag{35}$$

$$l_o \approx -\frac{1}{\pi}\sqrt{\frac{2}{1+c}} \arctan\left[\sqrt{\frac{1+c}{1+e}} \tan\left(\pi\sqrt{\frac{1+e}{2}}\right)\right]. \tag{36}$$

We also calculated the dependences of l_e and l_o on e numerically for different values of c. The results of this calculation are shown in Figure 3. The solid, dashed and dotted curves correspond to $c = 100$, $c = 200$ and $c = 300$ respectively. The comparison of numerical and analytical results show that Equation (35) gives l_e with accuracy better than 2% for the whole range of considered parameters. Equation (36) gives l_o with accuracy better than 5% for $100 \le c \le 300$ and $e \lesssim 0.5$. The accuracy for $e > 0.5$ is not very good. For example, the error is about 12% for $c = 100$ and $e = 0.8$. We can see from Figure 3 that the first even mode is leaky only when l is unrealistically small, while the first odd mode is leaky for realistic values of the fibril parameters.

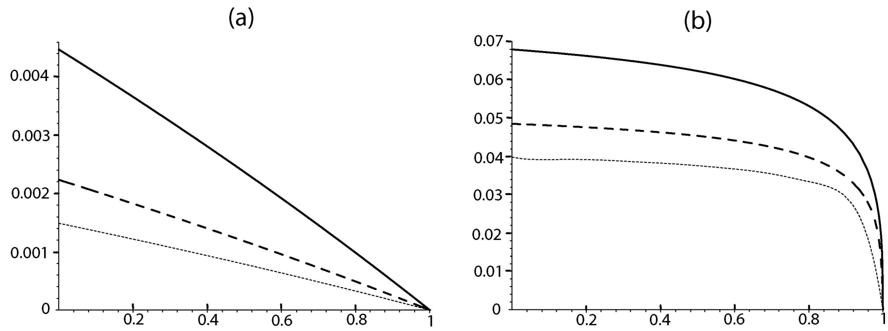

Figure 3. The dependences of l_e and l_o on e. Figures (a) and (b) correspond to l_e and l_o respectively. The solid, dashed and dotted curves correspond to $c = 100$, $c = 200$ and $c = 300$ respectively.

It is worth noting that leaky modes damp due to wave energy leakage from the oscillating tube. Hence, frequencies of leaky modes always have negative imaginary parts. In our calculations we found that frequencies of both leaky and non-leaky modes are real. This result is a consequence of approximations that we made. The ratio of the imaginary and real part of the frequency of a leaky mode is of the order of ϵ^2. Hence, to calculate the decrements of leaky modes we have to continue our calculations to higher order approximation with respect to ϵ.

To obtain more information about the properties of the first even mode and the first odd mode we solved the dispersion Equations (27) and (28) numerically. The typical density values for our configuration are $\rho_e \simeq \rho_{ie}$ and $\rho_{ic} \simeq 100\rho_e$. The cold part of a fibril is relatively short. Hence we choose the following values of physical parameters of equilibrium state to solve the dispersion Equations (27) and (28):

$$\frac{\rho_{ie}}{\rho_e} = 0.6,\ 1.2,\ 2,\ \frac{\rho_{ic}}{\rho_e} = 100,\ 150,\ 200,\ 300; \quad \frac{W}{L} = 0.05 \div 0.2. \tag{37}$$

Figures 4 and 5 show the non-dimensional frequencies Ω of the first even mode and the first odd mode respectively as functions of l for different values of c and e. Solid, dashed and dotted lines correspond to $e = 0.6$, $e = 1.2$ and $e = 2$ respectively. The plots reveal that the dependence of the frequency of the first even mode on the ratio ρ_{ie}/ρ_e is very weak, though this dependence is quite distinctive for the first odd mode when l is small.

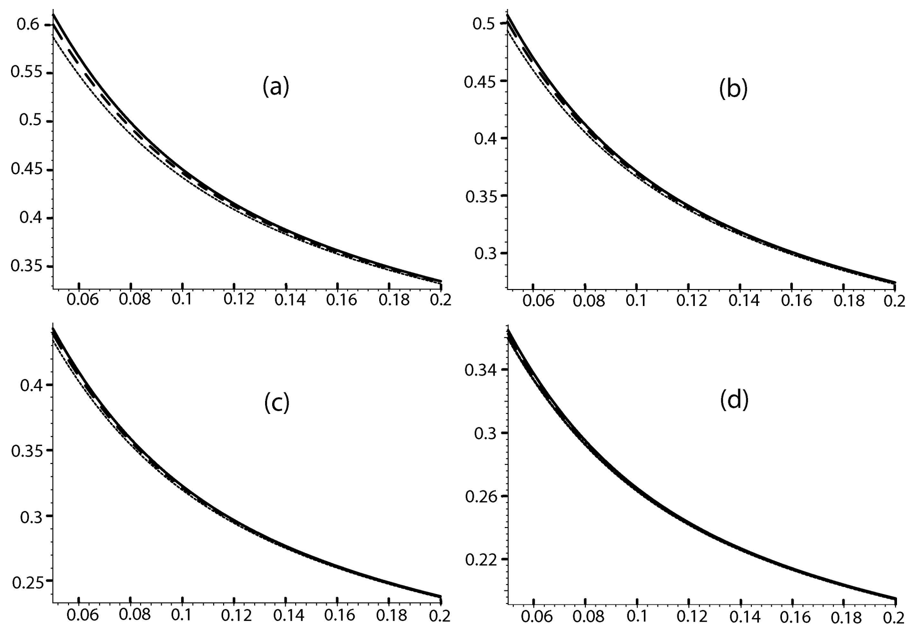

Figure 4. The dimensionless frequency Ω of the first even mode versus l. Figures (a), (b), (c) and (d) correspond to $c = 100, 150, 200, 300$ respectively.

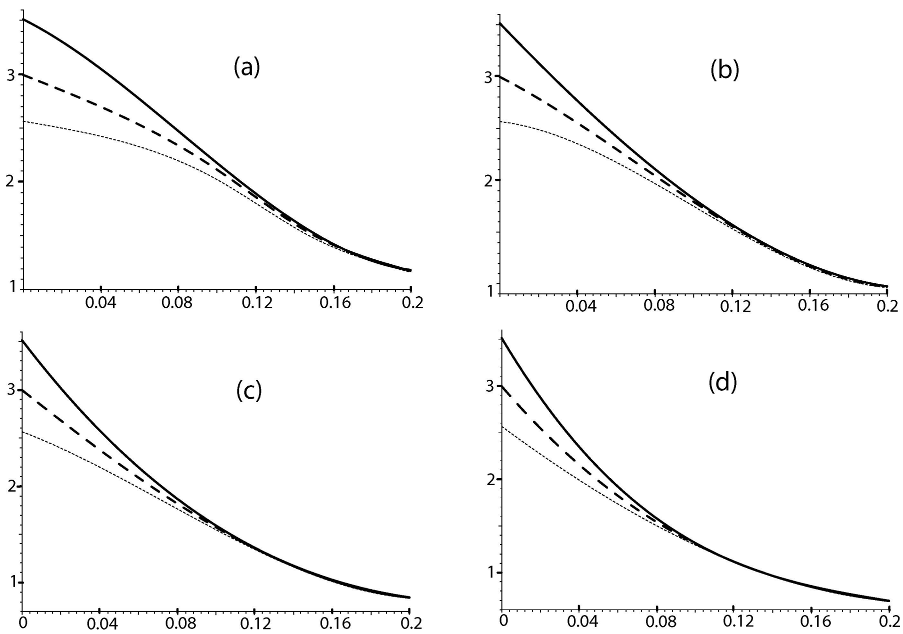

Figure 5. The dimensionless frequency Ω of the first odd mode versus l. Figures (a), (b), (c) and (d) correspond to $c = 100$, 150, 200, 300 respectively.

6. Summary and Conclusions

In this paper we considered non-axisymmetric ($m \neq 0$) oscillations of thin prominence fibrils. We used the cold plasma approximation and modeled a fibril by a straight magnetic tube with its ends frozen in a dense photospheric plasma. We took the density variation along the fibril into account and assumed that the equilibrium density inside and outside the tube varies only in the direction of the tube axis, while it is discontinuous at the tube boundary. Then using the method of asymptotic expansions with the ratio of the tube radius to the tube length, ϵ, as a small parameter, we derived the ordinary second order differential equation for the magnetic pressure at the tube boundary. Together with the zero boundary conditions that follow from the line-tying conditions at the tube ends this equation constitutes the Sturm–Liouville problem determining the frequencies of either normal or leaky modes of the tube oscillations.

As an example, we considered the same equilibrium as was studied by Diaz, Oliver, and Ballester (2002). In this equilibrium the plasma density is constant outside the tube, while it is piecewise constant inside: it is large in the central region and small in the two side regions. We derived the dispersion equations for even and odd modes and carried out the parametric study of these equations both analytically and numerically, restricting our analysis to the first even mode and the first odd mode.

It is instructive to compare the method developed in this paper with the one used by Diaz, Oliver, and Ballester (2002). These authors developed a very general method for studying magnetic tube oscillations that works for any value of ϵ. Using this method they obtained the dispersion function which is equal to an infinite determinant with the elements nonlinearly dependent on the dimensionless oscillation frequency Ω. The dimensionless oscillation frequencies of the wave modes of the tube oscillation are given by the zeros of the dispersion function. To calculate Ω (Diaz, Oliver, and Ballester, 2002) truncated the determinant. However, to have sufficient accuracy of calculations they had to deal with a determinant of a quite big order. In the case of piecewise constant background density the elements of the determinant can be calculated analytically, but even in this case the numerical analysis is quite involved. The situation is even more complicated when the background density varies continuously inside and outside the tube in the direction parallel to the tube. In that case, in general, the elements of the determinant can be calculated only numerically, so calculating the zeros of the determinant becomes a very complicated numerical problem.

The method that we developed in this paper works only for $\epsilon \ll 1$. This restriction is quite acceptable in applications to prominence fibrils because typically they are very thin with $\epsilon \lesssim 0.01$. On the other hand, our method is much simpler than that used by Diaz, Oliver, and Ballester (2002). To find the oscillation frequencies we only have to calculate the eigenvalues of the Sturm–Liouville problem (19), (21). In the case of piece-wise constant background density the dispersion function easily calculated analytically and takes a very simple form. When the background density is continuously stratified the eigenvalues of the Sturm–Liouville problem, in general, can be calculated only numerically. But this numerical calculation is trivial.

It is also worth to note that our method is only applicable to the fundamental harmonic with respect to the r-dependence. The reason is that (9) properly describes the dependence of P on r inside the tube only for the fundamental harmonic. This is clearly seen from the comparison of (9) with $m = 1$ and Figure 6 of Diaz, Oliver, and Ballester (2002), where the dependence of P on r is shown for the fundamental, first and second harmonics of the kink mode. However, this is not a very severe restriction because the fundamental harmonic is the most interesting for applications to solar prominences. The non-fundamental harmonics are much less interesting because they are leaky modes for almost all realistic values of fibril parameters and, what is even more important, their frequencies tend to infinity when $R/L \to 0$.

To illustrate the accuracy of the approximation provided by our method we compare our results with those obtained by Diaz, Oliver, and Ballester (2002). These authors calculated the periods of the first even and odd kink modes for $\rho_e = 8.37 \times 10^{-13}$ kg m^{-3}, $L = 5 \times 10^4$ km, $B = 5$ G, $R/L = 0.001$, $\rho_{ie}/\rho_e = 0.6$ and $\rho_{ic}/\rho_e = 200$. For $W/L = 0.1$ they obtained $\Omega = 0.32288$ (corresponding to period 33.2 min) for the first even mode, and $\Omega = 1.59153$ (corresponding to period

6.74 min) for the first odd mode. Our results are $\Omega \approx 0.324$ and $\Omega \approx 1.60$ respectively. For $W/L = 0.2$ Diaz, Oliver, and Ballester obtained $\Omega = 0.23837$ (corresponding to period 45.0 min) for the first even mode, and $\Omega = 0.84710$ (corresponding to period 12.7 min) for the first odd mode. Our results are $\Omega \approx 0.238$ and $\Omega \approx 0.847$ respectively. We see that the difference between our results and the results obtained by Diaz, Oliver, and Ballester (2002) is always less than 1%.

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