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MAXIMAL REGULAR ABSTRACT ELLIPTIC EQUATIONS AND APPLICATIONS

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Abstract: The oblique derivative problem is addressed for an elliptic operator differential equation with variable coefficients in a smooth domain. Several conditions are obtained, guaranteing the maximal regularity, the Fredholm property, and the positivity of this problem in vector-valued L_p-spaces. The principal part of the corresponding differential operator is nonselfadjoint. We show the discreteness of the spectrum and completeness of the root elements of this differential operator. These results are applied to anisotropic elliptic equations.

Keywords: boundary value problem, operator differential equation, completeness of root elements, Banach-valued function spaces, operator-valued multipliers, interpolation of Banach spaces, semigroup of operators

1. Introduction, Notation, and Background

In the recent years, the maximal regularity properties of boundary value problems (BVPs) for operator differential equations have found many applications to partial differential and pseudodifferential equations and various physical processes (see [1–19] for references). The main objective of the present paper is to discuss the maximal L_p regularity properties of BVPs for the elliptic operator differential equation with varying top coefficients

$$\begin{gathered} Lu = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} - A_\lambda(x)u(x) + \sum_{k=1}^{n} A_k(x)\frac{\partial u(x)}{\partial x_k} = f(x), \quad x \in G, \\ L_\Gamma u = \left[\alpha(x)\frac{\partial u}{\partial l} + \beta(x)u\right]_\Gamma = 0, \end{gathered} \tag{1}$$

where Γ is a boundary of a domain $G \subset \mathbb{R}^n$, l is nontangential direction on Γ, a_{ij}, α, and β are complex-valued functions, $A_\lambda(x) = A(x)+\lambda$, where A and A_k are possibly unbounded linear operators in a Banach space E.

Maximal regularity of partial differential operator equations in L_p-spaces was studied in [1, 4, 5, 15–19]. The results in [4] and [15–19] are restricted to the rectangular domains and the equations without mixed derivatives in the leading part. Moreover, the problems under study in [1] and [5] involve only bounded operator coefficients.

Unlike to the above articles, we study elliptic problems with unbounded operator coefficients in general domains with smooth boundaries.

We say that (1) is *maximal L_p-regular* (or *separable* in L_p) if:

(1) for all $f \in L_p(G;E)$ there exists a unique solution $u \in W_p^2(G;E(A),E)$ satisfying (1) a.e. on G;

(2) there exists a positive constant C independent of f such that

$$\sum_{i,j=1}^{n} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L_p(G;E)} + \|Au\|_{L_p(G;E)} \le C\|f\|_{L_p(G;E)}.$$

Let O be the operator generated by (1); i.e.,

$$D(O) = W_p^2(G;E(A),E,L_\Gamma) = \{u : u \in W_p^2(G;E) \cap L_p(G;E(A)),\ L_\Gamma u = 0\},\ Ou = Lu.$$

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We show that (1) is maximal L_p-regular, which implies that there is a bounded inverse of O from L_p to $W_p^2(G; E(A), E)$. Moreover, we prove that O is a positive operator and the generator of an analytic semigroup in L_p.

Since (1) involves unbounded operator coefficients, it becomes difficult to obtain global estimates for solutions of (1). Therefore, to show that O has a left inverse and its range coincides with L_p, we use the covering and flattening arguments, some formulas for solutions, results on the operator-valued Fourier multipliers, some abstract embedding theorems (Theorems A_1 and A_2) and the separability properties of local differential operators with constant coefficients (both on the plane and half-plane). Then by using these results along with the qualitative properties of some embedding operators we prove discreteness of the spectrum and the completeness of the root elements of O. By way of application we establish well-posedness for anisotropic elliptic equations in $L_{\mathbf{p}}$, $\mathbf{p} = (p_1, p)$ (i.e., the Lebesgue spaces with mixed norm) and L_p-separability for infinite systems of elliptic equations.

The paper is organized as follows: Section 2 collects definitions and background materials, embedding theorems of Sobolev–Lions spaces, maximal regularity properties for elliptic operator differential equations on the line and half-line and estimates of approximation numbers. In Section 3, the L_p-separability and results on the Fredholm property for (1) are presented. Finally, Sections 4 and 5 are devoted to spectral properties of O and some applications.

A Banach space E is an *UMD-space* if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int\limits_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy$$

is bounded in $L_p(R, E)$, $p \in (1, \infty)$ (e.g., see [20]). The UMD-spaces include, e.g., L_p- and l_p-spaces and as well the Lorentz spaces L_{pq}, $p, q \in (1, \infty)$.

Let $\mathbf{C}$ be the set of complex numbers and

$$S_\varphi = \{\lambda \in \mathbf{C},\ |\arg \lambda| \le \varphi\} \cup \{0\}, \quad 0 \le \varphi < \pi.$$

A linear operator A is *positive with bound* $M > 0$ in a Banach space E if $D(A)$ is dense in E and $\|(A + \lambda I)^{-1}\|_{L(E)} \le M(1 + |\lambda|)^{-1}$ with $\lambda \in S_\varphi, \varphi \in [0, \pi)$, where I is the identity operator in E and $L(E)$ is the space of all bounded linear operators in E. Sometimes instead of $A + \lambda I$ we will write $A + \lambda$ or A_λ. It is known that [21, § 1.15.1] there exist fractional powers A^θ of a positive operator A. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with the graph norm

$$\|u\|_{E(A^\theta)} = (\|u\|^p + \|A^\theta u\|^p)^{\frac{1}{p}}, \quad 1 \le p < \infty,\ -\infty < \theta < \infty.$$

Let E_1 and E_2 be two Banach spaces. Also $(E_1, E_2)_{\theta,p}$, $0 < \theta < 1$, $1 \le p \le \infty$, will denote the interpolation spaces that are defined by the K-method [21, § 1.3.1].

A set $W \subset B(E_1, E_2)$ is called *R-bounded* (see [5, 22]) if there is a constant $C > 0$ such that for all $T_1, T_2, \dots, T_m \in W$ and $u_1, u_2, \dots, u_m \in E_1$, $m \in N$,

$$\int\limits_0^1 \Big\| \sum_{j=1}^m r_j(y) T_j u_j \Big\|_{E_2} dy \le C \int\limits_0^1 \Big\| \sum_{j=1}^m r_j(y) u_j \Big\|_{E_1} dy,$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$.

Let $S(\mathbb{R}^n; E)$ denote the Schwartz class, i.e. the space of all E-valued rapidly decreasing smooth functions on $\mathbb{R}^n$. Let F stand for the Fourier transform. A function $\Psi \in C(\mathbb{R}^n; B(E_1, E_2))$ is called a *Fourier multiplier* from $L_p(\mathbb{R}^n; E_1)$ to $L_p(\mathbb{R}^n; E_2)$ if the map $u \to \Lambda u = F^{-1}\Psi(\xi)Fu$, $u \in S(\mathbb{R}^n; E_1)$ is well defined and extends to the bounded linear operator $\Lambda : L_p(\mathbb{R}^n; E_1) \to L_p(\mathbb{R}^n; E_2)$. The set of all multipliers from $L_p(\mathbb{R}^n; E_1)$ to $L_p(\mathbb{R}^n; E_2)$ will be denoted by $M_p^p(E_1, E_2)$. If $E_1 = E_2 = E$ we use the symbol $M_p^p(E)$.

Let $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, $\xi \in \mathbb{R}^n$, $\beta = (\beta_1, \beta_2, \dots, \beta_n)$, $\xi^\beta = \xi_1^{\beta_1}\xi_2^{\beta_2}\dots\xi_n^{\beta_n}$, $U_n = \{\beta = (\beta_1, \beta_2, \dots, \beta_n)$, $\beta_k \in \{0,1\}\}$.

Definition 1. A Banach space E is said to *satisfy the multiplier condition* if for every $\Psi \in C^{(n)}(\mathbb{R}^n; B(E))$ the R-boundedness of $\{\xi^\beta D_\xi^\beta \Psi(\xi) : \xi \in \mathbb{R}^n\backslash 0, \beta \in U_n\}$ implies that Ψ is a Fourier multiplier in $L_p(\mathbb{R}^n; E)$; i.e., $\Psi \in M_p^p(E)$ for all $p \in (1, \infty)$.

Definition 2. A φ-positive operator A is said to be an *R-positive in a Banach space E* if there exists $\varphi \in [0, \pi)$ such that $L_A = \{A(A+\xi)^{-1} : \xi \in S_\varphi\}$ is R-bounded.

A linear operator $A(x)$ is called *positive on E uniformly* in x if $D(A(x))$ is independent of x, $D(A(x))$ is dense in E, and

$$\|(A(x) + \lambda I)^{-1}\| \le \frac{M}{1+|\lambda|}$$

for all $\lambda \in S(\varphi)$, $\varphi \in [0, \pi)$.

Let $\sigma_\infty(E_1, E_2)$ denote the space of all compact operators from E_1 to E_2. If $E_1 = E_2 = E$ then we use the denotation $\sigma_\infty(E)$. Denote by $s_j(A)$ and $d_j(A)$ the approximation numbers (s-numbers) and Kolmogorov numbers (d-numbers) of A (e.g., see [21, § 1.16.1]). Put

$$\sigma_q(E_1, E_2) = \left\{A : A \in \sigma_\infty(E_1, E_2), \quad \sum_{j=1}^{\infty} s_j^q(A) < \infty, \quad 1 \le q < \infty\right\}.$$

Let E_0 and E be two Banach spaces with E_0 embedded continuously and densely into E. Let m be a natural number.

Let $W_p^m(G; E_0, E)$ stand for a function space of $u \in L_p(G; E_0)$ having the generalized derivatives $D_k^m u = \frac{\partial^m u}{\partial x_k^m}$ such that $D_k^m u \in L_p(G; E)$ which is endowed with the norm

$$\|u\|_{W_p^m(G;E_0,E)} = \|u\|_{L_p(G;E_0)} + \sum_{k=1}^{n} \|D_k^m u\|_{L_p(G;E)} < \infty.$$

We will called $W_p^m(G; E_0, E)$ a *Sobolev–Lions type space.* If $E_0 = E$ then $W_p^m(G; E_0, E)$ will be denoted by $W_p^m(G; E)$. Clearly,

$$W_p^m(G; E_0, E) = W_p^m(G; E) \cap L_p(G; E_0).$$

Let G be a domain in $\mathbb{R}^n$ with sufficiently smooth boundary Γ. The space $B_{p,p}^s(\Gamma; E_0, E)$ is defined as the trace space of $W_p^m(G; E_0, E)$ as in a scalar case (see [9, § 1.7.3] or [21, § 3.6.1]), i.e. for $E_0 = E = \mathbf{C}$ replacing $L_p(\mathbb{R}^{n-1})$ by $L_p(\mathbb{R}^{n-1}; E)$.

Theorem $\mathbf{A_1}$ [19]. *Suppose that*

(1) *E is a Banach space enjoying the multiplier condition with respect to $p \in (1, \infty)$ and A is an R-positive operator in E;*

(2) *$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ are n tuples of nonnegative integer numbers such that $\varkappa = \frac{|\alpha|}{m} \le 1$ and $0 < h \le h_0 < \infty$, $0 < \mu \le 1 - \varkappa$;*

(3) *$\Omega \in \mathbb{R}^n$ is a domain such that there is a bounded linear extension operator from $W_p^m(G; E(A), E)$ to $W_p^m(\mathbb{R}^n; E(A), E)$.*

Then the embedding $D^\alpha W_p^m(G; E(A), E) \subset L_p(G; E(A^{1-\varkappa-\mu}))$ is continuous and there exists a positive constant C_μ such that

$$\|D^\alpha u\|_{L_p(G;E(A^{1-\varkappa-\mu}))} \le C_\mu[h^\mu\|u\|_{W_p^m(G;E(A),E)} + h^{-(1-\mu)}\|u\|_{L_p(G;E)}]$$

for all $u \in W_p^m(G; E(A), E)$.

Theorem A$_2$. *Suppose that all conditions of Theorem* A$_1$ *are satisfied and G is a bounded domain in $\mathbb{R}^n$, $A^{-1} \in \sigma_\infty(E)$. Then the embedding $D^\alpha W_p^m(G; E(A), E) \subset L_p(G; E(A^{1-\varkappa-\mu}))$ is compact for $0 < \mu \le 1-\varkappa$.*

Put

$$E_j = W_p^{2(1-\theta_j)}(\mathbb{R}^{n-1}; (E(A), E)_{\theta_j,p}, E), \quad \theta_j = \frac{pj+1}{2p}.$$

Take $x^1 = (x_1, x_2, \dots, x_{n-1})$.

Theorem A$_3$. *$u \to u^{(j)}(x^1, 0)$ are surjective bounded linear operators from $W_p^2(R_+^n; E(A), E)$ to E_j.*

Proof. Indeed, $W_p^2(R_+^n; E(A), E) = W_p^2(R_+; E_0, E_1)$ where $E_0 = W_p^2(\mathbb{R}^{n-1}; E(A), E)$ and $E_1 = L_p(\mathbb{R}^{n-1}; E)$. By [21, § 3.6.1] the maps $u \to u^{(j)}(x^1, 0)$ are surjective bounded linear operators from $W_p^2(R_+^n; E(A), E)$ to $(E_0, E_1)_{\theta_j,p} = (W_p^2(\mathbb{R}^{n-1}; E(A), E), L_p(\mathbb{R}^{n-1}; E))_{\theta_j,p}$.

Since $W_p^2(\mathbb{R}^{n-1}; E(A), E) = W_p^2(\mathbb{R}^{n-1}; E) \cap L_p(\mathbb{R}^{n-1}; E(A))$; therefore,

$$(E_0, E_1)_{\theta_j,p} = (W_p^2(\mathbb{R}^{n-1}; E), L_p(\mathbb{R}^{n-1}; E))_{\theta_j,p} \cap (L_p(\mathbb{R}^{n-1}; E(A)), L_p(\mathbb{R}^{n-1}; E))_{\theta_j,p}.$$

By interpolation between $W_p^2(\mathbb{R}^{n-1}; E)$, $L_p(\mathbb{R}^{n-1}; E)$, and $L_p(\mathbb{R}^{n-1}; E(A))$ (e.g., see [23; 21 § 1.18]) we have

$$\big(W_p^2(\mathbb{R}^{n-1}; E), L_p(\mathbb{R}^{n-1}; E)\big)_{\theta_j,p} = W_p^{2(1-\theta_j)}(\mathbb{R}^{n-1}; E),$$

$$(L_p(\mathbb{R}^{n-1}; E(A)), L_p(\mathbb{R}^{n-1}; E))_{\theta_j,p} = L_p(\mathbb{R}^{n-1}; (E(A), E)_{\theta_j,p}).$$

From the above we infer the desired result.

First, consider the following operator differential equation on the whole $\mathbb{R}^n$

$$(L+\lambda)u = \sum_{k=1}^n a_k \frac{\partial^2 u(x)}{\partial x_k^2} + (A+\lambda)u = f(x), \quad x \in \mathbb{R}^n. \tag{2}$$

Put $L(\xi) = \sum_{k=1}^n a_k \xi_k^2$ for $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$.

Theorem A$_4$ [19]. *Let E be a Banach space satisfying the multiplier condition with respect to $p \in (1, \infty)$ and let A be an R-positive operator in E for $\varphi \in [0, \pi)$. Moreover,*

$$|L(\xi)| \ge M \sum_{k=1}^n |\xi_k|^2, \quad L(\xi) \in S(\varphi).$$

Then (2) has the unique solution $u \in W_p^2(\mathbb{R}^n; E(A), E)$ for $f \in L_p(\mathbb{R}^n; E)$, $|\arg \lambda| \le \varphi$ and the uniform coercive estimate holds

$$\sum_{k=1}^n \sum_{i=0}^2 |\lambda|^{1-\frac{i}{2}} \|D_k^i u\|_{L_p(\mathbb{R}^n;E)} + \|Au\|_{L_p(\mathbb{R}^n;E)} \le M\|f\|_{L_p(\mathbb{R}^n;E)}.$$

Consider the BVPs for the operator differential equation

$$(L+\lambda)u = \sum_{k=1}^n a_k D_k^2 u(x) + A_\lambda u = f(x), \quad x \in R_+^n, \tag{3}$$
$$L_k u = \sum_{j=1}^{m_k} \alpha_{kj} u^{(j)}(x^1, 0) = f_k, \quad k = 1, 2,$$

where $m_k \in \{0, 1\}$, a and α_{kj} are complex numbers, A is a possibly unbounded operator in E, and $A_\lambda = A + \lambda$.

Let ω_j, $j = 1, 2$, be the roots of an equation $a_n\omega^2 + 1 = 0$ and

$$L_0(\xi) = \sum_{j=1}^{n-1} a_j \xi_j^2,$$

$$F_k = (W_p^2(\mathbb{R}^{n-1}; E(A), E), L_p(\mathbb{R}^{n-1}; E))_{\theta_k, p}, \quad \theta_k = \frac{pm_k + 1}{2p}, \quad k = 1, 2.$$

By [2] and the trace Theorem A_3 we obtain

Theorem A_5. *Let E be a Banach space satisfying the multiplier condition with respect to $p \in (1, \infty)$ and let A be an R-positive operator in E for some $\varphi \in [0, \pi)$. Let $|\arg\ \omega_1 - \pi| \le \frac{\pi}{2} - \varphi$, $|\arg\ \omega_2| \le \frac{\pi}{2} - \varphi$, $\alpha_{km_k} \ne 0$, and*

$$|L_0(\xi)| \ge M \sum_{k=1}^{n-1} |\xi_k|^2, \quad L_0(\xi) \in S(\varphi).$$

Then the operator $u \to \{[L + \lambda]u, L_1 u, L_2 u\}$ is an isomorphism (algebraic and topological) from $W_p^2(R_+^n; E(A), E)$ onto $L_p(R_+^n; E) \times \prod_{k=1}^2 F_k$. Moreover, for $\lambda \in S(\varphi)$ and sufficiently large $|\lambda|$ the uniform coercive estimate holds

$$\sum_{k=1}^{n} \sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \left\| D_k^i u \right\|_{L_p(R_+^n; E)} + \|Au\|_{L_p(R_+^n; E)}$$
$$\le M \left[\|f\|_{L_p(R_+^n; E)} + \sum_{k=1}^{2} (\|f_k\|_{F_k} + |\lambda|^{1-\theta_k} \|f_k\|_E) \right].$$

Proof. Since $L_p(R_+^n; E) = L_p(R_+; L_p(\mathbb{R}^{n-1}; E))$, the problem (3) can be expressed as

$$Lu = a_n D_{x_n}^2 u(x_n) + (B + \lambda)u = f_k(y), \quad L_k u = f_k, \quad k = 1, 2,$$

where B is the differential operator in $L_p(\mathbb{R}^{n-1}; E)$ generated by (2). By [4, Theorem 3.1] B is R-positive in $L_p(\mathbb{R}^{n-1}; E)$. By [1, Theorem 4.5.2], $L_p(\mathbb{R}^{n-1}; E) \in UMD$ provided $E \in UMD$, $p \in (1, \infty)$. Moreover, in view of [22], $L_p(\mathbb{R}^{n-1}; E)$ satisfies the multiplier condition. Then by [2, Theorem 2] we get the claim.

Theorem A_6. *Let E_0 and E be Banach spaces with base. Suppose that*

$$s_j(I(E_0, E)) \sim j^{-\frac{1}{\gamma}}, \quad \gamma > 0, \quad j = 1, 2, \ldots, \infty.$$

Then

$$s_j(I(W_p^m(G; E_0, E), L_p(G; E))) \sim j^{-\frac{1}{\gamma + \varkappa}}, \quad \varkappa = \frac{m}{n}.$$

Proof. Let

$$F = L_p(G) \otimes E, F_0 = W_p^m(G; E) \cap L_p(G; E_0).$$

Since $F_0 = W_p^m(G) \otimes E \cap L_p(G) \otimes E_0$, where $E_1 \otimes E_2$ denotes the tensor product of E_1 and E_2, the embedding operator B can be expressed as

$$B = B_2 \otimes I_1 + I_2 \otimes B_1,$$

where B_1 and B_2 are the embedding operators from E_0 to E and from $W_p^m(G)$ to $L_p(G)$, respectively; I_1 and I_2 denote the identity operators in E and $L_p(G)$. Moreover, the finite-rank operators from F_0 to F can be represented as

$$K = K_2 \otimes I_1 + I_2 \otimes K_1.$$

Let $\{e_k\}, \{g_k(x)\}$, $k=1,2,\ldots,\infty$, be basis systems in E_0 and $W_p^m(G)$, respectively. Then the system $\{g_k \otimes e_j\}$ is a basis system in $W_p^m(G;E)$. By the properties of tensor products, for $u \in F_0$ we have

$$u = \sum_{i,j=1}^{\infty} a_{ij} g_i \otimes e_j.$$

By [21, § 1.16.1] and [21, § 3.8.1] we get

$$\begin{aligned} s_j(B(F_0,F)) &= \inf_{\dim K \le j} \sup_{\|u\| \le 1} \|(B-K)u\|_F \\ &\le \inf_{\dim K \le j} \sup_{\|u\| \le 1} \|[(B_2-K_2)\otimes I_1]u + [I_2 \otimes (B_1-K_1)]u\|_F \\ &\le \inf_{\dim K \le j} \sup_{\|u\| \le 1} \left\| \sum_{i,j=1}^{\infty} a_{ij}(B_1-K_1) g_i \otimes e_j \right\|_F \\ &+ \inf_{\dim K \le j} \sup_{\|u\| \le 1} \left\| \sum_{i,j=1}^{\infty} a_{ij}(B_1-K_1) e_j \otimes g_i \right\|_F \le C(j^{-1/\varkappa} + j^{-1/\gamma}) \le C j^{-\frac{1}{\varkappa+\gamma}}. \end{aligned} \tag{4}$$

Let us now estimate the d-numbers of the embedding operator B. Consider $u_k \in C^{\infty}(\sigma_k; E_0)$, $k = 1,2,\ldots,\infty$, such that

$$\|u_k\|_{F_0} \le 1, \quad \|u_k\|_F = 2^{-k\eta}, \quad \eta = \frac{1}{\varkappa+\gamma}.$$

Take C_ν, $\nu = 1,2,\ldots,N_\nu$, N_ν such that $\sum_{\nu=1}^{N_k} |C_\nu|^p = 1$. Put

$$\Phi_k = \left\{ u : u = \sum_{\nu=1}^{N_k} C_\nu \varphi_\nu,\ \varphi_\nu \in C^{\infty}(\sigma_k; E_0) \right\}.$$

Then $\|u\|_{F_0} \le 1$ and $\|u\|_F = 2^{-k\eta}$ for sufficiently large k.

Let O_k and O_{kl} stand for the embedding operators from Φ_k into F and F_0. Since $\dim \Phi_k = N_k$, by [21, § 3, Lemma 3] we have

$$d_{N_k-1}(O_k(\Phi_k, F)) = 1.$$

Since $O_k = BO_{kl}$, using the properties of d-numbers we get

$$d_{N_k-1}(O_k(\Phi_k,F)) \le \|O_{kl}\| d_{N_k-1}(B(\Phi_k,F)).$$

Therefore putting $N_k - 1 = j$ we find

$$d_j(B) \ge C j^{-\frac{1}{\varkappa+\gamma}}, \quad j = 1,2,\ldots,\infty. \tag{5}$$

Then by (4), (5) and in view of [21, § 3, Lemma 2] we obtain the claim.

By [14, Theorem 3] we have

Theorem A$_7$. *Suppose that*

(1) *E is an UMD-space;*

(2) *A is a densely defined unbounded operator in E with the property that for some λ in the resolvent set of A, the operator $R(\lambda A)$ is of class $\sigma_p(E)$, $p \in (1,\infty)$;*

(3) *$\gamma_1, \gamma_2, \ldots, \gamma_s$ are nonoverlapping differentiable arcs in the complex plane having a limiting direction at infinity and such that no adjacent pair of arcs forms the angle of $\frac{\pi}{p}$ at infinity;*

(4) *the resolvent of A satisfies the inequality $\|R(\lambda, A)\| = O(|\lambda|^{-1})$ as $\lambda \to \infty$ along each of these arcs γ_i.*

Then spA is the entire space E.

2. Partial Operator Differential Equations with Variable Coefficients

Consider the inhomogeneous problem (1), i.e.

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} - A_\lambda(x)u(x) + \sum_{k=1}^{n} A_k(x)\frac{\partial u(x)}{\partial x_k} = f(x), \quad x \in G,$$
$$L_\Gamma u = \left[\alpha(x)\frac{\partial u(x)}{\partial l} + \beta(x)u(x)\right]_\Gamma = g, \tag{6}$$

where the second equality is understood in the trace sense.

We start with obtaining some coercive estimate for strong solutions to (6).

Condition 1. *Suppose that* (1) $a_{ij} = a_{ji}$ *and there is* $\mu > 0$ *such that*

$$\mu^{-1}|\xi|^2 \le L_0(x,\xi) \le \mu|\xi|^2 \text{ for } x \in G,\ L_0(x,\xi) = \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j;$$

(2) $\Gamma \in C^{(2)}$ (*e.g., see* [8, § 6.2]), $\alpha, \beta \in C^{(1)}(\Gamma)$, $\alpha(x) \ne 0$, $\frac{\beta(x)}{\alpha(x)} \in S(\varphi)$.

Put $E_p = (E(A), E)_{\frac{1+p}{2p}}$.

Theorem 1. *Assume that Condition 1 is satisfied and*
(1) *E is a Banach space enjoying the multiplier condition with respect to $p \in (1,\infty)$;*
(2) *$A(x)$ is an R-positive operator in E uniformly in $x \in \overline{G}$, $A(x)A^{-1}(x^0) \in C(\overline{G}; B(E))$;*
(3) *for every $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that for a.e. $x \in G$ and for $u \in (E(A), E)_{\frac{1}{2},\infty}$*

$$\|A_k(x)u\| \le \varepsilon\|u\|_{(E(A),E)_{\frac{1}{2},\infty}} + C(\varepsilon)\|u\|.$$

Then for $u \in W_p^2(G; E(A), E)$ and for $\lambda \in S(\varphi)$ and sufficiently large $|\lambda|$ we have

$$\sum_{k=1}^{n}\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \|D_k^i u\|_{L_p(G;E)} \le M\Big[\|(L+\lambda)u\|_{L_p(G;E)} + \|L_\Gamma u\|_{B_{p,p}^{1-\frac{1}{p}}(\Gamma;E_p,E)}\Big]. \tag{7}$$

Proof. Let $G_1, G_2, \dots, G_N$ be domains in $\mathbb{R}^n$ and let $\varphi_1, \varphi_2, \dots, \varphi_N$ be a corresponding partition of unity; i.e., φ_j are smooth functions on R, $\sigma_j = \operatorname{supp}\varphi_j \subset G_j$, and $\sum_{j=1}^{N}\varphi_j(x) = 1$. Then for $u \in W_p^2(G; E(A), E)$ we have $u(x) = \sum_{j=1}^{N} u_j(x)$, where $u_j(x) = u(x)\varphi_j(x)$. Take $u \in W_p^2(G; E(A), E)$. Then from (6) we obtain

$$(L+\lambda)u_j = \sum_{k,i=1}^{n} a_{ki} D_{ki}^2 u_j(x) - A_\lambda(x)u_j(x) = f_j(x), \tag{8}$$
$$L_\Gamma u_j = g_j, \quad j = 1, 2, \dots, N. \tag{9}$$

where

$$f_j = f\varphi_j + \sum_{k,i=1}^{n} a_{ki}[\varphi_j D_{ki}^2 u + D_k u D_k \varphi_j + \varphi_j D_i u + u D_{ki}^2 \varphi_j]$$
$$-\sum_{k=1}^{n} \varphi_j A_k(x)\frac{\partial u(x)}{\partial x_k}, \quad j = 1, 2, \dots, N, \quad g_j = \left[g + \alpha\frac{\partial \varphi_j}{\partial l} u_j\right]_\Gamma. \tag{10}$$

Let $G_j \cap G = G_j$. Since the boundary Γ is sufficient smooth, there are (e.g., see [8, § 1.7.3] differentiable diffeomorphisms Ψ_j on the neighborhood of G_j transforming G_j to $\widetilde{G}_j$ with plane boundary and such that $L_\Gamma u_j$ become $\tilde{L}_\Gamma \tilde{u}_j = [\tilde{\alpha}(y)\frac{\partial \tilde{u}_j(y)}{\partial y_n} + \tilde{\beta}(y)\tilde{u}_j(y)]_{y_n=0}$, where $\tilde{\upsilon}(y) = \upsilon(\Psi_j(y))$ for $\upsilon \in W_p^2(G_j; E(A), E)$. For these transformations $W_p^2(G_j; E(A), E)$ is isomorphically mapped to $W_p^2(\widetilde{G}_j; E(A), E)$ and under these maps (8) transforms to

$$(L+\lambda)\tilde{u}_j = \sum_{k,,i=1}^{n} \tilde{a}_{kij} D_{ki}^2 \tilde{u}_j(y) - \widetilde{A}_{j\lambda}(y)\tilde{u}_j(y) = \tilde{f}_j(y).$$

Moreover, by Condition 1, there is a linear mapping that transforms the expression

$$\sum_{k,,i=1}^{n} \tilde{a}_{ki} D_{kij}^2 \tilde{u}_j(y) + \widetilde{A}_{j\lambda}(y)\tilde{u}_j(y) \quad \text{to} \quad \sum_{k}^{n} \tilde{a}_{kj} D_k^2 \tilde{u}_j(y) - \widetilde{A}_{j\lambda}(y)\tilde{u}_j(y), \quad \tilde{a}_{kj} > 0.$$

After redenoting y by x, $\widetilde{G}$ by G_j, $\tilde{a}_{kj}(y)$ by $a_{kj}(x)$, $\widetilde{A}_{j\lambda}(y)$ by $A_\lambda(x)$, and $\tilde{u}_j(y)$ by $u_j(x)$ etc., and freezing coefficients in the transformed equation (8) we obtain

$$\begin{aligned} &\sum_{k}^{n} a_{kj}(x_{j0}) D_k^2 u_j(x) + A_\lambda(x_{j0}) u_j(x) = F_j(x), \\ &L_\Gamma u_j = \left[\alpha(x_{j0})\frac{\partial u_j(y)}{\partial x_n} + \beta(x_{j0}) u_j(x)\right]_{x_n=0} = \Phi_j(x^1), \end{aligned} \tag{11}$$

where

$$F_j = f_j + [A(x_{j0}) - A(x)]u_j + \sum_{k}^{n} [a_k(x) - a_{ki}(x_{j0})] D_k^2 u_j(x), \tag{12}$$

$$B_j(x^1) = g_j(x^1) + \left[(\alpha(x_{j0}) - \alpha(x))\frac{\partial u_j}{\partial x_n} + (\beta(x_{j0}) - \beta(x)) u_j(x)\right]_{x_n=0}. \tag{13}$$

By Theorem A_5 we see that (11) has the unique solution $u_j \in W_p^2(G_j; E(A), E)$ and for $\lambda \in S(\varphi)$ and sufficiently large $|\lambda|$ the following coercive estimate holds:

$$\sum_{k=1}^{n} \sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \|D_k^i u_j\|_{G_j,p} + \|A u_j\|_{G_j,p} \le C \|F_j\|_{G_j,p} + \|\Phi_j\|_{B_{p,p}^{1-\frac{1}{p}}(\Gamma \cap G_j; E_p, E)}. \tag{14}$$

In a similar way, by Theorem A_4 we obtain estimates of type (14) for $G_j \subset G$. Whence, using properties of the smoothness of coefficients of (12) and (13) and by Theorems A_1 and A_3, choosing diameters of σ_j sufficiently small, we get

$$\|F_j\|_{G_j,p} \le \varepsilon \|u_j\|_{W_p^2(G_j; E(A), E)} + C(\varepsilon)\|u_j\|_{G_j,p}, \tag{15}$$

$$\|\Phi_j\|_{B_{p,p}^{1-\frac{1}{p}}(\Gamma \cap G_j; E_p, E)} \le C \|g_j\|_{B_{p,p}^{1-\frac{1}{p}}(\Gamma \cap G_j; E_p, E)} + \varepsilon \|u_j\|_{W_p^2(G_j; E(A), E)} + C(\varepsilon)\|u_j\|_{G_j,p}, \tag{16}$$

where ε is sufficiently small and $C(\varepsilon)$ is a continuous function. From (14)–(16) we get

$$\sum_{k=1}^{n} \sum_{i=0}^{1} |\lambda|^{1-\frac{i}{2}} \|D_k^i u_j\|_{G_j,p} + \|A u_j\|_{G_j,p} \le C \|f\|_{G_j,p} + \|g_j\|_{B_{p,p}^{1-\frac{1}{p}}(\Gamma \cap G_j; E_p, E)} + \varepsilon \|u_j\|_{W_p^2} + C(\varepsilon)\|u_j\|_{G_j,p}.$$

Choosing $\varepsilon < 1$ from the above inequality we have

$$\sum_{k=1}^{n}\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\left\|D_k^i u_j\right\|_{G_j,p} + \|Au_j\|_{G_j,p} \le C\big[\|f\|_{G_j,p} + \|u_j\|_{G_j,p} + \|g_j\|_{B_{p,p}^{1-\frac{1}{p}}(\Gamma\cap G_j;E_p,E)}\big]. \tag{17}$$

In a similar manner we also obtain (17) for the domains G_j lying entirely in G. Then by (17) for $u \in W_p^2(G; E(A), E)$ we have

$$\sum_{k=1}^{n}\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\|D_k^i u\|_p + \|Au\|_p \le C\big[\|(L+\lambda)u\|_p + \|u\|_p + \|g_j\|_{B_{p,p}^{1-\frac{1}{p}}}\big]. \tag{18}$$

Let $u \in W_p^2(G; E(A), E)$ be the solution to (6). Then for $\lambda \in S(\varphi)$ we get

$$\|u\|_p = \|(L+\lambda)u - Lu\|_p \le \frac{1}{\lambda}[\|(L+\lambda)u\|_p + \|u\|_{W_p^2}]. \tag{19}$$

Then by Theorem A_1, (18), and (19) for sufficiently large $|\lambda|$ and $u \in W_p^2(G; E(A), E)$ we get (7).

Now consider the BVP (1). Let O_λ denote the operator in $L_p(G; E)$ generated by (1), i.e.

$$D(O_\lambda) = W_p^2(G; E(A), E, L_\Gamma), \quad O_\lambda u = (L+\lambda)u.$$

Theorem 2. *Let all conditions of Theorem* 1 *be satisfied. Then for all $f \in L_p(G; E)$, $\lambda \in S(\varphi)$, and sufficiently large $|\lambda|$ there is a unique solution to* (1) *and the following uniform coercive estimate holds:*

$$\sum_{k=1}^{n}\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\left\|D_k^i u_j\right\|_{L_p(G;E)} \le M\|f\|_{L_p(G;E)}. \tag{20}$$

Proof. Let us show that for all $f \in L_{p,\gamma}(G; E)$ there is a unique solution $u \in W_p^2(G; E(A), E)$ to (1). The uniqueness of this problem follows from (7). So it suffices to prove that (1) has a solution $u \in W_p^2(G; E(A), E)$ for all $f \in L_p(G; E)$. Consider the smooth functions $g_j = g_j(x)$ with respect to the partition of unity $\varphi_j = \varphi_j(y)$ on G that equal one on $\operatorname{supp}\varphi_j$, where $\operatorname{supp} g_j \subset G_j$ and $|g_j(x)| < 1$. Let us construct for all j the functions u_j on $\Omega_j = G \cap G_j$ satisfying (1). First consider the case when G_j adjoins boundary points. The problem (1) can be expressed as

$$\sum_{k,i=1}^{n} a_{ki}(x_{j0})D_{ki}^2 u_j(x) + A_\lambda(x_j)u_j(x)$$
$$= g_j\left\{f + [A(x_{j0}) - A(x)]u_j + \sum_{k,i=1}^{n}[a_{ki}(x_{j0}) - a_{ki}(x)]D_{ki}^2 u_j - \sum_{k=1}^{n} A_k(x)\frac{\partial u_j(x)}{\partial x_k}\right\},$$
$$L_\Gamma u_j = 0, \quad j = 1, 2, \dots, N. \tag{21}$$

Consider the operators $O_{j\lambda}$ in $L_p(G_j; E)$ generated by BVP (21) when G_j partially belong to G. By Theorem A_5 for all $f \in L_p(G_j; E)$, $\lambda \in S(\varphi)$, and sufficiently large $|\lambda|$ we have

$$\sum_{k=1}^{n}\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\left\|D_k^i O_{j\lambda}^{-1}f\right\|_p + \left\|AO_{j\lambda}^{-1}f\right\|_p \le C\|f\|_p. \tag{22}$$

Extending u_j zero beyond $\operatorname{supp}\varphi_j$ and making the substitutions $u_j = O_{j\lambda}^{-1}\upsilon_j$ we obtain from (21) the operator equations

$$\upsilon_j = K_{j\lambda}\upsilon_j + g_j f, \quad j = 1, 2, \dots, N, \tag{23}$$

where $K_{j\lambda}$ are bounded linear operators in $L_p(G_j;E)$ defined as

$$K_{j\lambda} = g_j\left\{f + [A(x_{0j}) - A(x)]O_{j\lambda}^{-1} + \sum_{k,i=1}^{n}[a_{ki}(x_{j0}) - a_{ki}(x)]D_{ki}^2 O_{j\lambda}^{-1} - \sum_{k=1}^{n} A_k(x)\frac{\partial}{\partial x_k}O_{j\lambda}^{-1}\right\}.$$

By Theorem A_1, (22), the smoothness of the coefficients of the expression $K_{j\lambda}$, and condition (4) for $\lambda \in S(\varphi)$ and sufficiently large $|\lambda|$, we have $\|K_{j\lambda}\| < \varepsilon$ where ε is sufficiently small. Consequently, (23) have the unique solutions $\upsilon_j = [I - K_{j\lambda}]^{-1} g_j f$ and

$$\|\upsilon_j\|_p = \|[I - K_{j\lambda}]^{-1} g_j f\|_p \leq \|f\|_p.$$

Whence, $[I - K_{j\lambda}]^{-1} g_j$ are bounded linear operators from $L_p(G;E)$ to $L_p(G_j;E)$. Thus, we see that

$$u_j = U_{j\lambda} f = O_{j\lambda}^{-1}[I - K_{j\lambda}]^{-1} g_j f$$

are solutions to (21). By Theorem A_4 we can construct the solutions u_j for (21) for G_j lying entirely in G. Consider the linear operator $(U + \lambda)$ in $L_p(G;E)$ such that

$$(U + \lambda)f = \sum_{j=1}^{N} \varphi_j(y) U_{j\lambda} f.$$

It is clear from the constructions of U_j and (22) that $U_{j\lambda}$ are bounded linear operators from $L_p(G;E)$ to $W_p^2(G;E(A),E)$ and for $\lambda \in S(\varphi)$ and sufficiently large $|\lambda|$ we have

$$\sum_{k=1}^{n}\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \|D_k^i U_{j\lambda}^{-1} f\|_p + \|A U_{j\lambda}^{-1} f\|_p \leq C\|f\|_p. \tag{24}$$

Therefore $(U + \lambda)$ is a bounded linear operator from L_p to L_p. Applying O_λ to $u = \sum_{j=1}^{N} \varphi_j U_{j\lambda} f$ yields $O_\lambda u = f + \sum_{j=1}^{N} \Phi_{j\lambda} f$, where $\Phi_{j\lambda}$ are bounded linear operators from $L_{p,\gamma}(G;E)$ to $L_{p,\gamma}(G_j;E)$ defined as

$$\Phi_{j\lambda} = \sum_{k,i=1}^{n} a_{ki}\left[D_{ki}^2 \varphi_j U_{j\lambda} + \frac{\partial \varphi_j}{\partial x_k}\frac{\partial U_{j\lambda}}{\partial x_i} + \frac{\partial \varphi_j}{\partial x_i}\frac{\partial U_{j\lambda}}{\partial x_k}\right] + \sum_{i=1}^{n} \frac{\partial \varphi_j}{\partial x_i} A_i U_{j\lambda}.$$

By the embedding Theorem A_1 and (24), from the expression $\Phi_{j\lambda}$ we obtain that $\Phi_{j\lambda}$ are bounded linear operators from $L_p(G;E)$ to $L_p(G_j;E)$ and $\|\Phi_{j\lambda}\| < \varepsilon$. Therefore, the invertible bounded linear operator $\left(I + \sum_{j=1}^{N} \Phi_{j\lambda}\right)^{-1}$ is available. Whence, we see that for all $f \in L_p(G;E)$ BVP (1) have the unique solution

$$u(x) = O_\lambda^{-1} f = \sum_{j=1}^{N} \varphi_j O_{j\lambda}^{-1}[I - K_{j\lambda}]^{-1} g_j \left(I + \sum_{j=1}^{N} \Phi_{j\lambda}\right)^{-1} f,$$

i.e., we proved the claim.

Result 1. *Theorem 1 implies that O has the resolvent $(O + \lambda)^{-1}$ for $\lambda \in S(\varphi)$ and*

$$\sum_{k=1}^{n}\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \|D_k^i (O+\lambda)^{-1}\|_{B(L_p(G;E))} + \|A(O + \lambda)^{-1}\|_{B(L_p(G;E))} \leq C. \tag{25}$$

Remark 1. The estimate (20) and the embedding Theorem A_1 imply that under the conditions of Theorem 2

$$\sum_{i,j=1}^{n} \|\frac{\partial^2 u}{\partial x_i \partial x_j}\|_{L_p(G;E)} + \|Au\|_{L_p(G;E)} \leq C\|f\|_{L_p(G;E)}$$

for the solution to (1); i.e., (1) is separable in $L_p(G;E)$.

Remark 2. Result 1 implies that O is positive on $L_p(G;E)$. Moreover in view of (24) and by [21, § 1.14.5] O generates an analytic semigroup when $\varphi \in (\frac{\pi}{2}, \pi)$.

Remark 3. If $\alpha(x) \equiv 0$ then the claims of Theorems 2 and 3 hold for $g \in B_{p,p}^{2-\frac{1}{p}}(\Gamma; (E(A),E)_{\frac{1}{2p}}, E)$.

3. The Spectral Properties of Elliptic Differential Operators in Banach Spaces

Consider the differential operator O generated by BVP (21). It is clear to see that the principal part of this operator is nonselfadjoint. In this section we will address the spectral properties of O. In the following theorem we prove the Fredholm property of O.

Theorem 3. *Let all conditions of Theorem 1 hold and $A^{-1} \in \sigma_\infty(E)$. Then O is Fredholm from $W_p^2(G; E(A), E)$ into $L_p(G; E)$.*

Proof. Theorem 1 implies that $O+\lambda$ for sufficiently large $|\lambda|$ has a bounded inverse $(O+\lambda)^{-1}$ from $L_p(G; E)$ to $W_p^2(G; E(A), E)$; that is the operator $O+\lambda$ is Fredholm from $W_p^2(G; E(A), E)$ into $L_p(G; E)$. By Theorem A_2 the embedding $W_p^2(G; E(A), E) \subset L_p(G; E)$ is compact. Then by the perturbation theory of linear operators we find that O is Fredholm from $W_p^2(G; E(A), E)$ into $L_p(G; E)$.

Theorem 4. *Suppose that all conditions of Theorem 2 are satisfied. Let E be a Banach space with base and $s_j(I(E(A), E)) \sim j^{-\frac{1}{\nu}}$, $j = 1, 2, \ldots, \infty$, $\nu > 0$. Then*

(a)

$$s_j((O + \lambda)^{-1}(L_p(G; E))) \sim j^{-\frac{2}{2\nu+n}}; \tag{26}$$

(b) *the system of the root functions of O is complete in $L_{p,\gamma}(G; E)$.*

Proof. Let $I(E_0, E)$ denote the embedding operator from E_0 to E. By Result 1 there exists the resolvent $(O + \lambda)^{-1}$ bounded from $L_p(G; E)$ to $W_p^2(G; E(A), E)$. Thus, by Theorem A_6 the embedding operator $I(W_{p,\gamma}^l(G; E(A), E), L_{p,\gamma}(G; E))$ is compact and

$$s_j(I(W_p^2(G; E(A), E), L_p(G; E))) \sim j^{-\frac{2}{2\nu+n}}. \tag{27}$$

Since

$$(O + \lambda)^{-1}(L_p(G; E)) = (O + \lambda)^{-1}(L_p(G; E), W_p^2(G; E(A), E))I(W_p^2(G; E(A), E), L_p(G; E)); \tag{28}$$

from (27) and (28) we obtain (26). The estimate (25) and (27) imply that $O + \lambda_0$ is positive on $L_p(G; E)$ and

$$(O + \lambda_0)^{-1} \in \sigma_q(L_p(G; E)), \quad q > \frac{2}{2\nu + n}. \tag{29}$$

Then from (25), (29), and Theorem A_6 we obtain (b).

4. BVPs for Anisotropic Elliptic Equations

The Fredholm property for elliptic equations with parameters in smooth domains were studied in, e.g., [24, 25]; also for nonsmooth domains these questions were investigated, e.g., in [26, 27].

Let $\Omega \subset \mathbb{R}^n$ be an open connected set with compact C^{2m}-boundary $\partial\Omega$. Let us consider the BVPs in the cylindrical domain $\widetilde{\Omega} = G \times \Omega$ for the following anisotropic elliptic equation:

$$\begin{aligned} Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x, y)}{\partial x_i \partial x_j} + \sum_{k=1}^{n} d_k \frac{\partial u(x, y)}{\partial x_k} \\ + \sum_{|\alpha| \le 2m} a_\alpha(y) D_y^\alpha u(x, y) = f(x, y), \quad x \in G,\ y \in \Omega, \end{aligned} \tag{30}$$

$$L_\Gamma u = \left[\alpha(x) \frac{\partial u(x, y)}{\partial l} + \beta(x) u(x, y)\right]_\Gamma = 0, \tag{31}$$

$$B_j u = \sum_{|\beta| \le m_j} b_{j\beta}(y) D_y^\beta u(x, y) = 0, \quad x \in G,\ y \in \partial\Omega,\ j = 1, 2, \ldots, m. \tag{32}$$

Here Γ is the boundary of $G \in \mathbb{R}^n$, l is a nontangential direction on Γ and a_{ij}, α, and β are complex-valued functions on G and Γ, respectively, $D_j = -i\frac{\partial}{\partial y_j}$, $m_k \in \{0,1\}$, $y = (y_1, \ldots, y_n)$.

If $\widetilde{\Omega} = G \times \Omega$ and $\mathbf{p} = (p_1, p)$ then $L_{\mathbf{p}}(\widetilde{\Omega})$ will denote the space of all $\mathbf{p}$-summable scalar-valued functions with mixed norm (e.g., see [7, § 1]), i.e. the space of all measurable functions f on $\widetilde{\Omega}$ for which

$$\|f\|_{L_{\mathbf{p}}(\widetilde{\Omega})} = \left(\int\limits_G \left(\int\limits_\Omega |f(x,y)|^{p_1}\, dy\right)^{\frac{p}{p_1}} dx\right)^{\frac{1}{p}} < \infty.$$

Similarly, $W^{2,2m}_{\mathbf{p}}(\widetilde{\Omega})$ denotes the anisotropic Sobolev space with the corresponding mixed norm [26, § 10].

Theorem 5. *Assume that*

(1) *Condition* 1 *holds;*

(2) $a_\alpha \in C(\bar{\Omega})$ *for each* $|\alpha| = 2m$ *and* $a_\alpha \in [L_\infty + L_{r_k}](\Omega)$ *for each* $|\alpha| = k < 2m$ *with* $r_k \geq q$ *and* $2m - k > \frac{l}{r_k}$;

(3) $b_{j\beta} \in C^{2m-m_j}(\partial\Omega)$ *for all* j, β, *and* $m_j < 2m$, $\sum_{j=1}^m b_{j\beta}(y')\sigma_j \neq 0$, *for* $|\beta| = m_j$, $y' \in \partial G$, *where* $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{R}^n$ *is a normal to* $\partial\Omega$;

(4) *for* $y \in \bar{\Omega}$, $\xi \in \mathbb{R}^n$, $\lambda \in S(\varphi)$, $\varphi \in (0,\pi)$, $|\xi| + |\lambda| \neq 0$ *let* $\lambda + \sum_{|\alpha|=2m} a_\alpha(y)\xi^\alpha \neq 0$;

(5) *given* $y_0 \in \partial\Omega$ *in the local coordinates corresponding to* y_0, *the local BVP*

$$\lambda + \sum_{|\alpha|=2m} a_\alpha(y_0) D^\alpha \vartheta(y) = 0, \quad B_{j0}\vartheta = \sum_{|\beta|=m_j} b_{j\beta}(y_0) D^\beta u(y) = h_j, \quad j = 1, 2, \ldots, m,$$

has the unique solution $\vartheta \in C_0(R_+)$ *for all* $h = (h_1, h_2, \ldots, h_m) \in \mathbb{R}^m$, *and for* $\xi' \in \mathbb{R}^{n-1}$ *with* $|\xi'| + |\lambda| \neq 0$.

Then

(a) *for all* $f \in L_{\mathbf{p}}(\widetilde{\Omega})$, $|\arg\lambda| \leq \varphi$, *and sufficiently large* $|\lambda|$ *the problem* (29)–(31) *has the unique solution* u *that belongs to* $W^{2,2m}_{\mathbf{p}}(\widetilde{\Omega})$ *and the coercive uniform estimate holds*

$$\sum_{k=1}^n \sum_{i=0}^2 |\lambda|^{1-\frac{i}{2}} \left\| \frac{\partial^i u}{\partial x_k} \right\|_{L_{\mathbf{p}}(\widetilde{\Omega})} + \sum_{|\beta|=2m} \|D_y^\beta u\|_{L_{\mathbf{p}}(\widetilde{\Omega})} \leq C\|f\|_{L_{\mathbf{p}}(\widetilde{\Omega})};$$

(b) *problem* (30)–(32) *is Fredholm in* $L_{\mathbf{p}}(\widetilde{\Omega})$.

Proof. Put $E = L_{p_1}(\Omega)$. Then by [22, Theorem 3.6] part (1) of Theorem 2 is satisfied. Consider the operator A defined by

$$D(A) = W^{2m}_{p_1}(\Omega; B_j u = 0), \quad Au = \sum_{|\alpha| \leq 2m} a_\alpha(y) D^\alpha u(y).$$

Given $x \in \Omega$, consider the operators $A_k(x)u = d_k(x,y)u(y)$, $k = 1, 2, \ldots, n$. The problem (30)–(32) can be rewritten as (1), where $u(x) = u(x,.)$ and $f(x) = f(x,.)$ are functions with values in $E = L_{p_1}(\Omega)$. By [25] the problem

$$\lambda u(y) + \sum_{|\alpha| \leq 2m} a_\alpha(y) D_y^\alpha u(y) = f(y), \quad B_j u = \sum_{|\beta| \leq m_j} b_{j\beta}(y) D_y^\beta u(y) = 0, \quad j = 1, 2, \ldots, m,$$

has the unique solution for $f \in L_{p_1}(\Omega)$ and $\arg\lambda \in S(\varphi_0)$ as $|\lambda| \to \infty$. Moreover, in view of [5, Theorem 8.2] the differential operator A is R-positive on L_{p_1}. It is known that the embedding $W^{2m}_{p_1}(\Omega) \subset L_{p_1}(\Omega)$ is compact (e.g., see [21, Theorem 3.2.5]). Then by the interpolation properties of Sobolev spaces (e.g., see [21, § 4]) it is clear that (3) of Theorem 2 is fulfilled too. Then from Theorems 2 and 3 we obtain the claim.

5. BVPs for Infinite Systems of Elliptic Equations

Consider the following infinite systems of BVPs:

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u_m(x)}{\partial x_i \partial x_j} + (d_m(x)+\lambda)u_m(x)$$

$$+\sum_{k=1}^{n}\sum_{j=1}^{\infty} d_{kjm}(x)\frac{\partial u_j(x)}{\partial x_k} = f_m(x), \quad x \in G, \quad m = 1, 2, \ldots, \infty, \tag{33}$$

$$L_\Gamma u = \left[\alpha(x)\frac{\partial u_m(x)}{\partial l} + \beta(x)u_m(x)\right]_\Gamma = 0, \tag{34}$$

where Γ is the boundary of a domain $G \in \mathbb{R}^n$, l is a nontangential direction on Γ, and a_{ij}, α, and β are complex-valued functions on G and Γ, respectively.

Let $d(x) = \{d_m(x)\}$, $d_m > 0$, $u = \{u_m\}$, $Du = \{d_m u_m\}$, $m = 1, 2, \ldots, \infty$,

$$l_q(D) = \left\{u : u \in l_q, \|u\|_{l_q(d)} = \|Du\|_{l_q} = \left(\sum_{m=1}^{\infty}|d_m u_m|^q\right)^{\frac{1}{q}} < \infty\right\},$$

$x \in G$, $1 < q < \infty$. Let Q denote the differential operator in $L_p(G; l_q)$ generated by (33), (34). Let $B = B(L_p(G; l_q))$.

Theorem 6. *Assume that*

(1) *Condition* 1 *holds;*

(2) $d_j \in C(\overline{G}), d_{kmj} \in L_\infty(G)$ *and* $\max_k \sup_m \sum_{j=1}^{\infty} d_{kmj}(x) d_j^{-(\frac{1}{2}-\mu)} < M$ *for all* $x \in G$ *and* $0 < \mu < \frac{1}{2}$ *a.e. for* $x \in G$ *and* $1 < p < \infty$.

Then

(a) *For all* $f(x) = \{f_m(x)\}_1^\infty \in L_p(G; l_q)$, $\lambda \in S(\varphi)$, $\varphi \in (0, \pi)$, *and sufficiently large* $|\lambda|$ *the problem* (33), (34) *has the unique solution* $u = \{u_m(x)\}_1^\infty$ *belonging to* $W_p^2(G, l_q(D), l_q)$ *and the coercive estimate holds*

$$\sum_{k=1}^{n}\left[\int_G \left(\sum_{m=1}^{\infty}|D_k^2 u_m(x)|^q\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}} + \left[\int_G \left(\sum_{m=1}^{\infty}|d_m u_m(x)|^q\right) dx\right]^{\frac{1}{p}} \leq C\left[\int_G \left(\sum_{m=1}^{\infty}|f_m(x)|^q\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}}. \tag{35}$$

(b) *The resolvent* $(Q+\lambda)^{-1}$ *of* Q *exists for sufficiently large* $|\lambda| > 0$ *and*

$$\sum_{k=1}^{n}\sum_{j=0}^{2}|\lambda|^{1-\frac{j}{2}}\left\|D_k^j(Q+\lambda)^{-1}\right\|_B + \|A(Q+\lambda)^{-1}\|_B \leq M. \tag{36}$$

Proof. Indeed, assume that $E = l_q$, while A, and $A_k(x)$ are infinite matrices such that

$$A = [d_m(x)\delta_{jm}], \; A_k(x) = [d_{kjm}(x)], \; m, \quad j = 1, 2, \ldots, \infty.$$

It is clear that A is R-positive on l_q. Therefore, by Theorem 4 we infer that (33), (34) for all $f \in L_p(G; l_q)$, $\lambda \in S(\varphi)$, and sufficiently large $|\lambda|$ has the unique solution u in $W_p^2(G; l_q(D), l_q)$ and

$$\sum_{k=1}^{n}\|D_k^2 u\|_{L_p(G;l_q)} + \|Du\|_{L_p(G;l_q)} \leq C\|f\|_{L_p(G;l_q)}.$$

From above we obtain (35). The estimate (36) ensues from Result 1.

Remark 4. There are many positive operators on particular Banach spaces. Therefore, using particular Banach spaces and particular positive operators (i.e. pseudodifferential operators or finite or infinite matrices for instance) instead of E and A, respectively, from Theorems 2–4 we can obtain various classes of maximal regular BVPs for partial differential or pseudodifferential equations or finite or infinite systems of these equations.

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