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Nonparametric estimation of trend for stochastic differential equations driven by fractional Brownian motion

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Abstract Consider a stochastic process $\{X_t, 0 \leq t \leq T\}$ governed by a stochastic differential equation given by

$$dX_t = S(X_t)\, dt + \epsilon\, dW_t^H, \quad X_0 = x_0, \quad 0 \leq t \leq T$$

where $\{W_t^H, 0 \leq t \leq T\}$ is a standard fractional Brownian motion with known Hurst parameter $H \in (1/2, 1)$ and the function $S(.)$ is an unknown function. Suppose the process $\{X_t, 0 \leq t \leq T\}$ is observed over the interval $[0, T]$. We consider the problem of nonparametric estimation of trend function $S_t = S(x_t)$ by a kernel type estimator

$$\hat{S}_t = \frac{1}{\phi_\epsilon} \int_0^T G\left(\frac{\tau - t}{\phi_\epsilon}\right) dX_\tau$$

and study the asymptotic behaviour of the estimator as $\epsilon \to 0$. Here x_t is the solution of the differential equation given above when $\epsilon = 0$.

Keywords Stochastic differential equation · Trend · Nonparametric estimation · Kernel method · Small noise · Fractional Brownian motion

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1 Introduction

Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes have been studied earlier by several authors and comprehensive

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surveys of various methods of parametric and nonparametric estimation are given in Basawa and Prakasa Rao (1980), Prakasa Rao (1999) and Kutoyants (1984, 1994, 2004) among others. There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion (fBm) starting with the works of Le Breton (1998) and Kleptsyna and Le Breton (2002). In some of our recent work (cf. Prakasa Rao 2003, 2004, 2005, 2008), we studied statistical inference problems for classes of stochastic processes satisfying linear stochastic differential equations driven by a fractional Brownian motion and investigated the asymptotic properties of the maximum likelihood and the Bayes estimators for parameters involved in such processes. Prakasa Rao (2010) gives a survey of results in statistical inference for fractional diffusion processes, that is, processes governed by stochastic differential equations driven by a fractional Brownian motion.

Consider a stochastic process $\{X_t, 0 \leq t \leq T\}$ governed by a stochastic differential equation given by

$$dX_t = S(X_t)\,dt + \epsilon\,dW_t^H, \quad X_0 = x_0, \quad 0 \leq t \leq T \tag{1.1}$$

where $\{W_t^H, 0 \leq t \leq T\}$ is a standard fractional Brownian motion with known Hurst parameter $H \in (1/2, 1)$ and $S(.)$ is an unknown function. Suppose the process $\{X_t, 0 \leq t \leq T\}$ is observed over the interval $[0, T]$. We consider the problem of nonparametric estimation of trend function $S_t = S(x_t)$ by a kernel type estimator

$$\hat{S}_t = \frac{1}{\phi_\epsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\epsilon}\right) dX_\tau$$

and study the asymptotic behaviour of the estimator as $\epsilon \to 0$. Here x_t is the solution of the Eq. 1.1 when $\epsilon = 0$.

The main results are presented in three theorems. Under some smoothness conditions, the first one proves the mean square consistency of the estimator, the second theorem gives a bound on the the rate of convergence and the third theorem deals with the asymptotic normality of the estimator. Kutoyants (1994) investigated a similar problem for processes driven by the Brownian motion. Our methods are similar to those in Kutoyants (1994).

2 Preliminaries

Let $W^H = \{W_t^H, t \geq 0\}$ be a normalized fractional Brownian motion with known Hurst parameter $H \in (\frac{1}{2}, 1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0$, $E(W_t^H) = 0$ and

$$E(W_s^H W_t^H) = \frac{1}{2}\left[s^{2H} + t^{2H} - |s - t|^{2H}\right], \quad t \geq 0, s \geq 0.$$

Let us consider the stochastic differential equation

$$dX_t = S(X_t)\,dt + \epsilon\,dW_t^H, \quad X_0 = x_0, \quad 0 \leq t \leq T \tag{2.1}$$

where the function $S(.)$ is unknown. Suppose $\{x_t, 0 \leq t \leq T\}$ is the solution of the differential equation

$$\frac{dx_t}{dt} = S(x_t), x_0, 0 \leq t \leq T. \tag{2.2}$$

We have to estimate the function $S_t = S(x_t)$ based on the observation $\{X_t, 0 \leq t \leq T\}$. We assume that the function $S(x)$ satisfies the following condition which ensures the existence and uniqueness of the solution of Eq. 2.1:
(A_1) There exists $L > 0$ such that

$$|S(x) - S(y)| \leq L|x - y|, x, y \in R.$$

It is clear that the condition (A_1) implies that there exists a constant $M > 0$ such that $|S(x)| \leq M(1 + |x|), x \in R$ by choosing $M = \max(|S(0)|, L)$.

Existence and uniqueness of the solution of the stochastic differential Eq. 2.1 follow as a special case of the results in Nualart and Rascanu (2002).

Suppose the function $S(.)$ is bounded by a constant C. Since the function x_t satisfies the ordinary differential Eq. 2.2, it follows that

$$|S(x_t) - S(x_s)| \leq L|x_t - x_s| = L\left|\int_t^s S(x_r)dr\right| \leq LC|t - s|, t, s \in R.$$

Lemma 2.1 *Let the function $S(.)$ satisfy the conditions (A_1). Let X_t and x_t be the solutions of the Eqs. 2.1 and 2.2, respectively. Then, with probability one,*

$$(a)\ |X_t - x_t| < e^{Lt}\epsilon|W_t^H| \tag{2.3}$$

and

$$(b)\ \sup_{0\leq t\leq T} E(X_t - x_t)^2 \leq e^{2LT}\epsilon^2 T^{2H}. \tag{2.4}$$

Proof of (a): Let $u_t = |X_t - x_t|$. Then, by (A_1), we have

$$\begin{aligned} u_t &\leq \int_0^t |S(X_v) - S(x_v)|\, dv + \epsilon\ |W_t^H| \\ &\leq L\int_0^t u_v dv + \epsilon\ |W_t^H|. \end{aligned}$$

Applying the Gronwall's lemma (cf. Lemma 1.12, Kutoyants (1994), p. 26), it follows that

$$u_t \leq \epsilon|W_t^H|e^{Lt}.$$

Proof of (b): From (2.3), we have ,

$$\begin{aligned} E(X_t - x_t)^2 &\leq e^{2Lt}\epsilon^2 E(|W_t|^H)^2 \\ &= e^{2Lt}\epsilon^2 t^{2H}. \end{aligned} \tag{2.5}$$

Hence

$$\sup_{0\leq t\leq T} E(X_t - x_t)^2 \leq e^{2LT}\epsilon^2 T^{2H}. \tag{2.6}$$

3 Main results

Let $\Theta_0(L)$ denote the class of all functions $S(x)$ satisfying the condition (A_1) and uniformly bounded by the same constant C. Let $\Theta_k(L)$ denote the class of all functions $S(x)$ which are uniformly bounded by the same constant C and which are k-times differentiable with respect to x satisfying the condition

$$|S^{(k)}(x) - S^{(k)}(y)| \leq L|x-y|, x, y \in R.$$

Here $g^{(k)}(x)$ denotes the k-th derivative of $g(.)$ at x.

Let $G(u)$ be a bounded function with finite support $[A, B]$ satisfying the condition

(A_2) $G(u) = 0$ for $u < A$ and $u > B$, and $\int_A^B G(u)du = 1$.

Further suppose that the following conditions are satisfied by the function $G(.)$:

(i) $\int_{-\infty}^{\infty} G^2(u)du < \infty$;

(ii) $\int_{-\infty}^{\infty} u^{2(k+1)} G^2(u)du < \infty$, and

(iii) $\int_{-\infty}^{\infty} |G(u)|^{\frac{1}{H}} du < \infty$.

Following the procedure adapted in Kutoyants (1994), we define a kernel type estimator of the trend $S_t = S(x_t)$ as

$$\widehat{S}_t = \frac{1}{\varphi_\epsilon} \int_0^T G\left(\frac{\tau - t}{\varphi_\epsilon}\right) dX_\tau$$

where the normalizing function $\varphi_\epsilon \to 0$ with $\epsilon^2 \varphi_\epsilon^{-1} \to 0$ as $\epsilon \to 0$. Let $E_S(.)$ denote the expectation operator when $S(.)$ is the trend function.

Theorem 3.1 *Suppose that the trend function $S(x) \in \Theta_0(L)$ and the function $\varphi_\epsilon \to 0$ such that $\epsilon^2 \varphi_\epsilon^{-1} \longrightarrow 0$ as $\epsilon \to 0$. Further suppose that the condition (A_2) holds. Then, for any $0 < c \leq d < T$, the estimator $\widehat{S}_t$ is uniformly consistent, that is ,*

$$\lim_{\epsilon \to 0} \sup_{S(x) \in \Theta_0(L)} \sup_{c \leq t \leq d} E_S(|\widehat{S}_t - S(x_t)|^2) = 0.$$

In addition to the conditions (A_1) and (A_2), assume that

(A_3) $\int_{-\infty}^{\infty} u^j G(u)du = 0$ *for* $j = 1, 2, \ldots, k$; *and* $\int_{-\infty}^{\infty} |G(u)u^{k+1}|du < \infty$.

Theorem 3.2 *Suppose that the function $S(x) \in \Theta_{k+1}(L)$ and $\varphi_\epsilon = \epsilon^{\frac{1}{k-H+2}}$. Then, under the conditions (A_2), and (A_3),*

$$\limsup_{\epsilon \to 0} \sup_{S(x) \in \Theta_{k+1}(L)} \sup_{c \leq t \leq d} E_S(|\widehat{S}_t - S(x_t)|^2) \epsilon^{\frac{-2(k+1)}{k-H+2}} < \infty.$$

Remark Under the conditions stated in Theorem 3.2, it follows that the mean square error (MSE) of the estimator $\widehat{S}_t$ of the function $S(x_t)$ is of the order $O(\epsilon^{\frac{2(k+1)}{k-H+2}})$ as $\epsilon \to 0$ and the order of the MSE decreases as k increases.

The results stated above are analogues of the corresponding results in Kutoyants (1994) for nonparametric estimation of the trend for processes governed by the Brownian motion. We follow the techniques in Kutoyants (1994) for proving these results.

Theorem 3.3 *Suppose that the function $S(x) \in \Theta_{k+1}(L)$ and $\varphi_\epsilon = \epsilon^{\frac{1}{k-H+2}}$. Then, under the conditions (A_2) and (A_3), the asymptotic distribution of*

$$\epsilon^{\frac{-(k+1)}{k-H+2}}(\widehat{S}_t - S(x_t))$$

is Gaussian with the mean

$$m = \frac{S^{(k+1)}(x_t)}{(k+1)!}\int_{-\infty}^{\infty} G(u)u^{k+1}\,du$$

and the variance

$$\sigma^2 = H(2H-1)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(u)G(v)|u-v|^{2H-2}du\,dv$$

as $\epsilon \to 0$.

4 Proofs of theorems

Proof of Theorem 3.1: From (2.1) we have,

$$\begin{aligned}
E_S[(\widehat{S}_t - S(x_t))^2] &= E_S\Bigg\{\frac{1}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right)(S(X_\tau)-S(x_\tau))\,d\tau \\
&\quad + \frac{1}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right)S(x_\tau)d\tau - S(x_t) + \frac{\epsilon}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right)dW_\tau^H\Bigg\}^2 \\
&\le 3E_S\left[\frac{1}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right)(S(X_\tau)-S(x_\tau))d\tau\right]^2 \\
&\quad +3\left[\frac{1}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right)S(x_\tau)d\tau - S(x_t)\right]^2 \\
&\quad +\frac{3\epsilon^2}{\varphi_\epsilon^2}E_S\left[\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right)dW_\tau^H\right]^2 \\
&= I_1 + I_2 + I_3 \ \text{(say)}.
\end{aligned} \tag{4.1}$$

Note that

$$\begin{aligned}
I_1 &= 3E_S\left[\frac{1}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right)(S(X_\tau)-S(x_\tau))d\tau\right]^2 \\
&= 3E_S\left[\int_{-\infty}^{\infty} G(u)\left(S(X_{t+\varphi_\epsilon u}) - S(x_{t+\varphi_\epsilon u})\right)du\right]^2
\end{aligned}$$

$$\leq 3(B-A)\int_{-\infty}^{\infty} G^2(u)L^2 E_S\left(X_{t+\varphi_\epsilon u}-x_{t+\varphi_\epsilon u}\right)^2\, du \ \text{ (by using the condition } (A_1))$$

$$\leq 3(B-A)\int_{-\infty}^{\infty} G^2(u)\ L^2 \sup_{0\leq t+\varphi_\epsilon u\leq T} E_S\left(X_{t+\varphi_\epsilon u}-x_{t+\varphi_\epsilon u}\right)^2\, du$$

$$\leq C_1\epsilon^2 \ \text{ (by using (2.4))} \tag{4.2}$$

for some positive constant C_1 depending on H, T, L, and $B-A$. Furthermore

$$I_2 = 3\left[\frac{1}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right) S(x_\tau)d\tau - S(x_t)\right]^2$$

$$= 3\left[\int_{-\infty}^{\infty} G(u)\left(S(x_{t+\varphi_\epsilon u})-S(x_t)\right)\, du\right]^2$$

$$\leq C_1'\left[\int_{-\infty}^{\infty} |G(u)u|\varphi_\epsilon du\right]^2$$

$$\leq C_2\varphi_\epsilon^2\int_{-\infty}^{\infty} G^2(u)u^2\, du$$

$$\leq C_3\varphi_\epsilon^2 \ \text{ (by } (A_2)(ii)) \tag{4.3}$$

for some positive constant C_3 depending on T, L, C and $B-A$. Furthermore the last term tends to zero as $\epsilon \to 0$. In addition, for $\frac{1}{2} < H < 1$,

$$I_3 = \frac{3\epsilon^2}{\varphi_\epsilon^2} E_S\left(\int_0^T G\left(\frac{\tau-t}{\varphi_\epsilon}\right) dW_\tau^H\right)^2$$

$$\leq \frac{3\epsilon^2}{\varphi_\epsilon^2} C_4(2,H)\left[\int_0^T \left|G\left(\frac{\tau-t}{\varphi_\epsilon}\right)\right|^{\frac{1}{H}} d\tau\right]^{2H} \quad \text{(cf. Memin et al. 2001)}$$

$$\leq \frac{C_5\epsilon^2}{\varphi_\epsilon^2}\varphi_\epsilon^{2H} \text{ (by using } (A_2)(iii)\text{)}$$

$$= C_6\frac{\epsilon^2}{\varphi_\epsilon}\varphi_\epsilon^{2H-1} \tag{4.4}$$

for some positive constant C_6 depending on H and T. Theorem 3.1 is now proved by using the equations (4.1) to (4.4).

Proof of Theorem 3.2: By the Taylor's formula, for any $x \in R$,

$$S(y) = S(x) + \sum_{j=1}^{k} S^{(j)}(x)\frac{(y-x)^j}{j!} + [S^{(k)}(z) - S^{(k)}(x)]\frac{(y-x)^k}{k!}$$

for some z such that $|z - x| \leq |y - x|$.

Using this expansion, the Eq. 2.2 and the conditions in the expression I_2 defined in the proof of Theorem 3.1, it follows that

$$
\begin{aligned}
I_2 &\leq 3 \left[\int_{-\infty}^{\infty} G(u) \left(S(x_{t+\varphi_\epsilon u}) - S(x_t) \right) du \right]^2 \\
&= 3 \left[\sum_{j=1}^{k} S^{(j)}(x_t) \left(\int_{-\infty}^{\infty} G(u) u^j du \right) \varphi_\epsilon^j (j!)^{-1} \right. \\
&\quad \left. + \left(\int_{-\infty}^{\infty} G(u) u^k \left(S^{(k)}(z_u) - S^{(k)}(x_t) \right) du \right) \varphi_\epsilon^k (k!)^{-1} \right]^2
\end{aligned}
$$

for some z_u such that $|x_t - z_u| \leq |x_{t+\varphi_\epsilon u} - x_t| \leq C|\varphi_\epsilon u|$. Hence

$$
\begin{aligned}
I_2 &\leq C_7 L^2 \left[\int_{-\infty}^{\infty} |G(u) u^{k+1}| \varphi_\epsilon^{k+1} (k!)^{-1} du \right]^2 \\
&\leq C_8 (B - A)(k!)^{-2} \varphi_\epsilon^{2(k+1)} \int_{-\infty}^{\infty} G^2(u) u^{2(k+1)}\, du \leq C_9 \varphi_\epsilon^{2(k+1)} \qquad (4.5)
\end{aligned}
$$

for some positive constant C_9 depending on C, H, T, L and $B - A$. Combining the relations (4.2), (4.4) and (4.5), we get that there exists a positive constant C_{10} depending on H, T, L and $B - A$ such that

$$
\sup_{c \leq t \leq d} E_S |\widehat{S}_t - S(x_t)|^2 \leq C_{10} (\epsilon^2 \varphi_\epsilon^{2H-2} + \varphi_\epsilon^{2(k+1)} + \epsilon^2).
$$

Choosing $\varphi_\epsilon = \epsilon^{\frac{1}{k-H+2}}$, we get that

$$
\limsup_{\epsilon \to 0} \sup_{S(x) \in \Theta_{k+1}(L)} \sup_{c \leq t \leq d} E_S |\widetilde{S}_t - S(x_t)|^2 \epsilon^{-\frac{2(k+1)}{k-H+2}} < \infty.
$$

This completes the proof of Theorem 3.2.

Remark Choosing $\varphi_\epsilon = \epsilon^{\frac{1}{2-H}}$ and without assuming the condition (A_3), it can be shown that

$$
\limsup_{\epsilon \to 0} \sup_{S(x) \in \Theta_0(L)} \sup_{c \leq t \leq d} E_S |\widehat{S}_t - S(x_t)|^2 \epsilon^{-\frac{2}{2-H}} < \infty
$$

which gives a slower rate of convergence than the one obtained in Theorem 3.2.

Proof of Theorem 3.3: From (2.1), we obtain that

$$
\begin{aligned}
&\widehat{S}_t - S(x_t)\\
&= \Bigg[\frac{1}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau - t}{\varphi_\epsilon}\right)(S(X_\tau) - S(x_\tau))\ d\tau \\
&\quad + \frac{1}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau - t}{\varphi_\epsilon}\right) S(x_\tau) d\tau - S(x_t) + \frac{\epsilon}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau - t}{\varphi_\epsilon}\right) dW_\tau^H \Bigg] \\
&\quad = \Bigg[\int_{-\infty}^{\infty} G(u)(S(X_{t+\varphi_\epsilon} u) - S(x_{t+\varphi_\epsilon u}))\ du \\
&\quad + \int_{-\infty}^{\infty} G(u)(S(x_{t+\varphi_\epsilon u}) - S(x_t))\ du + \frac{\epsilon}{\varphi_\epsilon}\int_0^T G\left(\frac{\tau - t}{\varphi_\epsilon}\right) dW_\tau^H \Bigg].
\end{aligned} \tag{4.6}
$$

Let $\varphi_\epsilon = \epsilon^{\frac{1}{k-H+2}}$ and

$$
\widehat{S}_t - S(x_t) = R_1 + R_2 + R_3 \ \ \text{(say)}. \tag{4.7}
$$

By the Taylor's formula, for any $x \in R$,

$$
S(y) = S(x) + \sum_{j=1}^{k+1} S^{(j)}(x)\frac{(y-x)^j}{j!} + \left[S^{(k+1)}(z) - S^{(k+1)}(x)\right]\frac{(y-x)^{k+1}}{k+1!}
$$

for some z such that $|z - x| \leq |y - x|$.

Using this expansion, the Eq. 2.2, relation (4.7) and the condition (A_3), it follows that

$$
(R_2 - m)^2 = \left[\int_{-\infty}^{\infty} G(u)\left(S^{(k+1)}(z_u) - S^{(k+1)}(x_t)\right)\frac{(\varphi_\epsilon u)^{k+1})}{(k+1)!}\ du\right]^2
$$

for some z_u such that $|x_t - z_u| \leq |x_{t+\varphi_\epsilon u} - x_t| \leq C|\varphi_\epsilon u|$. Hence, by arguments similar to those given following the inequality (4.5), it follows that there exists a positive constant C_{11} such that

$$
\begin{aligned}
(R_2 - m)^2 &\leq C_{11}L^2C^2\left(\int_{\infty}^{\infty} G(u)u^{k+2}\frac{\varphi_\epsilon^{k+2}}{(k+1)!}\ du\right)^2 \quad (\text{by}(A_2)) \\
&\leq C_{12}\varphi_\epsilon^{2(k+2)} \ \ (\text{by} \ \ (A_3))
\end{aligned} \tag{4.8}
$$

for some positive constant C_{12} depending on T, L, H, C and $B - A$. Therefore

$$
\epsilon^{-\frac{2(k+1)}{k-H+2}}(R_2 - m)^2 \to 0 \quad \text{as } \epsilon \to 0.
$$

Furthermore

$$
0 \leq \epsilon^{-\frac{2(k+1)}{k-H+2}} E_S[R_1^2] = \epsilon^{-\frac{2(k+1)}{k-H+2}} O(\phi_\epsilon^{2(k+2)})
$$

by arguments similar to those given for proving the inequality (4.5). Hence

$$\epsilon^{-\frac{2(k+1)}{k-H+2}} E_S[R_1^2] \to 0 \quad \text{as } \epsilon \to 0.$$

In addition, it follows that $E_S[R_3^2]$ is finite by $(A_2)(iii)$ and the variance of the Gaussian random variable

$$\int_{-\infty}^{\infty} G(t) dW_t^H$$

is

$$H(2H-1) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(u)G(v)|u-v|^{2H-2}\, du\, dv$$

(cf. Prakasa Rao 2004). Combining these observations, an application of Slutsky's lemma proves Theorem 3.3.

Remark It would be interesting to find verifiable conditions which give the locally asymptotic minimax lower bound on the risks of all estimators of the function $S_t = S(x_t)$ in the sense of Hajek-Lecam following the techniques in Ibragimov and Khasminskii (1981) and Kutoyants (1994) and to check that the kernel type estimator proposed above is asymptotically efficient in the sense of achieving this asymptotic lower bound.

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References

Basawa I, Prakasa Rao BLS (1980) Statistical inference for stochastic processes. Academic Press, London

Ibragimov I, Khasminskii RZ (1981) Statistical estimation: asymptotic theory. Springer, New York

Kleptsyna M, Le Breton A (2002) Statistical analysis of the fractional Ornstein-Uhlenbeck process. Stat Infer Stoch Process 5:229–248

Kutoyants YA (1984) Parameter estimation for stochastic processes. Helderman, Berlin

Kutoyants YA (1994) Identification of dynamical systems with small noise. Kluwer, Dordrecht

Kutoyants YA (2004) Statistical inference for ergodic diffusion processes. Springer, London

Le Breton A (1998) Filtering and parameter estimation in a simple linear model driven by a fractional Brownian motion. Stat Prob Lett 38:263–274

Memin J, Mishura Y, Valkeila E (2001) Inequalities for the moments of Wiener integrals with respect to a fractional Brownian motion. Stat Prob Lett 51:197–206

Nualart D, Rascanu A (2002) Differential equations driven by fractional Brownian motion. Collect Math 53:55–81

Prakasa Rao BLS (1999) Statistical inference for diffusion type processes. Arnold, London and Oxford University Press, New York

Prakasa Rao BLS (2003) Parameter estimation for linear stochastic differential equations driven by fractional Brownian motion. Random Oper Stoch Eq 11:229–242

Prakasa Rao BLS (2004) Minimum L_1-norm estimation for fractional Ornstein-Uhlenbeck type process. Theor Prob Math Stat 71:181–189

Prakasa Rao BLS (2005) Berry-Esseen bound for MLE for linear stochastic differential equations driven by fractional Brownian motion. J Korean Stat Soc 34:281–295

Prakasa Rao BLS (2008) Parameter estimation for linear stochastic delay differential equations driven by fractional Brownian motion. Random Oper Stoch Eq 16:27–38

Prakasa Rao BLS (2010) Statistical inference for fractional diffusion processes. Wiley, Chichester