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DYNAMIC MECHANICAL PROPERTIES OF SOIL BASED ON FRACTIONAL-ORDER DIFFERENTIAL THEORY

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In this paper, a new viscoelastic constitutive model is proposed based on fractional-order differential theory, replacing the Newtonian dashpot of the classical Kelvin-Voigt model with the Abel dashpot. The analytic solutions for the fractional-order three-element model and classical three-element model are presented. The results estimated by the fractional-order three-element model correlate better with experimental data than those of the classical three-element model. The parameters of the fractional-order three-element model were further optimized using the nonlinear least squares method. The proposed fractional-order three element model was able to accurately describe the viscoelastic dynamic mechanical properties of soil during vibratory compaction.

Introduction

The road compaction effect will determine its quality, performance, and service life [1]. During the process of vibratory compaction, soil demonstrates elastic, plastic, and viscous characteristics [2], and acts like a viscoelastic material with properties of both an ideal solid and ideal fluid [3].The viscoelastic properties of soil have a significant influence on the final compaction effect; therefore, the dynamic mechanical properties of soil during the process of vibratory compaction should be fully considered.

Integer-order differential operators are often used in viscoelastic constitutive equations, and the kernel functions (creep modulus and relaxation modulus) of a viscoelastic model are generally a combination of exponential functions [4]. Therefore, simple constitutive relationships do not provide an adequate description of the actual deformation process of viscoelastic materials [5], and such models thus have difficulty accurately describing the dynamic mechanical properties of soil, which significantly limits their application. Fractional constitutive models of viscoelastic materials were first introduced by Gemant [6]. In the constitutive equations, integer-order differential operators are replaced by fractional-order differential operators. Extensive studies have since been performed in this field, and fractional calculus has shown promise in describing the rheological properties of viscoelastic and viscoplastic materials [7, 8]. Fractional calculus was previously used to describe memory-intensive and path-dependent phenomena, and fractional calculus-based models generally have fewer parameters in comparison with classical models [9-11]. The models have demonstrated powerful capabilities in capturing the static and dynamic stress-strain-time relationships of viscoelastic materials and have been applied in various fields over the past few decades [11-16]. Past research thus provides a theoretical basis for using fractional-order differential theory to describe the constitutive models and dynamic mechanical properties of soil during vibratory compaction. To date, research is still lacking and limited studies exist in the literature.

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This paper briefly describes the basic theory of fractional calculus and viscoelastic properties of soil. A Fractional-order three-element constitutive model of soil during vibratory compaction is presented, and the dynamic mechanical properties of soil are evaluated including the storage modulus, loss modulus, loss factor, and dynamic modulus. Several comparisons are made between the proposed model and classical three-element model. Finally, experimental data is used to verify the accuracy of the fractional-order three-element model and further optimization is explored.

Brief Introduction of Fractional Calculus

The basic operator of fractional calculus ${}_aD_t^\alpha$ [8, 17, 18] can be defined as

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{d^\alpha}{dt^\alpha} & R(\alpha) > 0 \\ 1 & R(\alpha) = 0 \\ \int_a^t d(\tau)^{-\alpha} & R(\alpha) < 0 \end{cases}, \tag{1} $$

where a and t are the lower and upper bounds of the operator respectively; α is the integral and differential order, which can be any complex number; and $R(\alpha)$ is the real part of α.

At present, there are numerous definitions of fractional calculus [8, 17-20], such as the Riemann-Liouville, Caputo, Grunwald-Letnikov, and Riesz definitions, which although different in form, can be used interchangeably and are essentially equivalent.

The Riemann-Liouville differential [21]

$$ {}_a^{RL}D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dt}\right)^m \int_a^t \frac{f(\tau)}{(t-\tau)^{1-(m-\alpha)}} d\tau \;\; (m = |\alpha| + 1, m-1 \le \alpha < m, t > a). \tag{2} $$

The left and the right fractional integrals are defined as

$$ D_{a^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \;\; (t > a, \alpha > 0); \tag{3} $$

$$ D_{b^-}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \;\; (t < b, \alpha > 0), \tag{4} $$

where Γ is the Gamma function and $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ is its integral definition, which meets the basic requirement of $\Gamma(z + 1) = z\Gamma(z)$.

Viscoelastic properties of soil

The stress-strain relationship of an ideal solid should satisfy Hooke's Law, $\sigma(t) - \varepsilon(t)$, and the stress-strain relationship of ideal fluid should satisfy Newton's law of viscosity [5], $\sigma(t) - d^1\varepsilon(t)/dt^1$. Moreover, $\sigma(t) - \varepsilon(t)$ can be written as $\sigma(t) - d^0\varepsilon(t)/dt^0$, such that the stress-strain relationship of an object with properties of both an ideal solid and ideal fluid can be defined as

$$ \sigma(t) = \xi d^u \varepsilon(t) / dt^u \quad (0 \le u \le 1), \tag{5} $$

where ξ and u are invariable parameters of the material, ξ is the equivalent of the modulus of elasticity for Hooke's law or dynamic viscosity coefficient for Newton's law of viscosity.

Before compaction, soil is saturated with the optimum moisture content; therefore, it is reasonable to assume that compacted soil has properties of both an ideal solid and ideal fluid, and that the stress-strain relationship of compacted soil can be expressed by Eq.(5).

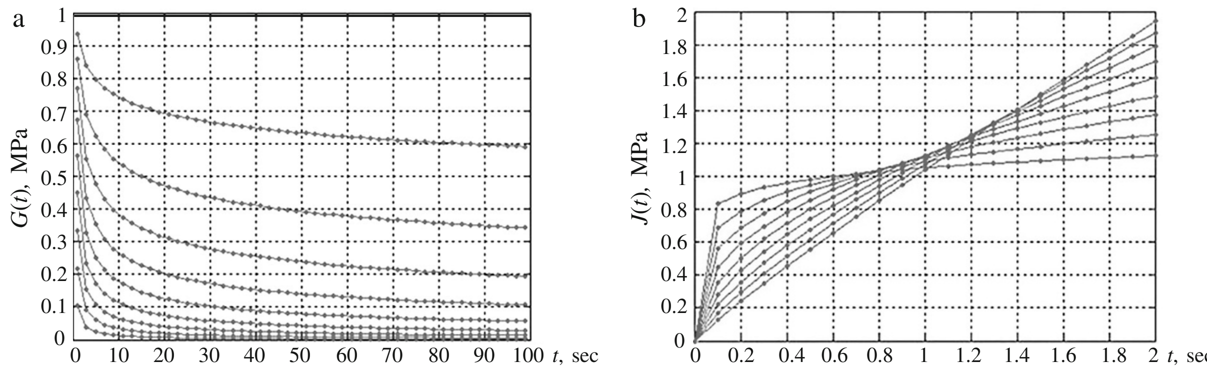

Fig.1. Relaxation modulus (a) and creep compliance (b) of Abel model as α increases from 0.1 to 0.9.

The constitutive relation of a Newtonian body (dashpot) states that stress is proportional to the first derivative of strain [22-24]. Therefore, the constitutive relation, relaxation modulus, and creep compliance of a Newtonian body under the uniaxial stress state are as follows:

$$\sigma = vD\varepsilon(t) = v\delta(t) * d\varepsilon(t) = \mathrm{G}(t) * d\varepsilon(t); \tag{6}$$

$$G(t) = v\delta(t); \tag{7}$$

$$J(t) = tH(t) / v, \tag{8}$$

where v is the dynamic viscosity coefficient of the fluid or fluid viscosity, $\delta(t)$ is the Dirac function, and $H(t)$ is the Heaviside step function.

When $v = 1$, the relaxation modulus $G(t)$ of a Newtonian body is proportional to $\delta(t)$, from Eq.(7). However, $\delta(t) = 0$ at every point except zero, and its integral over the entire domain is equal to 1. Beginning at time $t = 0$, the relaxation modulus $G(t)$ suddenly relaxes from infinity to zero, and from Eq.(8), the creep compliance $J(t)$ is related to $H(t)$ and t; when independent variable value $H(t) > 0$, the function value is 1, otherwise the function value is zero. Therefore, beginning at $t = 0$; $H(t)$ will always be equal to 1. Thus, J(t) is proportional to t.

Replacing the integer-order operator D in the constitutive relation of the Newtonian body with the fractional-order differential operator D^α [22-24], we can obtain the constitutive relation, relaxation modulus, and creep compliance for an Abel body under the uniaxial stress state

$$\sigma = vD^\alpha \varepsilon(t) = vI_\alpha(t) * d\varepsilon(t) = G(t) * d\varepsilon(t); \tag{9}$$

$$G(t) = vI_\alpha(t) = v / [\Gamma(1-\alpha)t^\alpha]; \tag{10}$$

$$J(t) = t^\alpha / [v\Gamma(1+\alpha)]. \tag{11}$$

When $v = 1$, the relaxation modulus $G(t)$ of the Abel body relaxes to zero according to the negative fraction power-law $t^{-\alpha}$ (Fig. 1a), and the creep compliance $J(t)$ of the Abel body increases according to the positive fraction power-law t^α (Fig. 1b); therefore, the Abel body can more closely describe the characteristics of viscoelastic materials [25].

Establishment of three-element constitutive model

The classical three-element model [26, 27] is formed by an elastic element (e_1) and classic Kelvin-Voigt model in series, and the classical Kelvin-Voigt model (k) is formed by an elastic element (e_2) and Newtonian dashpot (v) in parallel (Fig. 2).

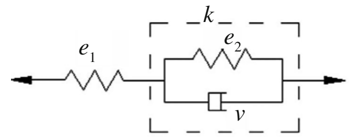

Fig.2. Classical three-element model.

The stress of the classical Kelvin-Voigt model is equal to the sum of stress of the elastic element (e_2) and Newtonian dashpot (v), and the strain is equal to each of the strains of e_2 and v:

$$\begin{cases} \sigma_k = \sigma_{e2} + \sigma_v \\ \varepsilon_k = \varepsilon_{e2} = \varepsilon_v \\ \sigma_{e2} = E_2 \varepsilon_{e2} \\ \sigma_v = v\dot{\varepsilon}_v \end{cases}. \tag{12}$$

The total stress of the classical three-element model is equal to each of the stresses of e_1 and classical Kelvin-Voigt model, and total strain is equal to the sum of the strains of e_1 and classical Kelvin-Voigt model:

$$\begin{cases} \sigma = \sigma_{e1} = \sigma_k \\ \varepsilon = \varepsilon_{e1} + \varepsilon_k \\ \sigma_{e1} = E_1 \varepsilon_{e1} \end{cases}. \tag{13}$$

From Eqs.(12) and (13), we obtain

$$\frac{E_1 v}{E_1 + E_2}\dot{\varepsilon} + \frac{E_1 E_2}{E_1 + E_2}\varepsilon = \frac{v}{E_1 + E_2}\dot{\sigma} + \sigma.$$

When $\beta = (E_1 + E_2)/v$, $E_\infty = E_1E_2/(E_1 + E_2)$, and $E_0 = E_1$, the viscoelastic constitutive equation of the classical three-element model is

$$E_0\beta^{-1}\dot{\varepsilon} + E_\infty\varepsilon = \beta^{-1}\dot{\sigma} + \sigma. \tag{14}$$

Taking the Fourier transform of Eq.(14), the complex modulus of the classical three-element model is

$$\begin{aligned} E^*(i\omega) &= \frac{\sigma(\omega)}{\varepsilon(\omega)} = \frac{E_0\beta^{-1}i\omega + E_\infty}{\beta^{-1}i\omega + 1} = \frac{E_0 i\omega + \beta E_\infty}{i\omega + \beta} = \\ &= \frac{\beta^2 E_\infty + E_0\omega^2 + i\beta\omega(E_0 - E_\infty)}{\beta^2 + \omega^2} = E' + iE''. \end{aligned} \tag{15}$$

The storage modulus (elastic section of the stress-strain relationship of the soil), loss modulus (viscous section of the stress-strain relationship of the soil), loss factor (ratio of loss modulus to storage modulus), and dynamic modulus (module value of complex modulus that reflects the ability of soil to resist deformation of soils during vibratory compaction) [28, 29] of the classical three-element model can be expressed as

$$E'(\omega) = \frac{\beta^2 E_\infty + E_0\omega^2}{\beta^2 + \omega^2}; \tag{16}$$

$$E''(\omega) = \frac{\beta\omega(E_0 - E_\infty)}{\beta^2 + \omega^2}; \tag{17}$$

$$\tan\phi = \frac{E''}{E'} = \frac{\beta\omega(E_0 - E_\infty)}{\beta^2 E_\infty + E_0\omega^2}; \tag{18}$$

$$\left|E^*(i\omega)\right| = \sqrt{\frac{(\beta^2 E_\infty + E_0\omega^2)^2 + \beta^2\omega^2(E_0 - E_\infty)^2}{(\beta^2 + \omega^2)^2}}. \tag{19}$$

Replacing the classical Kelvin-Voigt model in the classical three-element model with the fractional-order Kelvin-Voigt model, we obtain the fractional-order three-element model. The stress-strain relationship can be expressed as

$$\begin{cases} \sigma_k = \sigma_{e2} + \sigma_v \\ \varepsilon_k = \varepsilon_{e2} = \varepsilon_v \\ \sigma_{e2} = E_2\varepsilon_{e2} \\ \sigma_v = v(\dfrac{d^\alpha \varepsilon_v}{dt^\alpha}); \end{cases} \tag{20}$$

$$\begin{cases} \sigma = \sigma_{e1} = \sigma_k \\ \varepsilon = \varepsilon_{e1} + \varepsilon_k \ , \\ \sigma_{e1} = E_1\varepsilon_{e1} \end{cases} \tag{21}$$

where e_1 and e_2 represent the elastic element outside and inside the fractional-order Kelvin-Voigt model, respectively, k represents the fractional-order Kelvin-Voigt model, and v represents the Abel dashpot in the fractional-order Kelvin-Voigt model.

Using Eqs.(20) and (21), we can obtain the viscoelastic constitutive equation of the fractional-order three element model as

$$E_0\beta^{-1}\frac{d^\alpha\varepsilon}{dt^\alpha} + E_\infty\varepsilon = \beta^{-1}\frac{d^\alpha\sigma}{dt^\alpha} + \sigma. \tag{22}$$

Taking the Fourier transform of Eq.(22) yields

$$E_0\beta^{-1}(i\omega)^\alpha\varepsilon(\omega) + E_\infty\varepsilon(\omega) = \beta^{-1}(i\omega)^\alpha\sigma(\omega) + \sigma(\omega). \tag{23}$$

The complex modulus of the fractional-order three element model is

$$E^*(i\omega) = \frac{\sigma(\omega)}{\varepsilon(\omega)} = \frac{E_0\beta^{-1}(i\omega)^\alpha + E_\infty}{\beta^{-1}(i\omega)^\alpha + 1} = \frac{E_0(i\omega)^\alpha + \beta E_\infty}{(i\omega)^\alpha + \beta} = E' + iE''. \tag{24}$$

Since $i^a = \cos(\alpha\pi/2) + i\sin(\alpha\pi/2)$, Eq.(24) can be transformed into

$$E^*(i\omega) = \frac{\beta^2 E_\infty + E_0\omega^{2\alpha} + \beta\omega^\alpha(E_0 + E_\infty)\cos(\alpha\pi/2) + i\beta\omega^\alpha(E_0 - E_\infty)\sin(\alpha\pi/2)}{\beta^2 + \omega^{2\alpha} + 2\beta\omega^\alpha\cos(\alpha\pi/2)}. \tag{25}$$

Then, the storage modulus, loss modulus, loss factor, and dynamic modulus of the fractional-order three-element model can be expressed as

$$E'(\omega) = \frac{\beta^2 E_\infty + E_0\omega^{2\alpha} + \beta\omega^\alpha(E_0 + E_\infty)\cos(\alpha\pi/2)}{\beta^2 + \omega^{2\alpha} + 2\beta\omega^\alpha\cos(\alpha\pi/2)}; \tag{26}$$

$$E''(\omega) = \frac{\beta\omega^\alpha(E_0 - E_\infty)\sin(\alpha\pi/2)}{\beta^2 + \omega^{2\alpha} + 2\beta\omega^\alpha\cos(\alpha\pi/2)}; \tag{27}$$

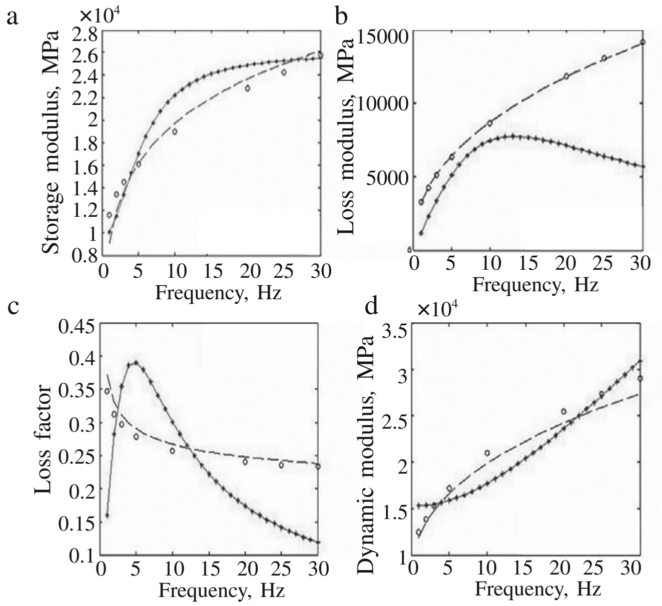

Fig. 3. Comparison of dynamic mechanical response curves of soil based on fractional order three-element model (–––), classical three-element model (–•–), and experiment data (○): a) storage modulus, b) loss modulus, c) loss factor, and d) dynamic modulus.

$$\tan\phi = \frac{E''}{E'} = \frac{\beta\omega^{\alpha}(E_0 - E_{\infty})\sin(\frac{\alpha\pi}{2})}{\beta^2 E_{\infty} + E_0\omega^{2\alpha} + \beta\omega^{\alpha}(E_0 + E_{\infty})\cos(\frac{\alpha\pi}{2})}; \tag{28}$$

$$\left|E^*(i\omega)\right| = \sqrt{\frac{(\beta^2 E_{\infty} + E_0\omega^2)^2 + \beta^2\omega^{2\alpha}(E_0^2 + E_{\infty}^2 + 2E_0E_{\infty}\cos\alpha\pi) + 2\beta\omega^{\alpha}(E_0 + E_{\infty})(\beta^2 E_{\infty} + E_0\omega^{2\alpha})\cos\frac{\alpha\pi}{2}}{(\beta^2 + \omega^{2\alpha} + 2\beta\omega^{\alpha}\cos\frac{\alpha\pi}{2})^2}}. \tag{29}$$

Estimation of dynamic mechanical properties of compacted soil using three-element model

The storage modulus, loss modulus, loss factor, and dynamic modulus of the fractional-order three-element model are compared with those obtained using the classical three-element model, as well as experimental data obtained using uniaxial compressive strength tests performed at a temperature of 25 C and pressure of 700 kPa. The results are illustrated in Fig.3.

Based on the results presented in Fig.3, the following observations can be made:

(1) Comparing the results estimated using the classic three-element model to the experimental data, curves of the loss modulus and loss factor are inconsistent with the experimental data and suggest the classical three-element model does not accurately reflect the actual dynamic mechanical properties of compacted soil.

(2) The results estimated using the proposed fractional-order three-element model are consistent with the experimental data, suggesting the model is more suitable than the classical three-element model for describing the characteristics of viscoelastic materials.

(3) The estimated curves based on the fractional-order three element model, including the storage modulus, loss modulus, and dynamic modulus of the compacted soil, increase rapidly; however, at small frequencies the loss factor rapidly decreases. The upward trends of storage modulus, loss modulus, and dynamic modulus slow down. Moreover, the downward trend of the loss factor also slows and tends to be stable at higher frequencies.

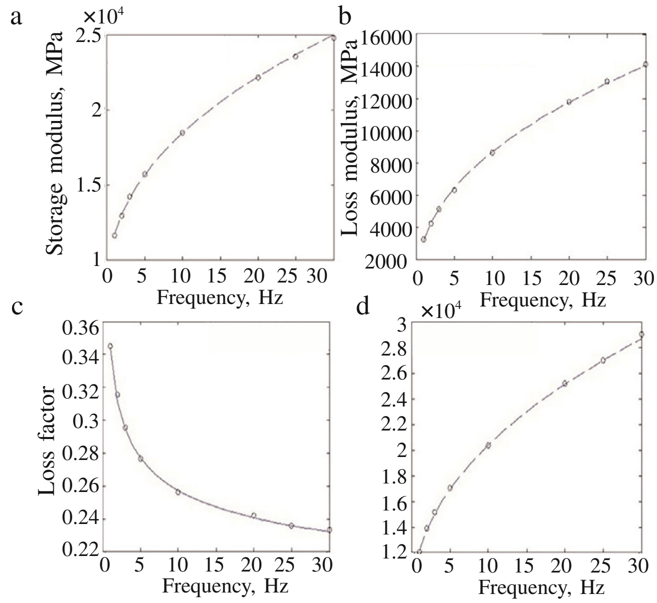

Fig. 4. Optimization of dynamic mechanical response fitting curve based on fractional-order three element model (– – –): a) storage modulus, b) loss modulus, c) loss factor, and d) dynamic modulus; ○) experiment data.

Although the results estimated using the fractional-order three element model are relatively consistent with the experimental data, some error exists, particularly in relation to the dynamic modulus curve. Therefore, to obtain better estimates that are closer to the actual dynamic mechanical properties of compacted soil, it is necessary to further optimize the fractional-order three-element model.

The fractional-order three-element model can be optimized using the nonlinear least squares method, which estimates $\beta = 636330$, $E_\infty = 7787.6$, $E_0 = 3.2014\times10^9$, and $\alpha = 0.4364$ from Eq.(26). The optimized curves are illustrated in Fig. 4.

Results based on the optimized fractional-order three-element model are extremely close to the experimental data; therefore, the model can more accurately describe the dynamic mechanical properties of soil during vibratory compaction compared with the classical three-element model and non-optimized model.

Conclusions

1. Rheological properties of viscoelastic and viscoplastic materials can be described by fractional calculus; moreover, fractional calculus-based models can reflect the real deformation process and thus have powerful capabilities in capturing the static and dynamic stress-strain-time relationship of viscoelastic materials.

2. Results estimated using the proposed fractional-order three-element model are better correlated with experimental data compared with those estimated using the classical three-element model.

3. The proposed fractional-order three-element model can describe the viscoelastic dynamic mechanical properties of soil during vibratory compaction more accurately than the classical three-element model.

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