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Σ-BOUNDED ALGEBRAIC SYSTEMS AND UNIVERSAL FUNCTIONS. II

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Abstract: Ershov algebras, Boolean algebras, and abelian p-groups are Σ-bounded systems, and there exist universal Σ-functions in hereditarily finite admissible sets over them.

Keywords: admissible set, Σ-definability, computability, universal Σ-function, Σ-bounded algebraic system, Ershov algebra, Boolean algebra, abelian p-group

This article continues [1] where we had introduced the concept of a Σ-bounded algebraic system and obtained a necessary and sufficient condition for the existence of universal Σ-functions in a hereditarily finite admissible set over a Σ-bounded system. In this article we prove that Ershov algebras, Boolean algebras, and abelian p-groups are Σ-bounded systems, and universal Σ-functions exist over them.

As regards the terminology and notation, we follow [2] for admissible sets, [3] for Ershov algebras, and [4–6] for groups. Recall the definition of a Σ-bounded algebraic system and some results of [1] needed below.

Definition 1. Given a locally finite and locally constructivizable algebraic system $\mathfrak{M}$ of signature σ_0 and a finite subset M_0, suppose the following:

1. The concept of a base is defined for every finite subset $X \subseteq M$. The predicate

$$\mathfrak{B}_0^{M_0}(X, Y) \rightleftharpoons \text{“the finite sequence } Y \in M^{<\omega} \text{ is a base for } X\text{”}$$

is a Δ-predicate of signature $\sigma_1(M_0)$ in $\langle \mathbb{HF}(\mathfrak{M}), M_0\rangle$. Given two bases Y^0 and Y^1 for X, $X \subseteq \langle Y^\varepsilon\rangle$, $\varepsilon = 0, 1$ and either $\mathfrak{B}_0^{M_0}(\operatorname{sp} Y^0, Y^1)$ or $\mathfrak{B}_0^{M_0}(\operatorname{sp} Y^1, Y^0)$ is true. A sequence Y is called a *base* whenever $\mathfrak{B}^{M_0}(Y) \rightleftharpoons \mathfrak{B}_0^{M_0}(\operatorname{sp} Y, Y)$ is true.

2. For every base Y a number $\chi^{M_0}(Y)$ is defined which is called the *characteristic* of Y, such that $\chi^{M_0}(Y)$ is a Σ-function of signature $\sigma_1(M_0)$ in $\langle \mathbb{HF}(\mathfrak{M}), M_0\rangle$. The set of all characteristics Ξ^{M_0} is a computable subset of ω. There exists a Δ-predicate $\operatorname{Cor}^{M_0}(z, Y, n)$ of signature $\sigma_1(M_0)$ such that

$$z \in \langle Y\rangle \Leftrightarrow \langle \mathbb{HF}(\mathfrak{M}), M_0\rangle \models \exists! n(n \neq 0 \ \&\ \operatorname{Cor}^{M_0}(z, Y, n)).$$

The number n is called the *coordinate* of z with respect to Y. If two elements are distinct then so are their coordinates.

3. Given two bases Y^ε of the same characteristic χ and finite subsystems $\mathfrak{M}^\varepsilon \supseteq \langle Y^\varepsilon\rangle$, $\varepsilon < 2$, there exist a base Y^2 and a subsystem $\mathfrak{M}^2 \supseteq \langle Y^2\rangle$ satisfying the following:

(1) $\chi = \chi(Y^2)$;

(2) there exists an embedding $\varphi_0^\varepsilon : \mathfrak{M}^\varepsilon \to \mathfrak{M}^2$ such that $\varphi^\varepsilon \restriction \langle M_0\rangle = \mathrm{id}$ and $\varphi^\varepsilon Y^\varepsilon = Y^2$, where the embedding $\varphi^\varepsilon : \mathbb{HF}(\mathfrak{M}^\varepsilon) \to \mathbb{HF}(\mathfrak{M}^2)$ naturally extends φ_0^ε.

In particular, every two bases of the same characteristic are of the same length.

4. For every partial function $f : \mathbb{HF}(\mathfrak{M}) \to \mathbb{HF}(\mathfrak{M})$ defined by a Σ-formula with parameters in M_0, if $u \in \mathbb{HF}(\mathfrak{M})$ and $u \in \delta f$ then there exists a base Y for $\operatorname{sp} u$ such that $\operatorname{sp} f(u) \subseteq \langle Y\rangle$.

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Then $\mathfrak{M}$ is called a *Σ-bounded algebraic system with respect to* M_0. If for every finite subset M_0 there exists a finite subset $M_0' \supseteq M_0$ such that $\mathfrak{M}$ is Σ-bounded with respect to M_0' then $\mathfrak{M}$ is called a *Σ-bounded algebraic system*.

Denote the set of all parameter-free Σ-formulas of signature $\sigma_1(M_0)$ by $\Sigma(\mathbb{HF}(\mathfrak{M}), M_0)$, the set of all functions in $\langle \mathbb{HF}(\mathfrak{M}), M_0 \rangle$ defined by the formulas in $\Sigma(\mathbb{HF}(\mathfrak{M}), M_0)$ by $F\Sigma(\mathbb{HF}(\mathfrak{M}), M_0)$, and the set of all unary functions in $F\Sigma(\mathbb{HF}(\mathfrak{M}), M_0)$ by $\mathfrak{F}^{M_0}$.

Theorem A [1, Corollary 4]. *If an algebraic system $\mathfrak{M}$ is Σ-bounded with respect to a finite subset M_0 of M then there exists a universal Σ-function $U^{M_0}(x, y) \in F\Sigma(\mathbb{HF}(\mathfrak{M}), M_0)$ for the family $\mathfrak{F}^{M_0}$ such that every $f \in \mathfrak{F}^{M_0}$ satisfies $\lambda y U^{M_0}(n, y) = f(y)$ for some n.*

Theorem B [1, Theorem 2]. *Given a Σ-bounded algebraic system $\mathfrak{M}$, in $\mathbb{HF}(\mathfrak{M})$ there exists a universal Σ-function with parameter A if and only if for every finite subset C with respect to which $\mathfrak{M}$ is Σ-bounded there is a finite subset C^1 such that for every finite subset X and every base Y_X^C there exists a base $Y_{X^*}^A$ for which $\langle Y_X^C \rangle \subseteq \langle Y_{X^*}^A \rangle$, where $X^* = C^1 \cup X$.*

1. Ershov Algebras

Here we prove the Σ-boundedness of every Ershov algebra $\mathfrak{A}$ and the existence of universal functions in $\mathbb{HF}(\mathfrak{A})$.

We consider Ershov algebras in the signature $\sigma_0 = \langle \cup, \cap, \setminus, 0 \rangle$ and Boolean algebras in the signature $\sigma_1 = \langle \cup, \cap, \setminus, 0, 1 \rangle$. Let us give some notation. Given a Ershov algebra $\mathfrak{A}$, take a finite subset A_0 of $\mathfrak{A}$. A sequence $\langle x_1, \dots, x_n \rangle$ is called *disjunctive* in $\mathfrak{A}$ whenever $x_i \neq 0$ for $1 \leq i \leq n$ are such that x_i and x_j are disjoint for all $i < j \leq n$. The expression $z = x_1 \sqcup \dots \sqcup x_n$ means that $z = x_1 \cup \dots \cup x_n$ and the sequence $\langle x_1, \dots, x_n \rangle$ is disjunctive. Put

$$\mathrm{At}(\mathfrak{A}) = \{a \in \mathfrak{A} \mid a \text{ is an atom in } \mathfrak{A}\}.$$

Given $a \in \mathfrak{A}$, put $\hat{a} = \{x \in \mathfrak{A} \mid x \leq a\}$ and $a^\perp = \{x \in \mathfrak{A} \mid x \cap a = 0\}$. Given $S \subseteq \mathfrak{A}$ denote by $\langle S \rangle \rightleftharpoons \langle S \rangle_{A_0}$ the subalgebra in $\mathfrak{A}$ generated by the set $S \cup A_0$. Refer to $a \in \mathfrak{A}$ as a *finite* element whenever it is the union of finitely many atoms, and denote their number by $|a|$; otherwise call a an *infinite* element.

Theorem 1. *Every Ershov algebra $\mathfrak{A}$ is a Σ-bounded algebraic system with respect to every finite subset $A_0 \subseteq A$.*

Proof. Since every Ershov algebra $\mathfrak{A}$ is locally constructivizable (see [7]) and locally finite, in order to prove the theorem it suffices to verify that conditions 1–4 in Definition 1 are fulfilled.

Enumerate the atoms $A_0' = \{a_1, \dots, a_s\}$ of $\langle A_0 \rangle$. For definiteness assume that $a_1, \dots, a_e$ are infinite, for $1 \leq e \leq s$, while $a_{e+1}, \dots, a_s$ are finite, and put $a = a_1 \cup \dots \cup a_s$ and $b = a_{e+1} \cup \dots \cup a_s$. If the subalgebra $\langle A_0 \rangle^\perp$ is finite then denote by a_{s+1} its greatest element; otherwise assume that $a_{s+1} = 0$.

1. A disjunctive sequence $Y = \langle y_1, \dots, y_q \rangle$ in $\mathfrak{A}$ is called a *base* for a subset X whenever the subalgebra generated by $X \cup \hat{b} \cup \hat{a}_{s+1}$ in $\langle \mathfrak{A}, A_0 \rangle$ coincides with $\langle Y \rangle$, and there exist numbers $\tilde{p}_0 = 0, p_1, \dots, p_{s+\delta}$ such that

$$a_i = y_{\tilde{p}_{i-1}+1} \cup \dots \cup y_{\tilde{p}_{i-1}+p_i},$$

where $\tilde{p}_i = \tilde{p}_{i-1} + p_i$ for $1 \leq i \leq s + \delta$ with $\delta = 0, 1$. If $\langle A_0 \rangle^\perp$ is finite then $\delta = 1$ and $q = \tilde{p}_{s+1}$; otherwise, $\delta = 0$ and $\tilde{p}_s \leq q$.

It is easy to see that the relation "Y is a base for X" is a binary Δ-predicate.

2. Define the characteristic of a base $Y = \langle y_1, \dots, y_q \rangle$. Put

$$\chi(Y) \rightleftharpoons \langle p_1, \dots, p_{s+\delta}, \alpha \rangle,$$

where $\alpha = q - \tilde{p}_{s+\delta}$.

It is easy to verify that the set of all characteristics

$$\Xi^{A_0} = \{\langle p_1, \dots, p_e, p_{e+1}, \dots, p_{s+\delta}, \alpha\rangle \mid p_i \in \omega,\ p_j = |a_j|,\ [(\delta = 0 \ \&\ \alpha \in \omega) \\ \vee(\delta = 1 \ \&\ \alpha = 0 \ \&\ p_{s+1} = |a_{s+1}|)],\ p_i > 0,\ 1 \le i \le e,\ e < j \le s,\ \delta = 0, 1\}$$

is computable.

Given $z \in \langle Y\rangle$, $z \neq 0$, there exists a unique sequence $\langle m_1, \dots, m_k\rangle$ of numbers such that $m_j < m_l$ for $1 \le j < l \le k$, and

$$z = y_{m_1} \cup \dots \cup y_{m_k}.$$

Refer to the number $n = [m_1, \dots, m_k]$ as the *coordinate* of z with respect to Y. Assume that the zero element has coordinate 0.

3. The fulfilment of this condition follows from the next lemma.

Lemma 1. *In a Ershov algebra $\mathfrak{A}$ take two bases Y^ε of the same characteristic*

$$\chi = \langle p_1, \dots, p_e, p_e, \dots, p_{s+\delta}, \alpha\rangle,$$

and finite subalgebras $\mathfrak{A}^\varepsilon \supseteq \langle Y^\varepsilon\rangle$ for $\varepsilon < 2$. Then there exist a base Y^2 of the same characteristic χ, a finite subalgebra $\mathfrak{A}^2 \supseteq \langle Y^2\rangle$, and embeddings $\varphi^\varepsilon : \mathfrak{A}^\varepsilon \to \mathfrak{A}^2$ such that $\varphi \restriction A_0 = \mathrm{id}$ and $\varphi^\varepsilon Y^\varepsilon = Y^2$.

The proof is given for $\delta = 0$. The case $\delta = 1$ is checked similarly. Suppose that the atoms of $\mathfrak{A}^\varepsilon$ are

$$z_1^\varepsilon, \dots, z_{r^\varepsilon}^\varepsilon,\ y_{\tilde{p}_e+1}, \dots, y_{\tilde{p}_s}$$

and they satisfy

$$y_j^\varepsilon = z^\varepsilon_{\tilde{t}^\varepsilon_{j-1}+1} \cup \dots \cup z^\varepsilon_{\tilde{t}^\varepsilon_{j-1}+t^\varepsilon_j}, \tag{1}$$

where $j = 1, \dots, \tilde{p}_e, \tilde{p}_s + 1, \dots, q$, and $\tilde{t}^\varepsilon_j = \tilde{t}^\varepsilon_{j-1} + t^\varepsilon_j$, $\tilde{t}^\varepsilon_0 = 0$, $t^\varepsilon_j \in \omega^+$, $\tilde{t}_{\tilde{p}_s} = \tilde{t}_{\tilde{p}_e}$, $\tilde{t}^\varepsilon_q \le r^\varepsilon$.

For every $j \in \{1, \dots, \tilde{p}_e\} \cup \{\tilde{p}_s+1, \dots, q\}$ put $\beta_j = \max\{t^0_j, t^1_j\}$, and then put $\beta = \max\{r^0 - \tilde{t}^0_q, r^1 - \tilde{t}^1_q\}$. It is easy to see that there exists a base $Y^2 = \langle y^2_1, \dots, y^2_{\tilde{p}_e}, y_{\tilde{p}_e+1}, \dots, y_{\tilde{p}_s}, y^2_{\tilde{p}_s+1}, \dots, y^2_q\rangle$ such that

(a) the element y^2_j is either infinite or contains at least β_j atoms;

(b) $a_i = y^2_{\tilde{p}_{i-1}+1} \cup \dots \cup y^2_{\tilde{p}_{i-1}+p_i}$ for $1 \le i \le e$;

(c) there exists a disjunctive sequence $\langle d_1, \dots, d_\beta\rangle$ such that $d_i \cap y^2_j = 0$ for all i and j with $1 \le i \le \beta$ and $1 \le j \le q$.

Then for $j \in \{1, \dots, \tilde{p}_e\} \cup \{\tilde{p}_s+1, \dots, q\}$ and $\varepsilon < 2$ there is a disjunctive sequence $c^\varepsilon_{\tilde{t}^\varepsilon_{j-1}+1}, \dots, c^\varepsilon_{\tilde{t}^\varepsilon_{j-1}+t^\varepsilon_j}$ of $\hat{y}^2_j$ satisfying

$$y^2_j = c^\varepsilon_{\tilde{t}^\varepsilon_{j-1}+1} \cup \dots \cup c^\varepsilon_{\tilde{t}^\varepsilon_{j-1}+t^\varepsilon_j}. \tag{2}$$

Also, for each $\varepsilon < 2$ there exists a disjunctive sequence $c^\varepsilon_{\tilde{t}^\varepsilon_q+1}, \dots, c^\varepsilon_{r^\varepsilon}$ such that $c_i \cap y^2_j = 0$ for all i and j with $\tilde{t}^\varepsilon_q < i \le r^\varepsilon$ and $1 \le j \le q$.

Denote by $\mathfrak{A}^2$ the subalgebra of $\mathfrak{A}$ generated by these sequences and the set $\{y_{\tilde{p}_e+1}, \dots, y_{\tilde{p}_s}\}$. It is easy to verify that there exist embeddings $\varphi^\varepsilon : \mathfrak{A}^\varepsilon \to \mathfrak{A}^2$ such that

$$\varphi^\varepsilon z^\varepsilon_{k^\varepsilon} = c^\varepsilon_{k^\varepsilon},\ 1 \le k^\varepsilon \le r^\varepsilon, \quad \varphi^\varepsilon y^\varepsilon_i = y^2_i,\ \tilde{p}_e < i \le \tilde{p}_s.$$

By (b), (1), and (2) this implies that

$$\varphi \restriction \langle A_0\rangle = \mathrm{id}, \quad \varphi^\varepsilon Y^\varepsilon = Y^2.$$

The proof of the lemma, as well as condition 3, is complete. □

In order to verify condition 4 we establish the following lemma.

Lemma 2. *Take finite subalgebras $B \subseteq C \subseteq D$, with $B \neq C$, of a Ershov algebra $\mathfrak{A}$, and an infinite element $b \in B$ of $\mathfrak{A}$. If b is an atom of B, but not an atom of C, then there exists an embedding $\varphi : D \to \mathfrak{A}$ such that $\varphi \restriction B = \mathrm{id}$ and $\varphi C \nsubseteq D$.*

PROOF. Take $b = (c_1 \sqcup c_2) \sqcup c_3$, where c_1 and c_2 are atoms of C, and c_3 is an element of C, which is possibly equal to zero. Write

$$c_\varepsilon = d_1^\varepsilon \sqcup \cdots \sqcup d_{n_\varepsilon}^\varepsilon, \quad \varepsilon = 1, 2,\ n_\varepsilon \geq 1, \tag{3}$$

where d_i^ε are atoms of D. Assume for definiteness that d_1^1 is an infinite element of $\mathfrak{A}$. Then there exist x_1, y_1, and x_i^1 for $1 \leq i \leq n_1$ such that

$$d_1^1 = x_1 \sqcup y_1, \quad x_1 = x_1^1 \sqcup \cdots \sqcup x_{n_1}^1, \quad x_2 = (c_1 \setminus x_1) \sqcup c_2. \tag{4}$$

For some x_i^2, $1 \leq i \leq n_2$, we have

$$x_2 = x_1^2 \sqcup \cdots \sqcup x_{n_2}^2. \tag{5}$$

Then there exists an embedding $\varphi : D \to \mathfrak{A}$ such that

$$\varphi d_i^\varepsilon = x_i^\varepsilon,\ \varepsilon = 1, 2,\ 1 \leq i \leq n_\varepsilon, \quad \varphi x = x \text{ for all } x \in D,\ x \cap (c_1 \sqcup c_2) = 0. \tag{6}$$

By (3)–(6), we have $\varphi c_\varepsilon = x_\varepsilon$, $\varphi(c_1 \sqcup c_2) = c_1 \sqcup c_2$, and $\varphi c_3 = c_3$; thus, $\varphi b = b$. Since $x_1 \notin D$, it follows that $\varphi c_1 \notin D$. $\square$

Lemma 3. *Take a finite subalgebra $D \subseteq \mathfrak{A}$ and $c_1, c_2 \in D$ such that $c_1 \cap c_2 = 0$, where c_2 is an infinite element. Then there exists an embedding $\varphi : D \to \mathfrak{A}$ such that φc_1 is an infinite element, $\varphi c_1 \notin D$, $\varphi(c_1 \sqcup c_2) = c_1 \sqcup c_2$, and $\varphi x = x$ for every $x \in D \cap (c_1 \sqcup c_2)^\perp$.*

PROOF. Write $c_i = d_1^i \sqcup \cdots \sqcup d_{n_i}^i$, $i = 1, 2$, where d_j^i are atoms of D for $1 \leq j \leq n_i$. We may assume that d_1^2 is an infinite element. Then there exist $x_1, \ldots, x_{n_1}, x_{n_1+1}$ in $\mathfrak{A}$ such that $d_1^2 = x_1 \sqcup \cdots \sqcup x_{n_1} \sqcup x_{n_1+1}$. Assume for definiteness that x_1 is an infinite element. Define the embedding $\varphi : D \to \mathfrak{A}$ by putting

$$\varphi d_j^1 = x_j, \quad \varphi d_1^2 = c_1, \quad \varphi\big(d_2^2\big) = d_2^2 \sqcup x_{n_1+1},$$

$$\varphi x = x \quad \text{for all } x \text{ such that } x \cap \big(c_1 \cup d_1^2 \cup d_2^2\big) = 0.$$

Hence,

$$\varphi c_1 = x_1 \sqcup \cdots \sqcup x_{n_1}, \quad \varphi(c_1 \sqcup c_2) = c_1 \sqcup c_2.$$

Consequently, φc_1 is infinite, and $\varphi c_1 \notin D$. $\square$

Lemma 4. *Take two subalgebras $C \subseteq D$ and some elements $c_\varepsilon < b$ for $\varepsilon = 0, 1$, and $b = c_0 \sqcup c_1$ of D. If c_0 is an atom of C and $D^\perp$ is an infinite algebra then there exist embeddings $\varphi_\varepsilon : D \to \mathfrak{A}$ such that*

$$\varphi_0(b) = \varphi_1(b), \quad \varphi_0(c_0) \notin \varphi_1(C), \quad \varphi x = x \textit{ for every } x \in b^\perp \cap D.$$

PROOF. Write $c_\varepsilon = d_1^\varepsilon \sqcup \cdots \sqcup d_{n_\varepsilon}^\varepsilon$, where d_i^ε are atoms of D for $1 \leq i \leq n_\varepsilon$. Take a disjunctive sequence $\{x_i^\varepsilon, z \mid 1 \leq n_\varepsilon,\ \varepsilon = 0, 1\}$ of elements of $D^\perp$, and put $x^\varepsilon = x_1^\varepsilon \sqcup \cdots \sqcup x_{n_\varepsilon}^\varepsilon$. Define the embeddings $\varphi_\varepsilon : D \to \mathfrak{A}$ by putting

$$\varphi_\varepsilon x = x \quad \text{for all } x \in b^\perp \cap D,$$

$$\varphi_0\big(d_i^0\big) = x_i^0, \quad \varphi_0\big(d_1^1\big) = x_1^1 \sqcup z, \quad \varphi_0\big(d_k^1\big) = x_k^1,$$

$$\varphi_1\big(d_1^0\big) = x_1^0 \sqcup z, \quad \varphi_1\big(d_j^0\big) = x_j^0, \quad \varphi_1\big(d_s^1\big) = x_s^1,$$

where $1 \leq i \leq n_0$, $2 \leq k \leq n_1$, $2 \leq j \leq n_0$, and $1 \leq s \leq n_1$. Then

$$\varphi_0(c_0) = x^0, \quad \varphi_0(c_1) = x^1 \sqcup z, \quad \varphi_1(c_0) = x^0 \sqcup z, \quad \varphi_1(c_1) = x^1.$$

Therefore, x^0 is an atom of $\varphi_0(C)$, while $x^0 \sqcup z$ is an atom of $\varphi_1(C)$. Consequently, $\varphi_0(c_0) \notin \varphi_1(C)$. $\square$

The fulfilment of condition 4 follows from the next lemma.

Lemma 5. *Take an algebra $\mathfrak{A}$ and the function $f : \mathbb{HF}(\mathfrak{A}) \to \mathbb{HF}(\mathfrak{A})$ whose graph is determined by some Σ-formula $\Phi(x, y, A_0)$. Then for all elements $u = \varkappa(X) \in \mathbb{HF}(\mathfrak{A})$, $\varkappa \in \mathbb{HF}(\omega)$, and bases Y of $\operatorname{sp} X$ we have*

$$\text{if } u \in \delta f \quad \text{then } \operatorname{sp} f(u) \subseteq \langle Y \rangle.$$

Proof. Assume on the contrary that

$$f(u) = \tau(Z) \rightleftharpoons v, \quad \operatorname{sp} Z \nsubseteq \langle Y \rangle, \tag{7}$$

where $\tau \in \mathbb{HF}(\omega)$ and Z is a sequence of elements of $\mathfrak{A}$.

Take

$$B = \langle Y \rangle, \quad C = \langle \operatorname{sp} Y \cup \operatorname{sp} Z \rangle. \tag{8}$$

Denote by D a finite subalgebra such that

$$C \subseteq D, \quad \mathbb{HF}(D) \models \Phi(u, v, A_0).$$

Take the atoms $b_1, \dots, b_m$ of B and consider all possible cases.

1. For some i the element b_i is infinite and is not an atom of C.

Then by Lemma 2 there exists an embedding $\varphi : D \to \mathfrak{A}$ such that $\varphi \restriction B = \mathrm{id}$ and $\varphi C \nsubseteq D$. Take the natural extension $\varphi^\sharp : \mathbb{HF}(D) \to \mathbb{HF}(\mathfrak{A})$ of φ. Then Lemma 6 yields

$$f(u) = f(\varphi^\sharp(u)) = f(\varkappa(\varphi X)) = \tau(\varphi Z), \quad \operatorname{sp} \varphi Z \nsubseteq C.$$

This contradicts (7) and (8).

2. For some i the element b_i is finite and is not an atom of C. Assume for definiteness that $i = 1$.

Here a few subcases are possible:

(a) $b_1 \le a$ for some $a \in \langle A_0 \rangle$.

We may assume that a is an atom of $\langle A_0 \rangle$. Since b_1 is not an atom of $\mathfrak{A}$, by the definition of Y the element a is infinite. Thus, $b_1 < a$, and for some i the element $b_i < a$ is infinite. By Lemma 3 there exists an embedding $\varphi : D \to \mathfrak{A}$ such that $\varphi \restriction \langle A_0 \rangle = \mathrm{id}$ and φb_1 is infinite. Then (7) and (8) imply that

$$f(\varkappa(\varphi X)) = \tau(\varphi Z), \quad \operatorname{sp} \varphi Z \nsubseteq \varphi B.$$

The subalgebras $\varphi B \subseteq \varphi C \subseteq \varphi D$ and the element φb_1 satisfy the condition of case 1. Thus, $\operatorname{sp} \varphi Z \subseteq \varphi C$, and we arrive at a contradiction.

(b) $b_1 \nleq a$ for every $a \in \langle A_0 \rangle$.

Then $b_1 \in \langle A_0 \rangle^\perp$ and $\langle A_0 \rangle^\perp$ is an infinite subalgebra. Suppose that there exists i such that b_i is infinite and $b_i \in \langle A_0 \rangle^\perp$. Then by Lemma 3 there exists an embedding $\varphi : D \to \mathfrak{A}$ such that $\varphi \restriction \langle A_0 \rangle = \mathrm{id}$, φb_1 is infinite, and φb_1 is not an atom of φC. Hence, as in case 2(a), we arrive at a contradiction.

Consequently, every atom of B lying in $\langle A_0 \rangle^\perp$ is finite. Since $b_1 \in \langle A_0 \rangle^\perp$, this implies that $B^\perp$ is an infinite subalgebra. We may assume that so is $D^\perp$. Indeed, suppose that $D^\perp$ is finite and take the atoms $d_1, \dots, d_n$ of D, where $d_1, \dots, d_e$ are all atoms of $B^\perp$. Since $D^\perp$ is finite, there exists $1 \le i \le e$ such that d_i is infinite. Suppose that d_1 is infinite. Then there exist x and y with $d_1 = x \sqcup y$. Suppose that x is infinite, and denote by D_0 the subalgebra generated by $B, y, d_2, \dots, d_e$. Define an embedding $\varphi : D \to D_0$ by putting

$$\varphi \restriction B = \mathrm{id}, \quad \varphi d_1 = y, \quad \varphi d_i = d_i, \quad 2 \le i \le e.$$

Since $x \in D_0^\perp$, it follows that $D_0^\perp$ is an infinite algebra.

Suppose that $b_1 = c_0 \sqcup c_1$, where c_0 is an atom of C. Lemma 4 implies that there exist embeddings $\varphi_0, \varphi_1 : D \to \mathfrak{A}$ such that $\varphi_0 \restriction B = \varphi \restriction B$ and $\varphi_0(c_0) \notin \varphi_1(C)$. Lemma 6 implies that $f(\varkappa(\varphi_0(X))) = f(\varkappa(\varphi_1(X)))$, and so $\operatorname{sp} \varphi_0(Z) = \operatorname{sp} \varphi_1(Z)$. Hence, $\varphi_0(C) \subseteq \varphi_1(C)$, which is a contradiction.

3. Cases 1 and 2 fail to hold.

Then b_i is an atom of C for every $1 \leq i \leq m$. Consequently, $C = B \oplus C_0$ for some subalgebra C_0. It follows from (7) that $C_0 \neq 0$. By the definition of Y the algebra $\langle A_0 \rangle^\perp$ is infinite. Assume that the following holds.

(B) There exists an infinite element $c \in C_0$.

We may assume that c is an atom of C_0. Then there exists an embedding $\varphi : D \to \mathfrak{A}$ such that $\varphi \restriction B = \mathrm{id}$ and $\varphi c < c$, and so $\varphi c \notin C$. Hence, as in case 1, we arrive at a contradiction.

Thus, all atoms of C_0 are finite. Consider the following possible cases one by one.

(C) There exists an infinite element $b_i \in \langle A_0 \rangle^\perp$.

Take an atom c of C_0. By Lemma 3 there exists an embedding $\varphi : D \to \mathfrak{A}$ such that $\varphi \restriction \langle A_0 \rangle = \mathrm{id}$, and an infinite element φc with $\varphi c \notin \varphi B$. Then for the subalgebra $\varphi B \subseteq \varphi C$ and the element φc case (B) holds, which is impossible.

(D) The subalgebra $C^\perp$ is infinite.

Take an atom c of C_0 and write $c = d_1 \sqcup \dots \sqcup d_n$, where d_i are atoms of D for $1 \leq i \leq n$. In $C^\perp$ choose x such that $x = x_1 \sqcup \dots \sqcup x_n$ for some x_i. Then there exists an embedding $\varphi : D \to \mathfrak{A}$ such that $\varphi \restriction B = \mathrm{id}$ and $\varphi c \notin C$, which is impossible.

Therefore, all possible cases lead to contradictions. The proof of the lemma is complete. $\square$

Therefore, we have verified conditions 1–4 of Definition 1 for all algebras $\mathfrak{A}$ and sets A_0. The proof of the theorem is complete. $\square$

Corollary 1. *Each Boolean algebra $\mathfrak{B}$ is a Σ-bounded algebraic system with respect to each finite subset $B_0 \subseteq B$.*

Indeed, every Boolean algebra can be regarded as an enrichment of a Ershov algebra by the symbol of the constant 1.

Theorems 1 and A, as well as Corollary 1, imply

Corollary 2. *Given a Ershov or Boolean algebra $\mathfrak{A}$, for every finite subset A_0 there exists a universal Σ-function $U^{A_0}(x, y) \in F\Sigma(\mathbb{HF}(\mathfrak{A}), A_0)$ for the family $\mathfrak{F}^{A_0}$ of functions such that every $f \in \mathfrak{F}^{A_0}$ satisfies $\lambda y U^{A_0}(n, y) = f(y)$ for some n.*

Corollary 3. *Given a Ershov algebra $\mathfrak{A}$, in $\mathbb{HF}(\mathfrak{A})$ there exists a universal Σ-function for the family of all unary Σ-functions.*

Proof. Put $A = \varnothing$ and define $A'_0 = \{a_1, \dots, a_e, a_{e+1}, \dots, a_s\}$ using A_0 as in the beginning of the proof of Theorem 1. Put

$$A_0^1 = \{a_1, \dots, a_e, a^1_{e+1}, \dots, a^1_{e+\alpha_{e+1}}, \dots, a^1_{(s+1)+1}, \dots, a^1_{(s+1)+\alpha_{s+1}}\},$$

where $a^1_{k+1}, \dots, a^1_{k+\alpha_k}$ are all atoms under a_k. It is easy to see that every finite subset X and bases $Y_X^{A_0}$ and $Y_{X^*}^{\varnothing}$, where $X^* = A_0^1 \cup X$, satisfy $\langle Y_X^{A_0} \rangle \subseteq \langle Y_{X^*}^{\varnothing} \rangle$. Then Theorem B, where we must replace C and C^1 by A_0 and A_0^1, yields the claim. $\square$

Similarly we can prove

Corollary 4. *Given a Boolean algebra $\mathfrak{B}$, in $\mathbb{HF}(\mathfrak{B})$ there exists a universal Σ-function for the family of all unary Σ-functions.*

2. Abelian p-Groups

In this section we prove the Σ-boundedness of every abelian p-group G and the existence of universal Σ-functions in $\mathbb{HF}(G)$.

Recall the necessary terminology and results of the theory of abelian p-groups. Take an abelian p-group G and a subgroup $G_0 \subseteq G$. The *order* of G_0 is the cardinality of G_0 denoted by $|G_0|$. The *period* $\mathrm{per}(G)$ is the smallest number p^m such that $p^m G = 0$; if this number fails to exist then $\mathrm{per}(G) = \omega$, and we say that G is an *unbounded* group. The period of the subgroup (x) is called the *order* of x and denoted by $|x|$. The *height* of $x \in G$, denoted by $h_G(x)$, is $\max\{p^n \mid x \in p^n G\}$; if this n fails to exist then $h_G(x) = \infty$.

Take a finite set $A_0 \subseteq G$. Given $X \subseteq G$ denote by $\langle X \rangle$ the subgroup generated by X in the group $\langle G, A_0 \rangle$, and by (X), in G. Put $G[p^n] = \{x \mid p^n x = 0\}$ and denote the cyclic group of order p^n by C_{p^n}, and the quasicyclic group by C_{p^∞}. Denote by G^α the direct sum of α copies of the group G. The *dimension* of G is the dimension of the vector space $G[p]$. A group G is called *divisible* whenever given $x \in G$ there exists y such that $x = py$. If G contains no divisible subgroups distinct from zero then it is called *reduced*.

Theorem C. *Each abelian group G is a direct sum, $G = R \oplus D$, of a reduced subgroup R and a divisible subgroup D.*

Theorem D (the first Prüfer theorem). *Each abelian p-group of finite period decomposes into a direct sum of cyclic subgroups.*

Theorem E (Prüfer–Kulikov). *If a servant subgroup A of an abelian group G has finite period then it appears in G as a direct summand.*

Theorem F. *All decompositions of an abelian p-group as direct sums of cyclic groups are isomorphic.*

The proof of Proposition 27.1 in [5, p. 139] implies

Proposition A. *Suppose that the period of an abelian p-group C is equal to p^n. Take $c \in C$ with $|c| = p^n$ and a subgroup $B \subseteq C$ such that $B \cap (c) = 0$. Then there exists a subgroup $E \supseteq B$ such that $C = E \oplus (c)$.*

Proposition B [6, p. 83]. *If a countable reduced abelian p-group G is unbounded then G has a direct summand that is an unbounded direct sum of cyclic groups.*

Theorem 2. *Suppose that an abelian p-group G satisfies at least one of the following:*
(1) *the reduced part R of G is unbounded;*
(2) *the divisible part D includes a subgroup $C^\omega_{p^\infty}$;*
(3) *there exists a subgroup $G_0 \subseteq G$ isomorphic to $C^\omega_{p^\alpha}$, where p^α is the period of G, and $\alpha \in \omega$, $\alpha > 0$.*
Then G is a Σ-bounded algebraic system with respect to every finite subset A_0.

In order to prove the theorem we will need the following lemma and proposition.

Lemma 6. *Take the same group G as in Theorem 2, and a finite subgroup $B \subseteq G$. Then for every number $p^n \le \operatorname{per}(G)$ there exists $c \in G$ of order p^n such that $B \cap (c) = 0$.*

Proof. It is easy to verify that there exists a countable subgroup H, with $B \subseteq H \subseteq G$, satisfying the hypotheses of Theorem 2. If H satisfies condition 1 then Proposition B implies that H has a direct summand H_0 which is a direct sum of cyclic groups of unbounded orders. Suppose that H satisfies condition 2. Then Theorem C yields $H = H_0 \oplus H_1$, where $H_0 \cong C^\omega_{p^\infty}$. However, if condition 3 is fulfilled then Theorem F yields $H = H_0 \oplus H_1$, where $H_0 \cong C^\omega_{p^\alpha}$. In all these cases we can choose the required element c in H_0. □

The next proposition generalizes Proposition 6 of [7].

Proposition 1. *Take the same group G as in Theorem 2, finite abelian p-groups B and C with $B \subseteq C$ and $\operatorname{per}(C) \le \operatorname{per}(G)$, and an embedding $\varphi : B \to G$. Then φ extends to an embedding $\psi : C \to G$ if and only if every $b \in B$ satisfies*

$$h_C(b) \le h_G(b'), \quad \text{where } \varphi b = b'. \tag{9}$$

Proof. *Necessity* is obvious.

Sufficiency can be proved by induction on the number of elements in C. Suppose that the period of C is equal to p^n and take $c \in C$ with $|c| = p^n$. Suppose that $(c) \cap B = 0$. Then by Proposition A there exists a subgroup $E \subseteq C$ such that $C = (c) \oplus E$ and $E \supseteq B$. By induction there exists an embedding $\psi_0 : E \to G$ with $\psi_0 \restriction B = \varphi$. By Lemma 6 there exists an element c' such that $(c') \cap E' = 0$ and $|c'| = p^n$, where $E' = \psi_0 E$. It is obvious that we can extend ψ_0 to $\psi : C \to G$ by putting $\psi c = c'$. Thus, we may assume that $(c) \cap B \neq 0$.

For every element c of order p^n denote by k_c the smallest number satisfying $p^{k_c} c \in B$. Take $c_0 \in C[p^n]$ with the smallest value of k_{c_0}. Put $c = c_0$ and $k = k_{c_0}$.

Suppose that

$$p^k c = b_0 \tag{10}$$

and show that there exists c' such that $p^k c' = b_0'$, where $\varphi b_0 = b_0'$, and for every $s < k$ we have $p^s c' \notin B'$.

Since the subgroup (c) is servant in C, it follows that $h_C(b_0) = k$. Thus, G contains g_0 with $p^k g_0 = b_0'$. Take the minimal number s satisfying $p^s g_0 \in B'$. If $s = k$ then $c' = g_0$ is the required element. Suppose that $s < k$. By Lemma 6 there exists $g_1 \in G$ such that $|g_1| = p^k$ and $(g_1) \cap G_0 = 0$, where $G_0 = \mathrm{gr}(B', g_0)$. Put $c' = g_0 + g_1$. Then $p^k c' = b_0'$. Verify that $p^s c' \notin B'$ for every $s < k$. Indeed, assume on the contrary that $p^s c' \in B'$ for some $s < k$. Then $p^s g_0 + p^s g_1 = b' \in B'$. Hence, $0 \neq p^s g_1 \in G_0$; this is a contradiction.

Put $H = \mathrm{gr}(B, c)$ and $H' = \mathrm{gr}(B', c')$. The defining relations of H are $p^k c = b$ and relations between the elements of B. The defining relations of H' are the same. Therefore, there exists an isomorphism $f : H \to H'$ with $f \restriction B = \varphi$.

By Theorem E there exists a subgroup $E \subseteq C$ such that

$$C = (c) \oplus E. \tag{11}$$

Take the projection $E_0 = \mathrm{pr}_E(B)$ of B onto the second coordinate of the decomposition in (11). Given $e \in E_0$, there are $b \in B$ and $s \in \omega$ such that

$$b = p^s \alpha c + e, \quad (\alpha, p) = 1. \tag{12}$$

Verify that

$$h_C(e) \leq h_G(e'), \tag{13}$$

where $e' = fe$. Suppose that $h_C(e) = r$ and check that either $e \in B$ or

$$r < s. \tag{14}$$

If $s \geq k$ then (10) and (12) imply that $e \in B$. Suppose that $s < k$ and check the validity of (14). Assume on the contrary that $r \geq s$. There exists $e_1 \in E$ with $e = p^r e_1$. By (12) this yields $b = p^s(\alpha c + p^{r-s} e_1)$. Since $(\alpha, p) = 1$ it follows that $c_1 = \alpha c + p^{r-s} e_1$ is of order p^n, and $k_{c_1} < k$. This contradicts the choice of c. Thus, (14) holds.

Since $f : H \to H'$ is an isomorphism, we have

$$b' = p^s \alpha c' + e'. \tag{15}$$

Verify that

$$h_G(e') \geq r. \tag{16}$$

If $e \in B$ then (16) follows from the hypotheses of the proposition and the definition of f. Suppose that $e \notin B$. Then (14) holds. Taking (12) into account, we have $h_C(e) = h_G(b) = r$. Consequently, $h_G(b') \geq r$. By (15) this yields $h_G(e') \geq \min\{h_G(b'), s\} \geq r$; i.e., we have established (13).

The embedding $\varphi_0 = f \restriction E_0$ of E_0 into G and the subgroups $E_0 \subseteq E$ satisfy condition (9) in the proposition. By induction there exists an embedding $\psi_0 : E \to G$ such that $\psi_0 \restriction E_0 = \varphi_0$. Verify that

$$\psi_0 E \cap (c') = 0. \tag{17}$$

Assume on the contrary that there exists $e \in E$, $e \neq 0$, such that

$$\psi_0 e = p^s c'. \tag{18}$$

We may assume that $|e| = p$ and $s \geq k$. Indeed, if $s < k$ then (18) implies that $\psi_0 p^{n-s-1} e = p^{n-1} c'$. Since $k \leq n-1$, this implies that we can take $p^{n-s-1}e$ as e. Therefore, $|e| = p$ and $s \geq k$. Show that we can assume that $e \in E_0$. Indeed, suppose that $e \notin E_0$, $h_C(e) = h_E(e) = m$, and $p^m e_1 = e$ for some $e_1 \in E$. The subgroup (e_1) is servant in E; thus, by Theorem E there exists a subgroup $E_1 \subseteq E$ such that $E = (e_1) \oplus E_1$.

Since $e \notin E_0$, it follows that $\mathrm{pr}_{(e_1)} B = 0$. By (11) we have $C = (c) \oplus (e_1) \oplus E_1$ and $B \subseteq (c) \oplus E_1$. By Lemma 6, in order to prove the proposition it suffices to embed the subgroup $(c) \oplus E_1$ into G. This is possible by induction. Thus, assume that $e \in E_0$. From (18) it follows that

$$\psi_0 e = \varphi_0 e = fe = p^s c' = p^{s-k} b_0'. \tag{19}$$

The definition of the isomorphism $f : H \to H'$ yields

$$f p^{s-k} b_0 = \varphi p^{s-k} b_0 = p^{s-k} b_0'. \tag{20}$$

From (19) and (20) we deduce that

$$e = p^{s-k} b_0.$$

Hence, $e \in (c) \cap E$, which is impossible. Thus, (17) holds. Therefore, the embeddings $f : (c) \to (c')$ and $\psi_0 : E \to G$ extend to an embedding $\psi : C \to G$, as required.

The proof of the proposition is complete. □

Proof of Theorem 2. It is proved in [7] that every abelian p-group G is locally constructivizable. Since G is locally finite, in order to prove the theorem it suffices to verify conditions 1–4 of Definition 1.

1. Take a finite subset $X \subseteq G$. By Theorem D the subgroup $\langle X \rangle$ decomposes into a direct sum of cyclic groups $\langle X \rangle = (y_1) \oplus \cdots \oplus (y_q)$. Call the sequence $Y = \langle y_1, \dots, y_q \rangle$ a *base* for X. By Theorem F all bases for X are of the same length. It is easy to verify that $\mathfrak{B}_0(X, Y)$ is a Δ-predicate in $\langle \mathbb{HF}(G), A_0 \rangle$.

2. For every base Y and every $z \in \langle Y \rangle$ there exists a unique sequence $\langle n_1, \dots, n_q \rangle$ of numbers with $n_j < |y_j|$ such that $z = n_1 y_1 + \cdots + n_q y_q$. Then $n = [n_1, \dots, n_q] + 1$ is called the *coordinate* of z with respect to Y. It is easy to verify that $\mathrm{Cor}(z, Y, n)$ is a Δ-predicate in $\langle \mathbb{HF}(G), A_0 \rangle$.

Suppose that $Y = \langle y_1, \dots, y_q \rangle$ satisfies $|y_j| = p^{m_j}$ and $\mathrm{Cor}(a_i, Y, n_i)$, where

$$A \rightleftharpoons \langle A_0 \rangle = (a_1) \oplus \cdots \oplus (a_e)$$

is some fixed decomposition. Then the sequence $\chi(Y) = \langle p^{m_1}, \dots, p^{m_q}, n_1, \dots, n_e \rangle$ is called the *characteristic* of Y. It is easy to verify that $\chi = \chi(Y)$ is a binary Σ-predicate in $\langle \mathbb{HF}(G), A_0 \rangle$.

The computability of the set of all characteristics follows from the next lemma. Suppose that $|a_i| = p^{l_i}$ for $1 \leq i \leq e$.

Lemma 7. *The sequence of numbers*

$$\xi = \langle p_1^{m_1}, \dots, p_q^{m_q}, n_1, \dots, n_e \rangle,$$

where $q \geq e$, $n_i = [s_{i1}, \dots, s_{iq}] + 1$, $s_{ij} = p^{r_{ij}} t_{ij}$, *and* $(t_{ij}, p) = 1$ *for* $1 \leq i \leq e$ *and* $1 \leq j \leq q$, *is a characteristic if and only if for all* $0 \leq \alpha_i \leq p^{l_i}$ *the following hold:*

(a) $\max\{m_j \mid 1 \leq j \leq q\} \leq \mathrm{per}(G)$;

(b) $0 \leq r_{ij} \leq m_j$ *and* $\max\{m_j - r_{ij} \mid 1 \leq j \leq q\} = l_i$;

(c) $\bigwedge_{j=1}^{q} \left[\sum_{i=1}^{e} \alpha_i s_{ij} \equiv 0 (\operatorname{mod} p^{m_j}) \right] \Leftrightarrow \bigwedge_{i=1}^{e} \left[\bigwedge_{j=1}^{q} (\alpha_i s_{ij} \equiv 0 (\operatorname{mod} p^{m_j})) \right]$;

(d) $\min \left\{ \exp \left(p, \sum_{i=1}^{e} \alpha_i s_{ij}\right) \mid 1 \leq j \leq q \right\} \leq h_G\left(\sum_{i=1}^{e} \alpha_i a_i \right)$.

Proof. *Necessity.* Suppose that a sequence ξ is the characteristic of Y. Then by definition

$$\langle Y \rangle = (y_1) \oplus \cdots \oplus (y_q), \ |y_j| = p^{m_j}, \quad a_i = s_{i1} y_1 + \cdots + s_{iq} y_q.$$

Verify (d) for instance. Given $x = \sum \alpha_i a_i$, we have $x = \sum_{j=1}^{q} \left(\sum_{i=1}^{e} \alpha_i s_{ij} \right) y_j$. Hence,

$$h_{\langle Y \rangle}(x) = \min \left\{ \exp \left(p, \sum_{i=1}^{e} \alpha_i s_{ij} \right) \mid 1 \leq j \leq q \right\}.$$

It is obvious that $h_{\langle Y \rangle}(x) \leq h_G(x)$, whence we obtain (d).

Sufficiency. Suppose that a sequence ξ satisfies (a)–(d). Define the group

$$B^\xi = \left(b_1^\xi\right) \oplus \cdots \oplus \left(b_q^\xi\right), \quad \left|b_j^\xi\right| = p^{m_j}.$$

Denote by A^ξ the subgroup generated by

$$a_i^\xi = s_{i1} b_1^\xi + \cdots + s_{iq} b_q^\xi, \quad 1 \le i \le e.$$

It is easy to verify that (b) and (c) imply that

$$A^\xi = \left(a_1^\xi\right) \oplus \cdots \oplus \left(a_e^\xi\right), \quad \left|a_i^\xi\right| = p^{l_i}.$$

Then there exists an isomorphism $\varphi^\xi : A^\xi \to A$ such that $\varphi^\xi a_i^\xi = a_i$. From (a) and (d) it follows that $\operatorname{per}(B) \le \operatorname{per}(G)$, and $h_{B^\xi}(x) \le h_G(\varphi^\xi x)$ for every $x \in A^\xi$.

Then by Proposition 1 there exists an isomorphic embedding $\psi^\xi : B^\xi \to G$ which extends φ^ξ. Hence, the sequence $Y = \left\langle \psi^\xi b_1^\xi, \dots, \psi^\xi b_q^\xi \right\rangle$ is of characteristic ξ. The proof of the lemma is complete. $\square$

The fulfilment of condition 3 follows from the next proposition which is also of interest in its own right.

Proposition 2. *Take the same group G as in Theorem 2, finite subgroups $A \subseteq B^\varepsilon \subseteq C^\varepsilon \subseteq G$, $\varepsilon < 2$, and an isomorphism φ of B^0 with B^1 satisfying $\varphi \restriction A = \mathrm{id}$. Then there exist isomorphic embeddings $\psi^\varepsilon : C^\varepsilon \to G$ such that $\psi^\varepsilon \restriction A = \mathrm{id}$ and $\psi^0 x^0 = \psi^1(\varphi x^0)$ for all $x^0 \in B^0$, while $\psi^0 B^0 \not\subseteq C^0 \cup C^1$.*

Proof. Firstly establish

Lemma 8. *On assuming the hypotheses of Proposition 2 there exist isomorphic embeddings $\psi^\varepsilon : B^\varepsilon \to G$ such that $\psi^\varepsilon \restriction A = \mathrm{id}$, $\psi^0 x^0 = \psi^1 x^1 \rightleftharpoons x^2$, and $h_{C^\varepsilon}(x^\varepsilon) \le h_G(x^2)$ for every element $x^0 \in B^0$, where $x^1 \rightleftharpoons \varphi x^0$ and $\psi^0 B^0 \not\subseteq C^0 \cup C^1$.*

Proof. Making t steps, we will construct finite subgroups B_t^α for $\alpha < 3$ and isomorphisms $\psi_t^\varepsilon : B_t^\varepsilon \to B_t^2$, where $B_t^\varepsilon \subseteq B^\varepsilon$, such that for every $x^0 \in B_t^0$ we have
(1^0) $\psi_t^\varepsilon \restriction A = \mathrm{id}$;
(2^0) $x^0 \in B_t^0 \Leftrightarrow x^1 \in B_t^1$;
(3^0) $\psi_t^0 x^0 = \psi_t^1 x^1 \rightleftharpoons x^2$;
(4^0) $\max\{h_{C^\varepsilon}(x^\varepsilon) \mid \varepsilon < 2\} \le h_G(x^2)$.

Step 0. $B_0^\alpha = 0$, $\psi_0^\varepsilon = \mathrm{id}$.

Assume that t steps were made.

Step $t+1$. Put $D_{t+1}^\varepsilon = \left\{x \in B^\varepsilon \mid x \notin B_t^\varepsilon,\ px \in B_t^\varepsilon\right\}$, $n_{t+1}^\varepsilon = \max\{h_{C^\varepsilon}(x) \mid x \in D_{t+1}^\varepsilon\}$, $H_{t+1}^\varepsilon = \left\{x \in D_{t+1}^\varepsilon \mid h_{c^\varepsilon}(x) = n_{t+1}^\varepsilon\right\}$, $E_{t+1}^\varepsilon = A \cap H_{t+1}^\varepsilon$.

In order to determine $\gamma < 2$ and b_{t+1}^γ consider the following possibilities:

1. $n_{t+1}^0 \ne n_{t+1}^1$.

Then take as γ some number satisfying $n_{t+1}^\gamma > n_{t+1}^{1-\gamma}$. If $E_{t+1}^\gamma \ne 0$ then take as b_{t+1}^γ an arbitrary nonzero $a \in E_{t+1}^\gamma$. Otherwise, put $b_{t+1}^\gamma = x$ for some $x \in H_{t+1}^\gamma$, $x \ne 0$.

2. $n_{t+1}^0 = n_{t+1}^1$.

If there exists $\varepsilon < 2$ such that $E_{t+1}^\varepsilon \ne 0$ then put $\gamma = \varepsilon$. Otherwise, put $\gamma = 0$. Choose b_{t+1}^γ as in case 1. Call b_{t+1}^γ a $(t+1)$-*high* element. Put $n \rightleftharpoons n_{t+1}^\gamma$ and $b_{t+1}^{1-\gamma} = \varphi^{-\gamma} b_{t+1}^\gamma$, where $\varphi^{-0} = \varphi$.

Now determine b_{t+1}^2. If $b_{t+1}^\gamma = a \in A$ then put $b_{t+1}^2 = a$. Suppose that $b_{t+1}^\gamma \notin A$ and $pb_{t+1}^\gamma \rightleftharpoons b_\gamma \in B_t^\gamma$. Consequently, $h_G\left(b_\gamma^2\right) \ge p^{n+1}$, where $b_\gamma^2 = \psi_t^\gamma b_\gamma$. Thus, G contains c and z satisfying

$$b_\gamma^2 = p^{n+1} c, \quad |z| = p, \quad h_G(z) \ge p^n, \tag{21}$$

$$(z) \cap \left(C^0 \cup C^1 \cup \{c\} \cup B_t^2\right) = 0. \tag{22}$$

Put $b^2_{t+1} = p^n c + z$. By (21) and (22),

$$pb^2_{t+1} = b^2_\gamma, \quad h_G\big(b^2_{t+1}\big) \geq p^n, \quad b^2_{t+1} \notin C^0 \cup C^1 \cup B^2_t.$$

Put $B^\alpha_{t+1} = B^\alpha_t + \big(b^\alpha_{t+1}\big)$ for $\alpha < 3$. It is easy to verify that there exists an isomorphism $\psi^\varepsilon_{t+1} : B^\varepsilon_{t+1} \to b^2_{t+1}$ such that $\psi^\varepsilon_{t+1} \restriction B^\varepsilon_t = \psi^\varepsilon_t$ and $\psi^\varepsilon_{t+1} b^\varepsilon_{t+1} = b^2_{t+1}$.

Step $t+1$ is complete, and we proceed to the next step.

In order to verify properties 1^0–4^0 at step $t+1$, we need

Lemma 9. *For all $\varepsilon, \delta < 2$, each step t, and all $c^\varepsilon_t \in B^\varepsilon_t$ and $d^\delta_t \in B^\delta \setminus B^\delta_t$ we have*

$$h_G\big(c^2_t\big) \geq h_{C^\delta}\big(d^\delta_t\big), \tag{23}$$

where $c^2_t \rightleftharpoons \psi^\varepsilon_t\big(c^\varepsilon_t\big)$.

The proof goes by induction on t. Take $c^\varepsilon_{t+1} \in B^\varepsilon_{t+1}$, $d^\delta_{t+1} \in B^\delta \setminus B^\delta_{t+1}$, and a $(t+1)$-high element b^γ_{t+1}. We may assume that $c^\varepsilon_{t+1} \notin B^\varepsilon_t$. Then the definition of B^ε_{t+1} implies that $c^\varepsilon_{t+1} = c^\varepsilon_t + mb^\varepsilon_{t+1}$ for some $c^\varepsilon_t \in B^\varepsilon_t$ and $0 < m < p$. Hence,

$$c^2_{t+1} = c^2_t + mb^2_{t+1}. \tag{24}$$

By the inductive assumption,

$$h_G\big(c^2_t\big) \geq h_{C^\delta}\big(d^\delta_{t+1}\big). \tag{25}$$

The definitions of b^γ_{t+1} and ψ^ε_{t+1} imply that

$$h_{C^\gamma}\big(b^\gamma_{t+1}\big) \geq h_{C^\delta}\big(d^\delta_{t+1}\big), \tag{26}$$

$$h_G\big(b^2_{t+1}\big) \geq h_{C^\gamma}\big(b^\gamma_{t+1}\big). \tag{27}$$

From (26) and (27) we deduce that

$$h_G\big(b^2_{t+1}\big) \geq h_{C^\delta}\big(d^\delta_{t+1}\big).$$

By (24) and (27) this yields the validity of (23) for $t+1$. The proof of Lemma 9 is complete. □

Let us verify properties 1^0–4^0 at step $t+1$ on assuming that they hold at step t. The validity of 2^0 and 3^0 follows directly from the construction. Let us establish 1^0. Take $a \in B^\varepsilon_{t+1} \setminus B^\varepsilon_t$. If $a = b^\gamma_{t+1}$ then $\psi^\varepsilon_{t+1} a = a$ by the construction of ψ^ε_{t+1}. Suppose that $a \neq b^\gamma_{t+1}$ and show that $a \in B^\gamma_{t+1}$. Indeed, suppose that $\varepsilon \neq \gamma$. Then $a = \varphi^{-\gamma} x^\gamma$ for some $x^\gamma \in B^\gamma_{t+1}$, where $\varphi^0 = \varphi$. The hypothesis $\varphi \restriction A = \mathrm{id}$ of the lemma yields $x^\gamma = a$, and so $a \in B^\gamma_{t+1}$. Then $a = c^\gamma_t + mb^\gamma_{t+1}$ for some $c^\gamma_t \in B^\gamma_t$ and $0 < m < p$. By construction $h_{C^\gamma}\big(c^\gamma_t\big) \geq h_{C^\gamma}\big(mb^\gamma_{t+1}\big)$. Hence, $h_{C^\gamma}(a) \geq h_{C^\gamma}\big(b^\gamma_{t+1}\big)$, which contradicts $b^\gamma_{t+1} \neq a$. Therefore, property 1^0 follows.

In order to obtain property 4^0, take $x^\varepsilon \in B^\varepsilon_{t+1} \setminus B^\varepsilon_t$. Then $x^\varepsilon = c^\varepsilon_t + mb^\varepsilon_{t+1}$ for $c^\varepsilon_t \in B^\varepsilon_t$ and $0 < m < p$. Therefore,

$$x^2 = c^2_t + mb^2_{t+1}. \tag{28}$$

The definitions of a $(t+1)$-high element and of ψ^ε_t imply that

$$h_{C^\varepsilon}(x^\varepsilon) \leq h_{C^\gamma}\big(mb^\gamma_{t+1}\big), \tag{29}$$

$$h_{C^\gamma}\big(mb^\gamma_{t+1}\big) \leq h_G\big(mb^2_{t+1}\big), \tag{30}$$

while Lemma 9 implies that $h_G\big(c^2_t\big) \geq h_{C^\gamma}\big(mb^\gamma_{t+1}\big)$. Taking (28) and (30) into account, we obtain

$$h_G(x^2) \geq h_{C^\gamma}\big(mb^\gamma_{t+1}\big).$$

By (29) this implies that $h_{C^\varepsilon}(x^\varepsilon) \leq h_G(x^2)$; therefore, 4^0 holds. The proof of Lemma 8 is complete. □

Resume the proof of Proposition 2. Suppose that the hypotheses of the proposition hold. Then by Lemma 8 there exist isomorphic embeddings $\psi^\varepsilon : B^\varepsilon \to G$ such that $\psi^\varepsilon \restriction A = \mathrm{id}$, $\psi^0 x^0 = \psi^1 x^1 \rightleftharpoons x^2$, and $h_{C^\varepsilon}(x^\varepsilon) \leq h_G(x^2)$ for every $x^0 \in B^0$, where $x^1 \rightleftharpoons \varphi x^0$. By Proposition 1 there exist isomorphic embeddings $f^\varepsilon : C^\varepsilon \to G$ extending ψ^ε, as required.

Proposition 2 is established, and so is condition 3. □

Corollary 5. *Take the same group G as in Theorem 2 and its finite subgroups $A \subseteq B \subseteq C$ with $B \neq A$. Then there exists an embedding $\psi : C \to G$ such that $\psi \restriction A = \mathrm{id}$ and $\psi B \nsubseteq C$.*

Indeed, put $B^\varepsilon = B$ and $C^\varepsilon = C$, and take $\varphi : B \to B$, $\varphi = \mathrm{id}$. Then all hypotheses of Proposition 2 hold. Thus, there exists an embedding $\psi : C \to G$, as required. □

4. Let us establish the last condition. Suppose that the graph of a function $f : \mathbb{HF}(G) \to HF(G)$ is determined by a Σ-formula $\Phi(x, y, A_0)$, where $A_0 \subseteq G$ and $u = \varkappa(X)$ with $f(u) = \tau(Z)$, $X, Z \in G^{<\omega}$. Put $A = (A_0 \cup \operatorname{sp} X)$ and $B = (A \cup \operatorname{sp} Z)$. Take a finite subgroup C of G such that

$$\mathbb{HF}(C) \models \Phi(u, \tau(Z), A_0), \quad C \supseteq B.$$

Suppose that $B \neq A$. Then by Corollary 5 there exists an embedding $\psi : C \to G$ such that $\psi \restriction A = \mathrm{id}$ and $\psi B \nsubseteq C$; i.e., $\operatorname{sp} \psi Z \nsubseteq \operatorname{sp} Z$. Take the natural extension $\psi^\# : \mathbb{HF}(C) \to \mathbb{HF}(C)$ of ψ defined as $\psi^\#(\varkappa(X)) = \varkappa(\psi X)$, where $\varkappa \in \mathbb{HF}(\omega)$. Lemma 6 of [1] yields

$$f(\psi^\#(\varkappa(X))) = f(\varkappa(\psi X)) = \tau(\psi Z).$$

Since $\psi X = X$, it follows that $f(\varkappa(\psi X)) = \tau(Z)$. Hence, $\tau(Z) = \tau(\psi Z)$; i.e., $\operatorname{sp} Z = \operatorname{sp} \psi Z$, which is impossible. Consequently, $B = A$ and $\operatorname{sp} Z \subseteq A$.

Therefore, condition 4 is established, and the proof of the theorem is complete. □

Theorems 2 and A imply

Corollary 6. *Take the same abelian p-group G as in Theorem 2. Given a finite subset A_0 there exists a universal Σ-function $U^{A_0}(x, y) \in F\Sigma(\mathbb{HF}(G), A_0)$ for the family of unary functions $\mathfrak{F}^{A_0}$ such that every function $f \in \mathfrak{F}^{A_0}$ satisfies $\lambda y U^{A_0}(n, y) = f(y)$ for some n.*

Theorem 3. *Take an abelian p-group that is the direct sum of a finite period and a finite-dimensional divisible group. Then it is Σ-bounded.*

Proof. By the first Prüfer theorem there exist $\alpha, \beta, \gamma \in \omega$, a cardinal λ, and subgroups G_0, G_1, and G_2 such that

$$G = G_0 \oplus G_1 \oplus G_2 \oplus D,$$

where $\operatorname{per}(G_0) < p^\alpha$, $G_1 \cong C_{p^\alpha}^\lambda$, $G_2 = (g_1) \oplus \dots \oplus (g_\beta)$ with $|g_i| > p^\alpha$ for $1 \le i \le \beta$, and $D \cong C_{p^\infty}^\gamma$ for $\lambda \ge \omega$.

Consider the case $\alpha > 0$. The proof for case $\alpha = 0$ is similar but simpler. Take a finite subset $A_0 \subseteq G$ containing $g_1, \dots, g_\beta$. In order to prove the theorem it suffices to establish that G is Σ-bounded with respect to A_0. To this end, we must verify conditions 1–4 of Definition 1.

1. Suppose that $Y = \langle y_1, \dots, y_q \rangle$, with $y_i = g_i$ for $1 \le i \le \beta$, is a base for a finite subset $X \subseteq G$ (with respect to A_0) if there exists a number m satisfying $p^m \ge \operatorname{per}(\langle X \rangle)$ and

$$H \rightleftharpoons \big(\langle X \rangle, D_m\big) = (y_1) \oplus \dots \oplus (y_q), \tag{31}$$

with $D_m \subseteq D$, $D_m \cong C_{p^m}^\gamma$, and $|y_i| \le p^\alpha$ for $\beta + 1 \le i \le e \rightleftharpoons q - \gamma$, while $|y_j| = p^m$ for $e + 1 \le j \le q$. This implies that $D_m = (y_{e+1}) \oplus \dots \oplus (y_q)$.

Observe that the decomposition (31) always exists since by Theorem E the subgroups G_2 and D_m are direct summands of H. It is easy to verify that $\mathfrak{B}_0(X, Y)$ is a Δ-predicate in $\langle \mathbb{HF}(G), A_0 \rangle$.

Take two bases Y^ε, $\varepsilon = 0, 1$, for X. Then there exist two numbers m^ε satisfying

$$H^\varepsilon = (\langle X \rangle, D_{m^\varepsilon}) = (g_1) \oplus \dots \oplus (g_\beta) \oplus \big(y_{\beta+1}^\varepsilon\big) \oplus \dots \oplus \big(y_q^\varepsilon\big), \tag{32}$$

where $m^\varepsilon \ge \operatorname{per}(\langle X \rangle)$ and $D_{m^\varepsilon} = \big(y_{e+1}^\varepsilon\big) \oplus \dots \oplus \big(y_q^\varepsilon\big) \subseteq D$ with $q = e + \gamma$. Suppose that $m^0 < m^1$. Then $D_{m^0} \subseteq D_{m^1}$; thus, $H^0 \subseteq H^1$. Therefore,

$$H^1 = \big(\langle X \rangle, D_{m^1}\big) \text{ and } p^{m^1} \ge p^{m^0} = \operatorname{per}(\langle Y^0 \rangle).$$

By (32) this implies that Y^1 is a base for Y^0. Therefore, condition 1 holds.

2. Take a base $Y = \langle y_1, \dots, y_q \rangle$, where $y_i = g_i$ for $1 \le i \le \beta$, $|y_j| = p^{m_j}$ with $m_j \le \alpha$ for $\beta + 1 \le j \le e = q - \gamma$, and $|y_k| = p^m$ for $e + 1 \le k \le q$. Then

$$\langle Y \rangle = (y_1) \oplus \dots \oplus (y_q), \quad D_m = (y_{e+1}) \oplus \dots \oplus (y_q).$$

Take $z \in \langle Y \rangle$. Then there exists a unique sequence $\bar{k} = \langle k_1, \dots, k_q \rangle$ of numbers such that

$$z = k_1 y_1 + \dots + k_q y_q,$$

where $k_s \le |y_s|$ for $1 \le s \le q$. The index $[\bar{k}]$ is called the *coordinate* of z with respect to Y. It is easy to verify that $\mathrm{Cor}(Z, Y, n)$ is a Δ-predicate in $\langle \mathbb{HF}(G), A_0 \rangle$.

Fix some decomposition $A = \langle A_0 \rangle = (a_1) \oplus \dots \oplus (a_r)$, with $a_i = g_i$ for $1 \le i \le \beta \le r$, and take some numbers n_i such that $\mathrm{Cor}(a_i, Y, n_i)$. Then the sequence

$$\chi(Y) = \langle |y_1|, \dots, |y_q|, n_1, \dots, n_r \rangle$$

is called a *characteristic* of Y. It is easy to verify that $\chi = \chi(Y)$ is a binary Δ-predicate in $\langle \mathbb{HF}(G), A_0 \rangle$.

The computability of the set of all characteristics follows from the next lemma. Suppose that $p^{l_i} = |g_i|$ and $|a_j| = p^{l_j}$, where $1 \le i \le \beta$ and $1 \le j \le r$.

Lemma 10. *A sequence*

$$\xi = \langle p^{m_1}, \dots, p^{m_q}, n_1, \dots, n_r \rangle$$

of numbers, where $q \ge r$, $n_i = [s_{i1}, \dots, s_{iq}] + 1$, $s_{ij} = p^{r_{ij}} t_{ij}$, *and* $(t_{ij}, p) = 1$ *for* $1 \le j \le q$ *and* $1 \le i \le r$, *is a characteristic if and only if for all* $0 \le \alpha_i \le p^{l_i}$, $1 \le i \le r$, *we have*

(a) $\max\{m_{\beta+1}, \dots, m_e\} \le \alpha$, $m_{e+1} = \dots = m_q \rightleftharpoons m$, $m \ge \max\{m_1, \dots, m_e\}$, *and* $p^{m_i} = |g_i|$ *for* $1 \le i \le \beta$;

(b) $0 \le r_{ij} \le m_j$ *and* $\max\{m_j - r_{ij} \mid 1 \le j \le q\} = l_i$;

(c) $\bigwedge_{j=1}^{q} \left[\sum_{i=1}^{r} \alpha_i s_{ij} \equiv 0 (\mathrm{mod}\, p^{m_j}) \right] \Leftrightarrow \bigwedge_{i=1}^{r} \left[\bigwedge_{j=1}^{q} (\alpha_i s_{ij} \equiv 0 (\mathrm{mod}\, p^{m_j})) \right]$,

(d) $\min\left\{ \exp\left(p, \sum_{i=1}^{r} \alpha_i s_{ij}\right) \mid \beta + 1 \le j \le e \right\} \le h_{G_0 \oplus G_1}\left(\sum_{i=1}^{r} \alpha_i a_i' \right)$,

where $a_i' \rightleftharpoons \mathrm{pr}_H(a_i)$ *is the projection of* a_i *onto the subgroup* $G' = G_0 \oplus G_1$.

Proof. Since G' satisfies the hypotheses of Theorem 2, Lemma 7 implies this lemma. □

Therefore, condition 2 holds.

3. Take two bases Y^ε, $\varepsilon = 0, 1$, of the same characteristic

$$\chi = \langle p^{m_1}, \dots, p^{m_e}, p^m, \dots, p^m, n_1, \dots, n_r \rangle$$

and finite subgroups

$$B^\varepsilon \supseteq (Y^\varepsilon). \tag{33}$$

By the definition of the base Y^ε of characteristic χ,

$$(Y^\varepsilon) = \left(y_1^\varepsilon\right) \oplus \dots \oplus \left(y_q^\varepsilon\right) = G_2 \oplus A^\varepsilon \oplus D_m, \tag{34}$$

where $A^\varepsilon = \left(y_{\beta+1}^\varepsilon\right) \oplus \dots \oplus \left(y_e^\varepsilon\right)$ and $D_m = \left(y_{e+1}^\varepsilon\right) \oplus \dots \oplus \left(y_q^\varepsilon\right) \subseteq D$. By (33) this implies that there exist subgroups $D^\varepsilon \subseteq D$ and $B_0^\varepsilon \subseteq B^\varepsilon$ such that B_0^ε is isomorphic to some subgroup of G' and

$$B^\varepsilon = G_2 \oplus B_0^\varepsilon \oplus D^\varepsilon, \tag{35}$$

so that, taking (33) into account, we have $D^\varepsilon \supseteq D_m$. Then according to (34) and (35) we may assume that, up to isomorphism,

$$A^\varepsilon \subseteq B_0^\varepsilon \subseteq G'. \tag{36}$$

Denote by n_i^ε the coordinate of $\mathrm{pr}_{A^\varepsilon}(a_i) \rightleftharpoons a_i'$. Since $\mathrm{Cor}(Y^\varepsilon, a_i, n_i)$, it follows that $n_i^0 = n_i^1 \rightleftharpoons n_i'$. The subgroup G' satisfies condition 3 of Theorem 2. It is easy to verify that $Y_0^\varepsilon = \langle y_{\beta+1}^\varepsilon, \dots, y_e^\varepsilon \rangle$ is a base for the group $\langle G', a_1', \dots, a_r' \rangle$ of characteristic

$$\chi' = \langle p^{m_{\beta+1}}, \dots, p^{m_e}, n_1', \dots, n_r' \rangle.$$

Then by Theorem 2 there exist a base $Y_0^2 = \langle y_{\beta+1}^2, \dots, y_e^2 \rangle$ of characteristic χ' and a subgroup $B_0^2 \subseteq G_0'$ such that there exist embeddings

$$\varphi_0^\varepsilon : B_0^\varepsilon \to B_0^2, \quad \varphi_0^\varepsilon Y_0^\varepsilon = Y_0^2.$$

Without restricting generality we may assume that $D^0 = D^1 \cong C_{p^n}^\gamma$ for some $n \geq m$. Put

$$Y^2 = \langle g_1, \dots, g_\beta, y_{\beta+1}^2, \dots, y_e^2, y_{e+1}^2, \dots, y_q^2 \rangle, \quad B^2 = G_2 \oplus B_0^2 \oplus D^0,$$

where $|y_i^2| = p^n$ and $e+1 \leq i \leq q$. It is easy to verify that there exist embeddings $\varphi^\varepsilon : B^\varepsilon \to B^2$ such that $\varphi^\varepsilon \restriction G_2 \oplus D^0 = \mathrm{id}$, $\varphi^\varepsilon Y^\varepsilon = Y^2$, and $\varphi^\varepsilon \restriction B_0^\varepsilon = \varphi_0^\varepsilon$, so that φ^ε are embeddings, as required.

Therefore, condition 3 holds.

In order to verify condition 4 we need

Lemma 11. *Every partial function $f : \mathbb{HF}(G) \to \mathbb{HF}(G)$ defined by a Σ-formula with parameters A_0 satisfies the following condition: given $u \in \delta f$ there is a base Y for $\mathrm{sp}\, u$ such that $\mathrm{sp}\, f(u) \subseteq \langle Y \rangle$.*

Proof. Suppose that the graph of f is defined by a Σ-formula $\Phi(x, y, A_0)$, and take

$$u = \varkappa(X),\ f(u) = \tau(Z),\ X, Z \in G^{<\omega},\ m = \mathrm{per}(A_0, \mathrm{sp}\, X, \mathrm{sp}\, Z). \tag{37}$$

Take a base $Y = \langle y_1, \dots, y_q \rangle$ for $\mathrm{sp}\, X$ with $|y_{\beta+1}| = \dots = |y_q| = p^m$, put

$$A = (Y), \quad B = (Y \cup \mathrm{sp}\, Z), \tag{38}$$

and denote by C a finite subgroup of G such that

$$B \subseteq C, \quad \mathbb{HF}(C) \models \Phi(u, \tau(Z), A_0). \tag{39}$$

Then some subgroups A_0', B_0, and C_0 isomorphic to $G' = G_0 \oplus G_1$ and D_0^1, $D^2 \subseteq D$ satisfy

$$A = A_0' \oplus G_2 \oplus D^1, \tag{40}$$

$$B = B_0 \oplus G_2 \oplus D^1, \tag{41}$$

$$C = C_0 \oplus G_2 \oplus D^2, \tag{42}$$

where $D^1 \cong C_{p^m}^\gamma$ and $D^2 \cong C_{p^n}^\gamma$ for some $n \geq m$. Verify that

$$A_0' \subseteq B_0 \subseteq C_0. \tag{43}$$

Take

$$x \in B_0. \tag{44}$$

By (39) this yields $x \in C$. By (42) for some elements $c_0 \in C_0$, $g \in G_2$, and $d \in D^2$ we have

$$x = c_0 + g + d. \tag{45}$$

Since $\mathrm{per}(B) = p^m$, it follows that $|x| \leq p^m$. Consequently, $|d| \leq p^m$; i.e., $d \in D^1$. From (41), (42), and (44) we deduce that $c_0 \in B_0$. Then (41) and (45) yield $g = d = 0$; i.e., $x = c_0 \in C$, and hence $B_0 \subseteq C_0$. Similarly, $A_0' \subseteq B_0$; i.e., (43) is established.

Up to isomorphism, we may assume that the subgroups $A_0' \subseteq B_0 \subseteq C_0$ are contained in G_0', which satisfies condition 3 of Theorem 2. Suppose that $A_0' \neq B_0$. Then by Corollary 5 there exists an embedding $\psi_0 : C_0 \to G_0'$ such that $\psi_0 \restriction A_0' = \mathrm{id}$ and $\psi_0 B_0 \not\subseteq C_0$. The embedding ψ_0 extends to an embedding $\psi : C \to G$ satisfying $\psi \restriction G_2 \oplus D^2 = \mathrm{id}$. Then $\psi \restriction A = \mathrm{id}$ and $\psi B \not\subseteq C$. By Lemma 6 of [1] from (37) and (38) we deduce that

$$f(u) = f(\varkappa(\psi X)) = \tau(\psi Z) \neq \tau(Z).$$

We arrive at a contradiction; i.e., $A_0' = B_0$. Consequently, $A = B$ and $\mathrm{sp}\, Z \in A = (Y)$.

The proofs of the lemma and the theorem are complete. □

Theorems 3 and A imply

Corollary 7. *Take the same abelian p-group G as in Theorem 3. Then for every finite subset $A_0 \supseteq \{g_1, \dots, g_\beta\}$ there exists a universal Σ-function $U^{A_0}(x, y) \in F\Sigma(\mathbb{HF}(G), A_0)$ for the family of unary functions $\mathfrak{F}^{A_0}$ such that every function $f \in \mathfrak{F}^{A_0}$ satisfies $\lambda y U^{A_0}(n, y) = f(y)$ for some n.*

Every abelian p-group G is the direct sum of its reduced and divisible parts. Thus, G satisfies the hypotheses of either Theorem 2 or Theorem 3. By Theorems 2 and 3 this implies

Corollary 8. *Every abelian p-group is a Σ-bounded algebraic system.*

Corollary 9. *Take an abelian p-group G. Then in $\mathbb{HF}(G)$ there exists a universal Σ-function for the family of all unary Σ-functions.*

Proof. Take a finite subset $A_0 \subseteq G$ with respect to which G is Σ-bounded, fix a decomposition $\langle A_0 \rangle = (a_1) \oplus \dots \oplus (a_e)$, some finite set X, and a base $Y_X^{A_0}$ for X with respect to A_0. Put $A_0^1 = \{a_1, \dots, a_e\}$ and $X^* = A_0^1 \cup X$.

If G satisfies the hypotheses of Theorem 2 then the subgroups $\langle Y_X^{A_0} \rangle$ and $\langle Y_{X^*}^{\varnothing} \rangle$ are generated by the same set X^*. Therefore, $\langle Y_X^{A_0} \rangle = \langle Y_{X^*}^{\varnothing} \rangle$.

If G satisfies the hypotheses of Theorem 3 then there exists m such that the subgroup H generated by the set $X \cup A_0^1 \cup D^m$ satisfies $H = (y_1) \oplus \dots \oplus (y_q)$, and $Y_X^{A_0} = \langle y_1, \dots, y_q \rangle$, where $D^m = D[p^m]$ and D is the divisible part of G. Then $Y_{X^*}^{\varnothing} \rightleftharpoons Y_X^{A_0}$ is a base for $X \cup A_0^1$ with respect to the empty set.

Therefore, in both cases the hypotheses of Theorem B hold. Then that theorem, where we must replace C and C^1 with A_0 and A_0^1, yields the claim. □

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