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LEVI QUASIVARIETIES[†)]

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Given a group-theoretic property $\mathcal{E}$, we say that a group G *possesses the property* $L(\mathcal{E})$ if the normal closure $(x)^G$ of every element x of G possesses the property $\mathcal{E}$. The property $L(\mathcal{E})$ is called a *Levi property* and was introduced in [1] under the influence of Levi's article [2] in which Levi gave a classification of groups with abelian normal closures of the form $(x)^G$. Observe that Levi properties are closely connected with the notion of Engel group. As regards nilpotent groups and their generalizations, these properties were studied rather thoroughly, for instance, in [1,3,4]. Study of Levi properties should be considered as a step towards studying the structure of groups covered by a system of normal subgroups.

It is natural to pass from Levi properties to Levi classes. Given a class $\mathcal{M}$ of groups, we denote by $L(\mathcal{M})$ the class of all groups G in which the normal closure $(x)^G$ of an arbitrary element x in G belongs to $\mathcal{M}$. The class $L(\mathcal{M})$ is said to be the *Levi class* of $\mathcal{M}$. It is shown in [5] that if $\mathcal{M}$ is a variety of groups then $L(\mathcal{M})$ is a variety too. It is also well known [6] that if $\mathcal{M}$ is the class of nilpotent groups of step ≤ 2 without elements of order 2 and order 5, then $L(\mathcal{M}) \subseteq \mathcal{N}_4$, where $\mathcal{N}_c$ is the variety of nilpotent groups of nilpotency class $\leq c$.

In the present article we demonstrate that if $\mathcal{M}$ is a quasivariety of groups then $L(\mathcal{M})$ is a quasivariety of groups as well. We find a condition whose fulfillment in the quasivariety $\mathcal{M}$ implies the inclusion $L(\mathcal{M}) \subseteq \mathcal{N}_3$. In particular, it turns out that if $\mathcal{M}$ is a minimal nonabelian quasivariety of nilpotent groups (for example, the quasivariety generated by a free nilpotent group of class 2) which has no groups of order 2 and order 5, then $L(\mathcal{M}) \subseteq \mathcal{N}_3$.

We recall that a group is called a 3-*Engel group* if the identity

$$(\forall x)(\forall y)([x,y,y,y]=1)$$

holds in it.

We need the following test for the membership of a finitely related group G in the quasivariety $q\mathcal{K}$. This *membership test* is a particular instance of Theorem 3 in [7] and reads: *a finitely related group G belongs to the quasivariety $q\mathcal{K}$ if and only if, for every nonidentity element $g \in G$, there exists a homomorphism φ from G into some group of $\mathcal{K}$ such that $\varphi(g) \neq 1$.*

Theorem 1. *If $\mathcal{M}$ is a quasivariety of groups then $L(\mathcal{M})$ is a quasivariety of groups as well.*

PROOF. It is clear that, together with each group, $L(\mathcal{M})$ contains all subgroups of the group. In view of Mal'tsev's theorem [8], it suffices to prove that the class $L(\mathcal{M})$ is closed under filtered products.

Suppose that $G_i \in L(\mathcal{M})$ $(i \in I)$, $\mathcal{D}$ is a filter over I, $G = \prod_{i\in I} G_i/\mathcal{D}$ is the filtered product of the groups G_i with respect to the filter $\mathcal{D}$, $f\mathcal{D} \in G$, and $f\mathcal{D} \neq 1$. Let $A_i = (f(i))^{G_i}$ be the normal closure of the element $f(i)$ in G_i and $A = \prod_{i\in I} A_i/\mathcal{D}$. Consider the mapping $\varphi : (f\mathcal{D})^G \to A$ defined as follows:

$$\varphi((f^{k_1}\mathcal{D})^{g_1\mathcal{D}} \dots (f^{k_s}\mathcal{D})^{g_s\mathcal{D}}) = h\mathcal{D},$$

where $h(i) = (f^{k_1}(i))^{g_1(i)} \dots (f^{k_s}(i))^{g_s(i)}$. It is easy to see that φ is an isomorphism of the group $(f\mathcal{D})^G$ onto the group $\varphi((f\mathcal{D})^G)$. Therefore, $(f\mathcal{D})^G \in \mathcal{M}$, which completes the proof of the theorem.

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Henceforth we use the following commutator identities which are valid in every group:

$$[xy, z] = [x, z]^y [y, z] = [x, z][x, z, y][y, z], \tag{1}$$

$$[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z], \tag{2}$$

$$[x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = 1, \tag{3}$$

$$[v, u^{-1}] = [v, u, u^{-1}]^{-1}[v, u]^{-1}, \tag{4}$$

$$[v, u, u^{-1}] = [v, u, u, u^{-1}]^{-1}[v, u, u]^{-1}, \tag{5}$$

$$[u^{-1}, v] = [v, u][v, u, u^{-1}]. \tag{6}$$

Lemma 1. *Let G be a 3-Engel group without elements of order 2 and order 5, $x_1, x_2, x_3 \in G$, $c(i,j,s,t) = [x_i, x_j, x_s, x_t]$, $z_1 = [x_3, x_2, [x_3, x_1]]$, $z_2 = [x_3, x_2, [x_2, x_1]]$, and $z_3 = [x_3, x_1, [x_2, x_1]]$. Then the following relations hold:*

$c(3,1,3,2) = c(3,1,2,3)z_1^{-1}$,
$c(2,1,3,2) = c(2,1,2,3)z_2^{-1}$,
$c(3,2,1,1) = c(2,1,1,3)^{-1}c(3,1,1,2)z_3^2$,
$c(3,2,1,3) = c(2,1,3,3)^{-1}c(3,1,2,3)$,
$c(3,2,3,1) = c(2,1,3,3)^{-1}c(3,1,2,3)z_1$,
$c(3,2,1,1) = c(2,1,1,3)^{-1}c(3,1,1,2)z_3^2$,
$c(1,3,1,2) = c(3,1,1,2)^{-1}$,
$c(1,3,2,1) = c(3,1,1,2)^{-1}z_3^{-1}$,
$c(1,3,2,2) = c(3,1,2,2)^{-1}$,
$c(1,3,2,3) = c(3,1,2,3)^{-1}$,
$c(3,2,2,1) = c(2,1,2,3)^{-1}c(3,1,2,2)z_2^2$,
$c(3,2,1,2) = c(2,1,2,3)^{-1}c(3,1,2,2)z_2$,
$c(2,1,3,1) = c(2,1,1,3)z_3^{-1}$.

PROOF. We will not carry out the proof of this lemma in detail. We only note that

$$c(3,1,1,2),\ c(3,1,2,2),\ c(3,1,2,3),\ c(2,1,1,3), \tag{7}$$

$$c(2,1,2,3),\ c(2,1,3,3),\ z_1,\ z_2,\ z_3 \tag{8}$$

are all basis commutators of weight 4 in three variables. The sought relations are obtained by a standard method, used in Hall's collection process (see, for instance, [9]), by means of (1)–(6). Two circumstances should be taken into account: (a) since by [6] every 2-generated 3-Engel group without elements of order 2 is nilpotent of class ≤ 3, the values of all commutators in two variables in the group G are equal to 1; (b) again by [6] every 3-Engel group without elements of order 2 and order 5 is nilpotent of class ≤ 4; hence, so is G.

Lemma 2. *Let G be a 3-Engel group without elements of order 2 and order 5, $x_1, x_2, x_3 \in G$, $c(i,j,s,t) = [x_i, x_j, x_s, x_t]$, $z_1 = [x_3, x_2, [x_3, x_1]]$, $z_2 = [x_3, x_2, [x_2, x_1]]$, and $z_3 = [x_3, x_1, [x_2, x_1]]$. Then the following relations hold:*

$$c(3,1,1,2)^3 c(2,1,1,3)^{-1} z_3^3 = 1, \tag{9}$$

$$c(3,1,2,2)^3 c(2,1,2,3)^{-2} z_2^3 = 1, \tag{10}$$

$$c(3,1,2,3)^4 c(2,1,3,3)^{-2} = 1, \tag{11}$$

$$c(2,1,1,3)^3 c(3,1,1,2)^{-1} z_3^{-3} = 1, \tag{12}$$

$$c(2,1,3,3)^3 c(3,1,2,3)^{-2} z_1^{-1} = 1, \tag{13}$$

$$c(2,1,2,3)^4 c(3,1,2,2)^{-2} z_2^{-4} = 1, \tag{14}$$

$$c(2,1,1,3)^{-2}c(3,1,1,2)^{-2} = 1, \tag{15}$$

$$c(2,1,3,3)^{-1}c(3,1,2,3)^{-2}z_1 = 1, \tag{16}$$

$$c(2,1,2,3)^{-2}c(3,1,2,2)^{-1}z_2 = 1. \tag{17}$$

PROOF. Let k, m, and n be arbitrary integers. Express the element

$$[x_3, x_3^k x_2^m x_1^n, x_3^k x_2^m x_1^n, x_3^k x_2^m x_1^n]$$

in terms of the elements (7) and (8). Using the fact that by [6] every 2-generated 3-Engel group without elements of order 2 is nilpotent of class ≤ 3 and that every 3-Engel group without elements of order 2 and order 5 is nilpotent of class ≤ 4 and involving Lemma 1, we infer that

$$\begin{aligned} 1 &= [x_3, x_3^k x_2^m x_1^n, x_3^k x_2^m x_1^n, x_3^k x_2^m x_1^n] \\ &= \left(c(3,1,1,2)^3 c(2,1,1,3)^{-1} z_3^3\right)^{n^2 m} \left(c(3,1,2,2)^3 c(2,1,2,3)^{-2} z_2^3\right)^{nm^2} \\ &\quad \times \left(c(3,1,2,3)^4 c(2,1,3,3)^{-2}\right)^{kmn}. \end{aligned} \tag{18}$$

Putting $k = 0$ in (18), we deduce that the relation

$$\left(c(3,1,1,2)^3 c(2,1,1,3)^{-1} z_3^3\right)^{n^2 m} \left(c(3,1,2,2)^3 c(2,1,2,3)^{-2} z_2^3\right)^{nm^2} = 1$$

holds for arbitrary integers m and n. By setting in this equality, first, $n = 1$ and $m = 1$, and next $n = 1$ and $m = 2$ and recalling that G contains no elements of order 2, we easily arrive at (9) and (10).

Now, (18) implies the equality

$$\left(c(3,1,2,3)^4 c(2,1,3,3)^{-2}\right)^{kmn} = 1,$$

whence we infer (11).

Equalities (12)–(14) are derived likewise from the relations

$$1 = [x_2, x_3^k x_2^m x_1^n, x_3^k x_2^m x_1^n, x_3^k x_2^m x_1^n],$$

and equalities (15)–(17), from the relations

$$1 = [x_1, x_3^k x_2^m x_1^n, x_3^k x_2^m x_1^n, x_3^k x_2^m x_1^n].$$

The lemma is proven.

Theorem 2. *Let K be a set of nilpotent groups of class 2 without elements of order 2 and order 5. Suppose that, in each group of K, the centralizer of every nonidentity element not belonging to the center of the group is an abelian subgroup. If $\mathcal{M} = qK$ then $L(\mathcal{M}) \subseteq \mathcal{N}_3$.*

PROOF. It is well known [10] that if every 3-generated subgroup of a given group is nilpotent of class ≤ 3 then the group itself is also nilpotent of class ≤ 3. Therefore, it suffices to demonstrate that every 3-generated group G in $L(\mathcal{M})$ belongs to $\mathcal{N}_3$.

Thus, take $G = gr(a, b, c)$. According to [11], G is 3-Engel. By setting $x_1 = a$, $x_2 = b$, and $x_3 = c$, from (11), (13), and (16) we easily deduce that

$$c(3,1,2,3)^2 = c(2,1,3,3), \quad z_1 = c(3,1,2,3)^4.$$

Hence,

$$[c, b, [c, a]] = [c, a, b, c]^4. \tag{19}$$

Choose the following elements in $(c)^G$:

$$x = [c, a, b], \quad y = [c, a], \quad z = [c, b], \quad c.$$

It follows from (19) that

$$[z, y] = [x, c]^4, \quad [x^4, y] = 1, \quad [x^4, z] = 1. \tag{20}$$

Suppose that $[z, y] \neq 1$. Since $(c)^G \in \mathcal{M}$, the membership test implies existence of a homomorphism $\varphi : (c)^G \to A$, with A a suitable group in K, for which $\varphi([z, y]) \neq 1$. However, $\varphi(y)$ and $\varphi(z)$ belong to the centralizer of the element $\varphi(x^4)$. Hence, $[\varphi(y), \varphi(z)] = 1$ by the hypothesis of the theorem. This contradiction means that $[z, y] = 1$. We have thus shown that

$$[c, b, [c, a]] = 1. \tag{21}$$

By setting $x_3 = b$, $x_2 = a$, and $x_1 = c$, in a similar way we infer that $[b, a, [b, c]] = 1$; i.e.,

$$[c, b, [b, a]] = 1. \tag{22}$$

By setting $x_3 = a$, $x_2 = b$, and $x_1 = c$, by analogy to (20) we obtain $[a, b, [a, c]] = 1$; i.e.,

$$[c, a, [b, a]] = 1. \tag{23}$$

Again assuming that $x_3 = c$, $x_2 = b$, and $x_1 = a$, from the system of (9)–(17) and (21)–(23) we now easily deduce that $c(i, j, k, l) = 1$ for all values $c(i, j, k, l)$ of the commutators involved in this system. This means that $G = gr(a, b, c)$ is a nilpotent group of class ≤ 3. The theorem is proven.

Consider groups with the following presentation in the variety $\mathcal{N}_2$:

$$H_{ps} = gr(x, y \parallel [x, y]^p = x^{p^s} = y^{p^s} = 1), \quad s \in \mathbb{N},$$
$$H_p = gr(x, y \parallel [x, y]^p = 1).$$

Let $F_2(\mathcal{N}_2)$ be a free nilpotent group of class 2 and rank 2. These groups have the following property [7]: the collection

$$qH_{ps}, \ qH_p, \ ,qH_{22}, \ qF_2(\mathcal{N}_2), \quad p \neq 2, \quad p \text{ is a prime number}, \tag{24}$$

exhausts the full list of quasivarieties of nilpotent groups all whose proper subquasivarieties contain only abelian groups. Since the groups in this list satisfy the conditions of Theorem 2, we come to the following

Corollary. *If $\mathcal{M}$ is one of the quasivarieties* (24) *then* $L(\mathcal{M}) \subseteq \mathcal{N}_3$.

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