Siberian Mathematical Journal, Vol. 51, *No.* 3, *pp.* 479–490, 2010
Original Russian Text Copyright © 2010 *Maksimova L. L.*

JOINT CONSISTENCY IN EXTENSIONS OF THE MINIMAL LOGIC

L. L. Maksimova UDC 510.64

Abstract: Analogs of Robinson's theorem on joint consistency are found which are equivalent to the weak interpolation property (WIP) in extensions of Johansson's minimal logic J. Although all propositional superintuitionistic logics possess this property, there are J-logics without WIP. It is proved that the problem of the validity of WIP in J-logics can be reduced to the same problem over the logic Gl obtained from J by adding the tertium non datur. Some algebraic criteria for validity of WIP over J and Gl are found.

Keywords: minimal logic, interpolation, joint consistency

In [1] A. Robinson proved a theorem on joint consistency for the classical predicate logic. The theorem says that, for every two consistent theories of different languages, if their intersection is complete in the common language then the union of these theories is consistent. Various analogs of this theorem were later investigated also for other logics (see [2]).

Robinson's theorem turned out equivalent to an interpolation theorem by W. Craig [3] for the classical first order logic. Craig's theorem became a source of many studies on the interpolation problem in classical and nonclassical theories [2, 4]. At present, interpolation is considered as a standard property of logics alongside consistency, completeness, etc. For the intuitionistic predicate logic and for Johansson's minimal logic, the interpolation problem was proved by K. Schütte [5]. A semantic proof of the interpolation theorem in the intuitionistic predicate logic was found by D. Gabbay [6]. In the same book interrelations between interpolation and joint consistency in intuitionistic theories were dealt with.

In this article we consider a weak variant of the interpolation property in the minimal logic and its extensions. The minimal logic introduced by I. Johansson [7] has the same positive fragment as the intuitionistic logic but the minimal logic has no special axioms for the absurdity constant. Unlike of the classical and intuitionistic logics, the minimal logic admits the nontrivial theories that contain some proposition together with its negation. Some semantic interpretation of the minimal logic was proposed in [8], where completeness theorems were proved for this logic and a few of its extensions.

The original definition of interpolation admits different analogs, equivalent in the classical logic but inequivalent in other logics. It is proved in [6] that the full version of Robinson's joint consistency theorem fails in the intuitionistic predicate logic. However, a slightly weaker variant RCP$''$ of this theorem is valid, and the Craig interpolation property (CIP) is equivalent to this variant of Robinson's theorem in all extensions of the intuitionistic predicate logic.

This paper addresses WIP that was introduced in [9]. We prove that in all extensions of the minimal logic, WIP is equivalent to a weak version WRP of the Robinson property. It is demonstrated in [9] that all propositional superintuitionistic logics possess WIP but this cannot be extended to all superintuitionistic predicate logics. Since only finitely many superintuitionistic logics have CIP [10], WIP and WRP are not equivalent to CIP and RCP$''$ over the intuitionistic logic. Especially they are not equivalent over the minimal logic J. Here we introduce one more analog of the Robinson property, JCP, and prove its equivalence to WIP and WRP. We note that WIP is nontrivial in the propositional extensions of

---

The author was supported by the Russian Foundation for Basic Research (Grant 09–01–00090a), the Leading Scientific Schools of the Russian Federation (Grant NSh–3606.2010.1), and the Russian Federal Agency for Education (Grant 2.1.1.419)

†) To Yuriĭ Leonidovich Ershov.

Novosibirsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 51, No. 3, pp. 604–619, May–June, 2010. Original article submitted February 1, 2010.

the minimal logic: both families of J-Logics with WIP and without WIP have the cardinality of the continuum.

In Sections 4–6 the joint consistency property is reduced to some properties of varieties of J-algebras. In Section 5 an algebraic equivalent of WIP is found; more exactly, a weak amalgamation property is introduced and its equivalence to WIP is proved. In Section 6 an easier algebraic criterion of weak amalgamability is found. In addition, it is proved that the WIP problem in J-logics can be reduced to the extensions of a logic Gl which are obtained from J by adding the tertium non datur. In Section 7 a useful classification of logics over Gl is found, and some logics with CIP and logics without WIP are listed.

1. Joint Consistency and Interpolation

If $\mathbf{p}$ is a list of nonlogical symbols then by $A(\mathbf{p})$ we denote the formula whose all nonlogical symbols are in $\mathbf{p}$, and by $\mathscr{F}(\mathbf{p})$, the set of all these formulas.

Let L be a logic, with $\vdash_L$ the consequence relation of L. Let $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ be disjoint lists of nonlogical symbols, while $A(\mathbf{p},\mathbf{q})$ and $B(\mathbf{p},\mathbf{r})$ are formulas. The *Craig interpolation property* (CIP) and the *deductive interpolation property* (IPD) are defined as follows:

CIP: If $\vdash_L A(\mathbf{p},\mathbf{q}) \to B(\mathbf{p},\mathbf{r})$ then there is a formula $C(\mathbf{p})$ such that $\vdash_L A(\mathbf{p},\mathbf{q}) \to C(\mathbf{p})$ and $\vdash_L C(\mathbf{p}) \to B(\mathbf{p},\mathbf{r})$.

IPD: If $A(\mathbf{p},\mathbf{q}) \vdash_L B(\mathbf{p},\mathbf{r})$ then there is a formula $C(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L C(\mathbf{p})$ and $C(\mathbf{p}) \vdash_L B(\mathbf{p},\mathbf{r})$.

In [9] the *weak interpolation property* was introduced:

WIP: If $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$ then there is a formula $A'(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$.

In the classical logic these properties are equivalent. In all normal modal logics we have

$$\text{CIP} \Rightarrow \text{IPD} \Rightarrow \text{WIP}\,.$$

The reverse arrows are in general not valid. CIP and IPD are equivalent in the intuitionistic logic and its extensions.

In the classical predicate logic CIP is equivalent to the following joint consistency property by Robinson:

RCP: Let T_1 and T_2 be two consistent L-theories in the languages $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively. If $T_1 \cap T_2$ is a complete L-theory in the common language $\mathscr{L}_1 \cap \mathscr{L}_2$ then $T_1 \cup T_2$ is L-consistent.

The same equivalence holds in all classical modal logics [4]. Here by an L-theory we mean a set of formulas which contains all theorems of the logic L and is closed under the modus ponens. A theory is said to be *consistent* if it does not contain the absurdity constant $\bot$. An L-theory T in the language $\mathscr{L}$ is called *complete* in $\mathscr{L}$ if either $A \in T$ or $\neg A \in T$ for every formula $A \in \mathscr{L}$, where $\neg A = A \to \bot$.

It is proved in [9] that in the classical modal logics RCP is equivalent to the following property:

RCP′: Let T_1 and T_2 be two L-theories in the languages $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively. Put $\mathscr{L}_0 = \mathscr{L}_1 \cap \mathscr{L}_2$ and $T_{i0} = T_i \cap \mathscr{L}_0$. If the theory $T_{10} \cup T_{20}$ in the common language $\mathscr{L}_0$ is L-consistent then $T_1 \cup T_2$ is also L-consistent.

2. J-Logics and Joint Consistency

Investigation of the interpolation property and joint consistency in the intuitionistic logic was carried out by D. Gabbay [6]. He proved that in the extensions of the intuitionistic predicate logic CIP is equivalent to a weaker version RCP″ of the Robinson property, and the general form of Robinson's theorem fails in the intuitionistic first order logic. The notion of an intuitionistic theory was defined as a pair (T, F), with T a set of "true" formulas and F a set of "false" formulas. So the general Robinson property needed to keep all true and all false formulas of both theories but it was not always possible. RCP″ required an additional condition $F_1 \subseteq F_2$, in particular, F_1 must be in the common language. By analogy with RCP′, in [9] a weaker version WRP of the Robinson property was defined, where each theory was identified with its set of "true" formulas like the theories in the classical logic. These theories were called *open*. It was proved that, in the case of superintuitionistic logics, WRP is equivalent to WIP and is much weaker than CIP. Moreover, all propositional superintuitionistic logics possess WIP.

In the present paper we consider extensions of Johansson's minimal logic. An axiomatization of the minimal logic JQ has as its postulates all axiom schemes of the intuitionistic predicate logic not containing the absurdity constant.

Let L be any extension of the minimal logic by new axiom schemes. If Γ is a set of formulas and A is a formula then we write $\Gamma \vdash_L A$, whenever A is deducible from $\Gamma \cup L$ by the modus ponens: $A, A \to B / B$. Then the deduction lemma holds.

Lemma 2.1. $\Gamma, A \vdash_L B \iff \Gamma \vdash_L A \to B$.

We define an *open L-theory* of the language $\mathscr{L}$ as a set T of formulas of this language closed under deducibility $\vdash_L$. Then an open L-theory T is the same as the theory $(T, \varnothing)$ in the sense of Gabbay [6]. A set $T \subseteq \mathscr{L}$ is called *complete* in $\mathscr{L}$ if either $A \in T$ or $\neg A \in T$ for every formula $A \in \mathscr{L}$. A set T is called *L-consistent* if $T \nvdash_L \bot$. It is easy to see that any L-consistent set which is complete in $\mathscr{L}$ is an open L-theory. The following lemma proved by Lindenbaum for the classical logic holds for every J-logic L.

Lemma 2.2. *Each L-consistent open L-theory can be extended to a complete L-consistent open L-theory.*

Proof. Note that if a set of formulas T is L-consistent then either $T \cup \{A\}$ or $T \cup \{\neg A\}$ is also L-consistent for every formula A. Indeed, if both sets $T \cup \{A\}$ and $T \cup \{\neg A\}$ are inconsistent then by the deduction theorem $T \vdash_L A \to \bot$ and $T \vdash_L (A \to \bot) \to \bot$; hence, $T \vdash_L \bot$.

Thus the statement easily follows from Zorn's lemma since each union of a chain of L-consistent sets is L-consistent too. $\square$

We define the two variants of the Robinson property for open theories as the *joint consistency property* (JCP) and the *weak Robinson property* (WRP).

JCP: Let T_1 and T_2 be two L-consistent open L-theories of the languages $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively. If $T_1 \cap T_2$ is a complete L-theory in the common language $\mathscr{L}_1 \cap \mathscr{L}_2$ then $T_1 \cup T_2$ is L-consistent.

WRP: Let T_1 and T_2 be two open L-theories of the languages $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively. Put $\mathscr{L}_0 = \mathscr{L}_1 \cap \mathscr{L}_2$ and $T_{i0} = T_i \cap \mathscr{L}_0$. If $T_{10} \cup T_{20}$ is L-consistent in the common language then $T_1 \cup T_2$ is L-consistent.

For open L-theories in extensions of J the equivalence of WIP and WRP holds, and JCP and WRP are equivalent as well.

The following theorem is an analog of the theorem by D. Gabbay for superintuitionistic logics [6, Theorem 8.32].

Theorem 2.3. *WIP and WRP are equivalent for each (predicate or propositional) extension L of the minimal logic.*

Proof. Assume that L has WIP and prove WRP. Let T_1 and T_2 be two open L-theories of the languages $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively. Put $\mathscr{L}_0 = \mathscr{L}_1 \cap \mathscr{L}_2$ and $T_{i0} = T_i \cap \mathscr{L}_0$. Suppose that $T_1 \cup T_2$ is L-inconsistent. Then there are formulas $A \in T_1$ and $B \in T_2$ such that $A, B \vdash_L \bot$. By WIP there exists a formula C in the common language $\mathscr{L}_0$ such that $A \vdash_L C$ and $C, B \vdash_L \bot$. Then $T_1 \vdash_L C$ and $T_2 \vdash_L C \to \bot$, and so $T_{10} \cup T_{20}$ is L-inconsistent.

Conversely, assume that L have WRP. Let A and B be arbitrary formulas of the languages $\mathscr{L}_1$ and $\mathscr{L}_2$ and $A, B \vdash_L \bot$. Denote by T_1 an open theory of the language $\mathscr{L}_1$ with the only axiom A; and by T_2, an open theory of the language $\mathscr{L}_2$ with the axiom B. Then $T_1 \cup T_2$ is L-inconsistent. By WRP $T_{i1} \cup T_{i2}$ is L-inconsistent in the common language $\mathscr{L}_0 = \mathscr{L}_1 \cap \mathscr{L}_2$, where $T_{i0} = T_i \cap \mathscr{L}_0$; i.e., $T_{10} \cup T_{20} \vdash_L \bot$. Therefore there is a formula C in the common language $\mathscr{L}_0$ such that $C \in T_{10}$ and $C, T_{20} \vdash_L \bot$. By the definition of T_i and T_{i0}, $A \vdash_L C$ and $C, B \vdash_L \bot$. $\square$

Corollary 2.4. *If a (predicate or propositional) extension of the minimal logic has* CIP *then it has* WRP.

Proof. By the deduction theorem it is clear that CIP implies WIP. So the statement is immediate from Theorem 2.3. $\square$

Theorem 2.5. WRP *and* JCP *are equivalent for every extension of* J.

Proof. Assume that L has WRP. Let two L-consistent open L-theories T_1 and T_2 of the languages $\mathscr{L}_1$ and $\mathscr{L}_2$ be given, where $T_0 = T_1 \cap T_2$ is a complete L-theory in the common language $\mathscr{L}_0 = \mathscr{L}_1 \cap \mathscr{L}_2$. It is clear that T_0 is L-consistent. We show that $T_{i0} = T_i \cap \mathscr{L}_0 = T_0$ for $i = 1, 2$. Indeed, it is evident that $T_0 \subseteq T_{i0}$. Suppose that $C \in T_{i0}$, $C \notin T_0$. Since T_0 is complete, we obtain $\neg C \in T_0 \subseteq T_i$, and T_i is inconsistent contrary to the condition.

Thus, $T_{10} = T_{20} = T_0$. Hence, $T_{10} \cup T_{20} = T_0$ is consistent. By WRP $T_1 \cup T_2$ is consistent, and JCP is proved.

Conversely, let L have JCP and prove WRP. Let T_1 and T_2 be two open L-theories of the languages $\mathscr{L}_1$ and $\mathscr{L}_2$. Put $\mathscr{L}_0 = \mathscr{L}_1 \cap \mathscr{L}_2$ and $T_{i0} = T_i \cap \mathscr{L}_0$. Assume that $T_{10} \cup T_{20}$ is L-consistent in the common language. We will prove that $T_1 \cup T_2$ is L-consistent.

By Lemma 2.2 $T_{10} \cup T_{20}$ can be extended to a complete L-consistent open theory T_0 of the language $\mathscr{L}_0$. Put $T_1' = \{A \in \mathscr{L}_1 \mid T_1 \cup T_0 \vdash_L A\}$ and prove that $T_1' \cap \mathscr{L}_0 = T_0$. Indeed, take $C \in T_1' \cap \mathscr{L}_0$. Then $T_1 \cup T_0 \vdash_L C$ and by the deduction theorem $T_1 \vdash A \to C$ for some formula $A \in T_0$. Therefore $(A \to C) \in T_{10} \subseteq T_0$, $C \in T_0$, and so the equality is proved.

By analogy $T_2' \cap \mathscr{L}_0 = T_0$, where $T_2' = \{A \in \mathscr{L}_2 \mid T_2 \cup T_0 \vdash_L A\}$. Since $T_0 \nvdash_L \perp$ and $\perp \in \mathscr{L}_0$, the theories T_1' and T_2' are consistent. Moreover, their intersection T_0 is complete. By JCP the set $T_1' \cup T_2'$ is L-consistent, and so the subset $T_1 \cup T_2$ of it is L-consistent. $\square$

We note that the results of this section concern consistent theories. Remember that in J-logics, unlike superintuitionistic logics, the inconsistency of a theory does not imply its triviality, and the consistent theories constitute not a great part of nontrivial theories.

3. Propositional J-Logics

In this section we consider propositional J-logics.

In [10] a full description of superintuitionistic logics with CIP was found. There are only finitely many these logics. All positive logics with the interpolation property are described in [11], where there was also started the study of this property in extensions of Johansson's minimal logic.

The language of J contains the symbols $\&, \vee, \to, \perp, \top$ as primitives; the negation is defined as the abbreviation $\neg A = A \to \perp$; $(A \leftrightarrow B) = (A \to B)\&(B \to A)$. The set of all formulas is denoted by For. A formula is said to be *positive* if it contains no occurrences of $\perp$. The logic J can be given by the calculus that has the same axiom schemes as the positive intuitionistic calculus Int^+ and the modus ponens as the only inference rule. By a J-*logic* we mean any set of formulas containing all axioms of J and closed under substitution and the modus ponens. Put

$$\mathrm{Int} = \mathrm{J} + (\perp \to p), \quad \mathrm{Cl} = \mathrm{Int} + (p \vee \neg p), \quad \mathrm{Neg} = \mathrm{J} + \perp.$$

A logic is *nontrivial* if it differs from the set of all formulas For. A J-logic is called *superintuitionistic* if it contains the intuitionistic logic Int, and *negative* if it extends Neg; L is said to be *paraconsistent* if L contains neither Int nor Neg. We can prove that a logic is negative if and only if L is not contained in Cl. Given a J-logic L, by $E(L)$ we denote the family of all J-logics containing L.

It is proved in [9] that all propositional superintuitionistic logics possess WIP. Obviously, all negative logics also have this property.

Theorem 3.1. *For every* J*-logic* L *the following are equivalent:*
(1) L *has* WIP,
(2) $L \cap L_1$ *has* WIP *for every negative logic* L_1,
(3) $L \cap \mathrm{Neg}$ *has* WIP.

Proof. $1 \Rightarrow 2$: Let L have WIP, $L_1 \vdash \bot$, and

$$L \cap L_1 \vdash A(\mathbf{p}, \mathbf{q}) \to (B(\mathbf{p}, \mathbf{r}) \to \bot).$$

Then

$$L \vdash A(\mathbf{p}, \mathbf{q}) \to (B(\mathbf{p}, \mathbf{r}) \to \bot).$$

Since L has WIP, there exists a formula $C(\mathbf{p})$ such that

$$L \vdash A(\mathbf{p}, \mathbf{q}) \to C(\mathbf{p}) \text{ and } L \vdash C(\mathbf{p}) \to (B(\mathbf{p}, \mathbf{r}) \to \bot).$$

From the former condition, it follows that

$$L \cap L_1 \vdash A(\mathbf{p}, \mathbf{q}) \to C(\mathbf{p}) \vee \bot.$$

The latter condition implies

$$L \vdash C(\mathbf{p}) \vee \bot \to (B(\mathbf{p}, \mathbf{r}) \to \bot).$$

Moreover,

$$L_1 \vdash C(\mathbf{p}) \vee \bot \to (B(\mathbf{p}, \mathbf{r}) \to \bot).$$

Therefore,

$$L \cap L_1 \vdash C(\mathbf{p}) \vee \bot \to (B(\mathbf{p}, \mathbf{r}) \to \bot).$$

Thus, $C(\mathbf{p}) \vee \bot$ is an interpolant of the given formula in $L \cap L_1$.

$2 \Rightarrow 3$: Obvious.

$3 \Rightarrow 1$: Let $L \cap \mathrm{Neg}$ have WIP and

$$L \vdash A(\mathbf{p}, \mathbf{q}) \to (B(\mathbf{p}, \mathbf{r}) \to \bot).$$

Then

$$L \cap \mathrm{Neg} \vdash A(\mathbf{p}, \mathbf{q}) \to (B(\mathbf{p}, \mathbf{r}) \to \bot).$$

There is an interpolant $C(\mathbf{p})$ of this formula in $L \cap \mathrm{Neg}$. Clearly, $C(\mathbf{p})$ is an interpolant in L too. □

Corollary 3.2. *Every propositional* J*-logic containing* $\mathrm{J} + (\bot \vee (\bot \to p))$ *possesses* WIP.

Proof. Every logic containing $\mathrm{J} + (\bot \vee (\bot \to p))$ can be represented as an intersection of a negative and a superintuitionistic logic. As we already noted, all superintuitionistic logics have WIP. Hence the statement follows from Theorem 3.1. □

Corollary 3.2 cannot be extended to the class of all J-logics. The picture changes when we consider extensions of the logic

$$\mathrm{Gl} = \mathrm{J} + (p \vee (p \to \bot)) = \mathrm{J} + (p \vee \neg p).$$

It it proved in [12] that this logic has CIP. In the last section of this paper it is shown that not all extensions of Gl have WIP. Theorem 6.3 says that the weak interpolation problem in J-logics can be reduced to the same problem in the extensions of Gl.

Let us consider the logic Gl in more detail. The well-known Glivenko theorem says that a formula of the form $\neg A$ is valid in Int if and only if it is a tautology of the classical logic Cl. We note that there is an analogous correspondence between J-logics and extensions of Gl. The proposition below follows from [13, Proposition 6.1.3].

Proposition 3.3. $L + (p \vee \neg p) \vdash \neg A \iff L \vdash \neg A$ *for every* J*-logic* L *and any formula* A.

4. Algebraic Semantics

The algebraic semantics for extensions of the minimal logic is built using the so-called J-*algebras*; i.e., the algebras $\mathbf{A} = \langle A; \&, \vee, \to, \bot, \top \rangle$ satisfying the conditions:

$\langle A; \&, \vee, \to, \bot, \top \rangle$ is a lattice with respect to & and $\vee$ with a greatest element $\top$,

$z \le x \to y \iff z \& x \le y$,

$\bot$ is an arbitrary element of A.

A J-algebra is called a *Heyting algebra* or a *pseudoboolean algebra* if $\bot$ is the least element of A, and a *negative algebra* if $\bot$ is the greatest element of A. The one-element J-algebra $\mathbf{E}$ is said to be *degenerate*; it is the unique J-algebra that is a negative algebra and a Heyting algebra at the same time.

A J-algebra $\mathbf{A}$ is said to be *nondegenerate* if it contains at least two elements; $\mathbf{A}$ is *well-connected*, or *strongly compact*, if $x \vee y = \top \Leftrightarrow (x = \top \text{ or } y = \top)$ for all $x, y \in \mathbf{A}$. An element Ω of $\mathbf{A}$ is said to be the *second greatest element*, or an *opremum*, of $\mathbf{A}$, if it is the greatest among the elements of $\mathbf{A}$ different from $\top$. By B_0 we denote the two-element boolean algebra.

Recall that a nondegenerate algebra $\mathbf{A}$ is said to be *subdirectly irreducible* if $\mathbf{A}$ cannot be represented as a subdirect product of factors different from $\mathbf{A}$. An algebra $\mathbf{A}$ is *finitely indecomposable* if $\mathbf{A}$ cannot be represented as a subdirect product of finitely many factors different from $\mathbf{A}$.

It is proved in [14] that there is a one-to-one correspondence between the congruences on an implicative lattice and the filters of this lattice. The same is true for J-algebras. The congruence

$$x \sim_\nabla y \rightleftharpoons (x \to y) \& (y \to x) \in \nabla$$

is associated with a filter ∇. We put $\mathbf{A}/\nabla \rightleftharpoons \mathbf{A}/{\sim_\nabla}$. If Θ is a congruence then $\nabla(\Theta) \rightleftharpoons \{x \mid x\Theta\top\}$ is a filter, and $\sim_{\nabla(\Theta)}$ equals Θ.

A filter ∇ is said to be *prime* if ∇ cannot be represented as an intersection of finitely many filters different from ∇. The following lemmas known for Heyting algebras (see, for example, [10]) are easily extended to J-algebras.

Lemma 4.1. *For every* J-*algebra* $\mathbf{A}$

(a) $\mathbf{A}$ *is finitely indecomposable if and only if the unit filter* $\nabla = \{\top\}$ *is prime, i.e.* $\mathbf{A}$ *is well-connected;*

(b) $\mathbf{A}$ *is subdirectly irreducible if and only if* $\mathbf{A}$ *has an opremum.*

Lemma 4.2. *For every* J-*algebra* $\mathbf{A}$ *and every filter* ∇ *of* $\mathbf{A}$ *the following are equivalent:*

(a) ∇ *is a prime filter,*

(b) $\mathbf{A}/\nabla$ *is finitely indecomposable.*

The proof of the following lemma is similar to that of the corresponding lemma for Heyting algebras in [15].

Lemma 4.3. *Let* Φ *be a filter in a* J-*algebra* $\mathbf{A}$ *not containing an element* b. *Then there exist a subdirectly irreducible* $\mathbf{C}$ *with an opremum* Ω *and a homomorphism* $f : \mathbf{A} \to \mathbf{C}$ *such that* $f(x) = \top$ *for all* $x \in \Phi$ *and* $f(b) = \Omega$. *In particular, if* $a \le b$ *does not hold in* $\mathbf{A}$, *then there exist a subdirectly irreducible* $\mathbf{C}$ *with an opremum* Ω *and a homomorphism* $f : \mathbf{A} \to \mathbf{C}$ *such that* $f(a) = \top$ *and* $f(b) = \Omega$.

Recall a construction from [12]. If $\mathbf{A} = \langle A; \&, \vee, \to, \bot, \top \rangle$ is a negative algebra and $\mathbf{B} = \langle B; \&, \vee, \to, \bot, \top \rangle$ is a Heyting algebra then we define the new J-algebra $\mathbf{A} \uparrow B$ as follows: the universe of the new algebra is $C = A \cup B'$, where B' is isomorphic to B, $A \cap B' = \{\bot_\mathbf{A}\} = \{\bot_{\mathbf{B}'}\}$, and C is partially ordered by the relation

$$x \le_\mathbf{C} y \Leftrightarrow [(x \in A \text{ and } y \in B'), \text{ or } (x, y \in A \text{ and } x \le_\mathbf{A} y), \text{ or } (x, y \in B' \text{ and } x \le_{\mathbf{B}'} y)].$$

As a consequence, $\bot_\mathbf{C} = \bot_\mathbf{A} = \bot_{\mathbf{B}'}$, $\top_\mathbf{C} = \top_{\mathbf{B}'}$.

Thus **A** and **B** can be considered as intervals of the partially ordered set **C**. It follows from the definition that **A** and **B** are sublattices of **C**, and the operation $\to$ satisfies the conditions

$$x \to_{\mathbf{C}} y = \begin{cases} \top, & \text{if } x \leq_{\mathbf{C}} y, \\ x \to_{\mathbf{A}} y, & \text{if } x, y \in \mathbf{A},\ x \not\leq_{\mathbf{A}} y, \\ x \to_{\mathbf{B}'} y, & \text{if } x, y \in \mathbf{B}', \\ y, & \text{if } x \in \mathbf{B}',\ y \in \mathbf{A} - \{\top_{\mathbf{A}}\}. \end{cases}$$

In particular, every negative algebra **A** is representable as $\mathbf{A} \uparrow \mathbf{E}$ and a Heyting algebra **B** as $\mathbf{E} \uparrow \mathbf{B}$. A J-algebra is called *well-composed* if it is of the form $\mathbf{A} \uparrow \mathbf{B}$ for some suitable negative algebra **A** and a Heyting algebra **B**.

From the definition we easily obtain

Lemma 4.4. 1. *The algebra* **B** *is a subalgebra of* $\mathbf{C} = \mathbf{A} \uparrow \mathbf{B}$.

2. *The algebra* **A** *is a homomorphic image of* $\mathbf{A} \uparrow \mathbf{B}$ *under the homomorphism* $f(z) = z \,\&\bot$.

3. **A** *is a subalgebra of* **C** *if and only if* **B** *is a degenerate algebra.*

A special part in this paper belongs to well-composed algebras of the form $\mathbf{A} \uparrow B_0$, where B_0 is the two-element boolean algebra. Given a negative algebra **A** we define

$$\mathbf{A}^{\Lambda} = \mathbf{A} \uparrow B_0.$$

Evidently, all J-algebras $\mathbf{A}^{\Lambda}$ are subdirectly irreducible and have $\bot$ as their oprema.

Lemma 4.5. *Let* **A** *and* **B** *be negative algebras, and let* **C** *be a* J*-algebra.*

1. *A mapping* $\alpha : \mathbf{A}^{\Lambda} \to \mathbf{B}^{\Lambda}$ *is a monomorphism if and only if its restriction* α^l *to* **A** *is a monomorphism of* **A** *to* **B**.

2. *For every homomorphism* $h : \mathbf{A}^{\Lambda} \to \mathbf{C}$ *just one of the conditions holds:*

(a) $h(\bot) = \top_{\mathbf{C}}$, *the algebra* **C** *is negative and the restriction* h^l *of* h *to* **A** *is a homomorphism of* **A** *to* **C***;*

(b) $h(\bot) \neq \top_{\mathbf{C}}$ *and* h *is an isomorphism of* $\mathbf{A}^{\Lambda}$ *to* **C**.

3. *If* **C** *is not a negative algebra then there exists a homomorphism of* **C** *onto a suitable algebra of the form* $\mathbf{C}_1^{\Lambda}$.

Proof. 1. This is immediate from the definition.

2. If $h(\bot) = \top_{\mathbf{C}}$ then $\bot_{\mathbf{C}} = h(\bot) = \top_{\mathbf{C}}$. Let $h(\bot) \neq \top_{\mathbf{C}}$. Then for all $x, y \in \mathbf{A}$, $x \not\leq y$, we have $h(x) \to h(y) = h(x \to y) \leq h(\bot) < \top_{\mathbf{C}}$, and so $h(x) \not\leq h(y)$.

3. Follows from Lemma 4.3. □

It is well known that the family of J-algebras is a variety and there is a one-to-one correspondence between the logics containing J and the varieties of J-algebras. If A is a formula and **A** is an algebra then we say that the formula A is *valid* in **A** and write $\mathbf{A} \models A$ if the identity $A = \top$ holds in **A**. Write $\mathbf{A} \models L$ instead of $(\forall A \in L)(\mathbf{A} \models A)$.

To each logic $L \in E(\mathrm{J})$ there corresponds the variety of J-algebras

$$V(L) = \{\mathbf{A} \mid \mathbf{A} \models L\}.$$

Each logic is characterized by the variety $V(L)$. If $V(L)$ is generated by an algebra **A** then we sometimes write $L = L\mathbf{A}$.

If $L \in E(\mathrm{Int})$ then $V(L)$ is some variety of Heyting algebras, and if $L \in E(\mathrm{Neg})$ then $V(L)$ is some variety of negative algebras.

Given $L_1 \in E(\mathrm{Neg})$ and $L_2 \in E(\mathrm{Int})$, denote by $L_1 \uparrow L_2$ the logic that is characterized by all algebras of the form $\mathbf{A} \uparrow \mathbf{B}$, where $\mathbf{A} \models L_1$, $\mathbf{B} \models L_2$. By $L_1 \Uparrow L_2$ we denote the logic that is characterized by the class of algebras of the form $\mathbf{A} \uparrow \mathbf{B}$, where **A** is a finitely indecomposable algebra in $V(L_1)$ and $\mathbf{B} \in V(L_2)$.

As an example we consider the logic $\mathrm{Gl} = \mathrm{J} + (p \vee \neg p)$.

Proposition 4.6. *The logic* $\mathrm{Gl} = \mathrm{J} + (p \vee \neg p)$ *coincides with* $\mathrm{Neg} \uparrow \mathrm{Cl}$ *and is generated by the class* $\{\mathbf{A}^\Lambda \mid \mathbf{A}$ *is a negative algebra*$\}$.

Proof. Note that in any algebra of the form $\mathbf{A} \uparrow \mathbf{B}$, where $\mathbf{B}$ is a boolean algebra, the formula $p \vee \neg p$ is valid. Therefore, $\mathrm{Neg} \uparrow \mathrm{Cl} \supseteq \mathrm{Gl}$. On the other hand, if the formula $(p \vee \neg p)$ is valid in a subdirectly irreducible J-algebra, then by Lemma 4.1 this algebra either is negative itself or is of the form $\mathbf{A}^\Lambda$ for a suitable negative algebra $\mathbf{A}$. Hence the opposite inclusion holds. Moreover, each negative algebra $\mathbf{A}$ is a homomorphic image of the algebra $\mathbf{A}^\Lambda$; consequently, the logic is generated by the above-mentioned class of algebras. □

In [12, Corollary 3.5(2)] we found an axiomatization for the logics of the form $L \uparrow \mathrm{Cl}$ and $L \Uparrow \mathrm{Cl}$, where L is a negative logic. The following was proved there:

Proposition 4.7. *For every negative logic* L
$L \uparrow \mathrm{Cl} = \mathrm{Gl} + \{(\bot \to A) \mid A \in L\}$,
$L \Uparrow \mathrm{Cl} = \mathrm{Gl} + \{(\bot \to A) \mid A \in L\} + ((\bot \to p \vee q) \to (\bot \to p) \vee (\bot \to q))$.

By analogy with Proposition 4.6 it is not difficult to show that the logic $L \uparrow \mathrm{Cl}$ is generated by the class $\mathbf{A}^\Lambda$, where $\mathbf{A} \in V(L)$, and the logic $L \Uparrow \mathrm{Cl}$ by a class $\mathbf{A}^\Lambda$, where $\mathbf{A}$ is a finitely indecomposable algebra in $V(L)$.

5. Weak Amalgamation

In this section an algebraic equivalent of WIP will be found.

Recall [11] that a J-logic possesses CIP if and only if the variety $V(L)$ has the amalgamation property (AP). In the case of J-algebras AP is equivalent to the superamalgamation property (SAP). We recall the necessary definitions.

Let V be a class of algebras invariant under isomorphisms. The class V is *amalgamable* if V satisfies the following condition AP for all algebras $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in V.

AP: If $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$ then there exist $\mathbf{D}$ in V and monomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$.

A triple $(\mathbf{D}, \delta, \varepsilon)$ is called an *amalgam* for $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$.

Say that a class V has the *superamalgamation property* (SAP) if for all algebras $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in V the condition AP is satisfied and, moreover, in $\mathbf{D}$ the following hold:

$$\delta(x) \le \varepsilon(y) \iff (\exists z \in \mathbf{A})(x \le z \text{ and } z \le y), \delta(x) \ge \varepsilon(y) \iff (\exists z \in \mathbf{A})(x \ge z \text{ and } z \ge y).$$

We find an algebraic equivalent of WIP. We define the *weak amalgamation property* for a class V of J-algebras.

WAPJ: For all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ and monomorphisms $\beta : \mathbf{A} \to \mathbf{B}$ and $\gamma : \mathbf{A} \to \mathbf{C}$ there exist an algebra $\mathbf{D}$ in V and homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta\beta(x) = \varepsilon\gamma(x)$ for all $x \in \mathbf{A}$, where $\bot \ne \top$ in $\mathbf{D}$, whenever $\bot \ne \top$ in $\mathbf{A}$.

A variety of J-algebras is said to be *weakly amalgamable* if it has WAPJ.

Note that this definition differs from the definition of weak amalgamation, WAP, considered in [16]. WAP is a particular case of WAPJ.

We note that if a class V is closed under isomorphisms then WAPJ is equivalent to the following condition:

For all $\mathbf{B}, \mathbf{C} \in V$ with a common subalgebra $\mathbf{A}$, there exist $\mathbf{D}$ in V and homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$, where $\bot \ne \top$ in $\mathbf{D}$, whenever $\bot \ne \top$ in $\mathbf{A}$.

We prove that for the varieties of J-algebras WIP is equivalent to WAPJ. To prove this, we apply the methods of [4, 10]. Representation of algebras by generators and defining relations [17] is used.

If $\mathbf{x}$ is a set of variables then we denote by $F(\mathbf{x})$ the set of all formulas built by using the variables of $\mathbf{x}$.

Let an algebra **A** be given, which is generated by a set X. With any element $a \in X$ a variable p_a is associated. Put $\mathbf{x} = \{p_a \mid a \in X\}$. Define the canonical valuation $v_0(p_a) = a$. On the set $F(\mathbf{x})$ we define the relation

$$A =_{\mathbf{A}} B \iff \mathbf{A} \models v_0(A) = v_0(B).$$

Then $=_{\mathbf{A}}$ is a congruence on $F(\mathbf{x})$, and there exists a monomorphism φ_0 between $F(\mathbf{x})/=_{\mathbf{A}}$ and **A** such that $\varphi_0(A/_{=_{\mathbf{A}}}) = v_0(A)$ for any $A \in F(\mathbf{x})$. Given $A, B \in F(\mathbf{x})$, we denote

$$D^+(\mathbf{A}, X) = \{A(\mathbf{x}) \mid \mathbf{A} \models v_0(A(\mathbf{x})) = \top\}.$$

If X equals **A** then we put

$$D^+(\mathbf{A}) = D^+(\mathbf{A}, X).$$

If $\mathbf{B} \models D^+(\mathbf{A})[v]$ for some algebra **B** and valuation v then the mapping $h(a) = v(p_a)$ is a homomorphism of **A** to **B**.

Given a class K, the class of finitely generated algebras of K is denoted by $FG(K)$. We prove the following

Theorem 5.1. *Let L be a* J-*logic. Then the following are equivalent:*
(1) *L has* WIP,
(2) *$V(L)$ has* WAPJ,
(3) *$FG(V(L))$ has* WAPJ.

Proof. 1 ⇒ 2: Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V(L)$ and let **A** be a common subalgebra of **B** and **C**. Put $\mathbf{x} = \{p_a \mid a \in \mathbf{A}\}$, $\mathbf{y} = \{p_a \mid a \in \mathbf{B} - \mathbf{A}\}$, and $\mathbf{z} = \{p_a \mid a \in \mathbf{C} - \mathbf{A}\}$. On the set $F(\mathbf{x}, \mathbf{y}, \mathbf{z})$ define the relation

$$A \Theta B \iff D^+(\mathbf{B}), D^+(\mathbf{C}) \vdash_L (A \leftrightarrow B).$$

Then Θ is a congruence on $F(\mathbf{x}, \mathbf{y}, \mathbf{z})$. Put

$$\mathbf{D} = F(\mathbf{x}, \mathbf{y}, \mathbf{z})/\Theta$$

and

$$g(b) = p_b/\Theta \text{ for } b \in \mathbf{B}, \quad h(c) = p_c/\Theta \text{ for } c \in \mathbf{B}.$$

Then g and h are homomorphisms from **B** and **C** respectively to **D**, where $g(a) = h(a)$ for all $a \in \mathbf{A}$.

Suppose that $\bot = \top$ in **D**. Then $D^+(\mathbf{B}), D^+(\mathbf{C}) \vdash_L \bot$. It follows that there are finite subsets $\Gamma \subseteq D^+(\mathbf{B})$ and $\Delta \subseteq D^+(\mathbf{C})$ such that $\Gamma, \Delta \vdash_L \bot$. Denote by $B(\mathbf{x}, \mathbf{y})$ the conjunction of all formulas in Γ, and by $C(\mathbf{x}, \mathbf{z})$ the conjunction of all formulas in Δ. Then

$$B(\mathbf{x}, \mathbf{y}), C(\mathbf{x}, \mathbf{z}) \vdash_L \bot.$$

It follows from WIP that there exists a formula $A(\mathbf{x})$ such that

$$B(\mathbf{x}, \mathbf{y}) \vdash_L A(\mathbf{x}) \text{ and } A(\mathbf{x}), C(\mathbf{x}, \mathbf{z}) \vdash_L \bot.$$

Using the valuation $v(p_b) = b$ for $b \in \mathbf{B}$, we infer that $\mathbf{B} \models \Gamma(\mathbf{x}, \mathbf{y})[v]$, and so $\mathbf{A} \models A(\mathbf{x})[v]$. Put $v'(p_c) = c$ for $c \in \mathbf{C}$. Then $v(p_a) = v'(p_a)$ for $a \in \mathbf{A}$. Hence, $\mathbf{C} \models A(\mathbf{x})[v']$. Further, $\mathbf{C} \models D^+(\mathbf{C})[v']$. Therefore, $\mathbf{C} \models C(\mathbf{x}, \mathbf{z})[v']$ and $\mathbf{C} \models \bot$, and so $\bot = \top$ in **C** and **A**.

2 ⇒ 3: Obvious.

3 ⇒ 1: Assume that $FG(V(L))$ has WAPJ. We prove that L has WIP.

Let $B(\mathbf{x}, \mathbf{y}), C(\mathbf{x}, \mathbf{z}) \vdash_L \bot$. It is clear that the lists **x**, **y**, and **z** can be taken finite. We denote

$$\Gamma(\mathbf{x}) = \{A(\mathbf{x}) \mid B(\mathbf{x}, \mathbf{y}) \vdash_L A(\mathbf{x})\}$$

and show that $\Gamma(\mathbf{x}), C(\mathbf{x}, \mathbf{z}) \vdash_L \bot$.

Assume the contrary. On $F(\mathbf{x},\mathbf{z})$ define the relation

$$t\Theta t' \iff \Gamma(\mathbf{x}), C(\mathbf{x},\mathbf{z}) \vdash_L (t \leftrightarrow t').$$

Then Θ is a congruence and $\mathbf{C} = F(\mathbf{x},\mathbf{z})/\Theta \in FG(V(L))$, where $\mathbf{C} \not\models \bot = \top$.

On $F(\mathbf{x},\mathbf{y})$ define the relation

$$t\Phi t' \iff B(\mathbf{x},\mathbf{y}) \vdash_L (t \leftrightarrow t').$$

Put $\mathbf{B}_1 = F(\mathbf{x},\mathbf{y})/\Phi$, and let $\mathbf{A}_1$ be a subalgebra of $\mathbf{B}_1$ generated by $\mathbf{x}/\Phi$. Note that for all formulas $t, t' \in F(\mathbf{x})$ such that $(t,t') \in \Phi$, $(t \leftrightarrow t') \in \Gamma(\mathbf{x})$ and so $t\Theta t'$. It follows that there exists a homomorphism of $\mathbf{A}_1$ onto a subalgebra $\mathbf{A}$ of $\mathbf{C}$ generated by $\mathbf{x}/\Theta$. Since the variety of J-algebras has the congruence extension property (CEP), this homomorphism can be extended to a homomorphism f of the whole algebra $\mathbf{B}_1$ onto $\mathbf{B} = f(\mathbf{B}_1)$. Moreover, $f(\mathbf{A}_1) = \mathbf{A}$ and $f(x/\Phi) = x/\Theta$ for all $x \in \mathbf{x}$.

Since $\mathbf{A}$ is a subalgebra of both $\mathbf{B}$ and $\mathbf{C}$, all algebras are finitely generated and $\bot \neq \top$ in $\mathbf{C}$, we see by WAPJ that there exist a finitely generated algebra $\mathbf{D} \in FG(V(L))$ and two homomorphisms $g : \mathbf{B} \to \mathbf{D}$ and $h : \mathbf{C} \to \mathbf{D}$ such that $g(a) = h(a)$ for all $a \in \mathbf{A}$. Moreover, $\bot \neq \top$ in $\mathbf{A}$, and so the same holds in $\mathbf{D}$.

We define the valuation v' in $\mathbf{D}$: $v'(u) = g(f(u/\Phi))$ for $u \in \mathbf{x} \cup \mathbf{y}$ and $v'(u) = h(u/\Theta)$ for $u \in \mathbf{x} \cup \mathbf{z}$. The definition is correct since $g(f(x/\Phi)) = h(x/\Theta)$ for $x \in \mathbf{x}$.

By the definition of $\mathbf{B}$ we obtain $\mathbf{D} \models B(\mathbf{x},\mathbf{y})[v']$, and by the definition of $\mathbf{C}$ we have $\mathbf{D} \models C(\mathbf{x},\mathbf{z})[v']$. At last, $\mathbf{D} \not\models \bot = \top$. This contradicts the condition $B(\mathbf{x},\mathbf{y}), C(\mathbf{x},\mathbf{z}) \vdash_L \bot$. The theorem is proved. $\square$

For modal logics, an algebraic equivalent of WIP was found in [9].

6. Criteria for WIP

We find an easier criterion for the validity of WIP in J-logics. Moreover, we show that the consideration of WIP in J-logics can be reduced to the study of extensions of the logic $\mathrm{Gl} = \mathrm{J} + (p \vee \neg p)$.

Remember the denotation from Section 4: given a negative algebra $\mathbf{A}$, put

$$\mathbf{A}^{\Lambda} = (\mathbf{A} \uparrow B_0),$$

where B_0 is the two-element boolean algebra. Given a J-logic L, we define the class

$$\Lambda(L) = \{\mathbf{A}^{\Lambda} \mid \mathbf{A} \text{ is a negative algebra and } \mathbf{A}^{\Lambda} \in V(L)\}.$$

It is easily seen that the following holds:

Lemma 6.1. *The class $\Lambda(L)$ is empty if and only if L is a negative logic.*

Proof. If L is not negative then $\bot \neq \top$ in some algebra $\mathbf{A} \in V(L)$. Then B_0 is a subalgebra of $\mathbf{A}$ and belongs to $\Lambda(L)$. The converse is obvious. $\square$

Theorem 6.2. *Let L be a J-logic. Then L has WIP if and only if $\Lambda(L)$ is empty or amalgamable.*

Proof. $\Rightarrow$: Let L have WIP. Then $V(L)$ has WAPJ by Theorem 3.1. Suppose that $\Lambda(L)$ is nonempty. Assume that the algebras $\mathbf{A}^{\Lambda}$, $\mathbf{B}^{\Lambda}$ and $\mathbf{C}^{\Lambda}$ belong to $\Lambda(L)$ and $\beta : \mathbf{A}^{\Lambda} \to \mathbf{B}^{\Lambda}$, $\gamma : \mathbf{A}^{\Lambda} \to \mathbf{C}^{\Lambda}$ are monomorphisms. By WAPJ there exist $\mathbf{D}$ in $V(L)$ and homomorphisms $\delta : \mathbf{B}^{\Lambda} \to \mathbf{D}$ and $\varepsilon : \mathbf{C}^{\Lambda} \to \mathbf{D}$ such that $\delta\beta = \varepsilon\gamma$ and $\bot \neq \top$ in $\mathbf{D}$. We can consider the algebra $\mathbf{D}$ to be subdirectly irreducible with an opremum $\bot$. Then $\mathbf{D}$ is of the form $\mathbf{D}_1^{\Lambda}$ and belongs to $\Lambda(L)$. If $x, y \in \mathbf{B}^{\Lambda}$ and $x \not\leq u$ then $x \to y \leq \bot$, $\delta(x) \to \delta(y) = \delta(x \to y) \leq \bot < \top$, and $\delta(x) \not\leq \delta(y)$. Hence δ is a monomorphism. By analogy, ε is a monomorphism.

$\Leftarrow$: If $\Lambda(L)$ is empty then L is negative by Lemma 6.1 and so L has WIP.

Assume that $\Lambda(L)$ is nonempty and amalgamable. We prove that $V(L)$ has WAPJ.

Suppose that we are given $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V(L)$ and monomorphisms $\beta : \mathbf{A} \to \mathbf{B}$ and $\gamma : \mathbf{A} \to \mathbf{C}$, where $\bot \neq \top$ in $\mathbf{A}$. Then there exists a filter ∇ in $\mathbf{A}$ maximal among the filters not containing $\bot$. We can extend

$\beta(\nabla)$ to a filter Φ in $\mathbf{B}$ maximal among the filters of $\mathbf{B}$ not containing $\top$. Then $\Phi \cap \beta(\mathbf{A}) = \beta(\nabla)$ and $\perp$ is an opremum of $\mathbf{B}/\Phi$. Analogously, we can extend $\gamma(\nabla)$ to a filter Ψ in $\mathbf{C}$ maximal among the filters of $\mathbf{C}$ not containing $\top$. Then $\Psi \cap \gamma(\mathbf{A}) = \gamma(\nabla)$ and $\perp$ is an opremum of $\mathbf{C}/\Psi$. Therefore $\mathbf{A}_1 = \mathbf{A}/\nabla$, $\mathbf{B}_1 = \mathbf{B}/\Phi$, and $\mathbf{C}_1 = \mathbf{C}/\Psi$ are subdirectly irreducible and belong to $\Lambda(L)$; there are monomorphisms $\beta_1 : \mathbf{A}_1 \to \mathbf{B}_1$ and $\gamma_1 : \mathbf{A}_1 \to \mathbf{C}_1$ such that $\beta(x)/\Phi = \beta_1(x/\nabla)$ and $\gamma(x)/\Psi = \gamma_1(x/\nabla)$ for $x \in \mathbf{A}$.

By the amalgamability of $\Lambda(L)$ there exist $\mathbf{D} \in \Lambda(L)$ and monomorphisms $\delta : \mathbf{B}_1 \to \mathbf{D}$, $\varepsilon : \mathbf{C}_1 \to \mathbf{D}$ such that $\delta\beta_1 = \varepsilon\gamma_1$. Then $\delta'(y) = \delta(y/\Phi)$ and $\varepsilon'(z) = \varepsilon(z/\Psi)$ are the required homomorphisms from $\mathbf{B}$ and $\mathbf{C}$ to $\mathbf{D}$. $\square$

As a consequence, the problem of validity of WIP in J-logics can be reduced to considering the extensions of the logic $\mathrm{Gl} = \mathrm{J} + (p \vee (p \to \perp))$.

Theorem 6.3. *A* J*-logic L has WIP if and only if $L + \mathrm{Gl}$ has WIP.*

Proof. Note that $\Lambda(L + \mathrm{Gl}) = \Lambda(L)$. Indeed, it is evident that $\Lambda(L + \mathrm{Gl}) \subseteq \Lambda(L)$. On the other hand, if $\mathbf{A} \in \Lambda(L)$ then $\mathbf{A} \models (p \vee (p \to \perp))$, and so $\mathbf{A} \in \Lambda(L + \mathrm{Gl})$. Thus the statement follows from Theorem 6.2. $\square$

7. Logics with CIP over Gl

Theorem 6.3 reduces consideration of WIP in J-logics to studying extensions of Gl. Moreover, Theorem 6.2 indicates the role of classes $\Lambda(L)$ in this investigation. The following proposition shows that these classes split the family of Gl-logics into intervals. It gives a useful classification of logics over Gl which supplies a classification of J-logics given in [18].

Proposition 7.1. *Let a* J*-logic L_0 be generated by the class $\Lambda(L_0)$. Then L_0 contains* Gl *and for all $L \in E(\mathrm{Gl})$ the equivalence holds:*

$$\Lambda(L) = \Lambda(L_0) \iff \mathrm{Neg} \cap L_0 \subseteq L \subseteq L_0.$$

Proof. It is clear that the formula $p \vee (p \to \perp)$ is valid in all algebras of $\Lambda(L_0)$, and so Gl is contained in L_0. We prove the equivalence.

$\Rightarrow$: Let $\Lambda(L) = \Lambda(L_0)$. Then $\Lambda(L_0) \subseteq V(L)$ and $V(L_0) \subseteq V(L)$. Hence, $L \subseteq L_0$. Prove that $\mathrm{Neg} \cap L_0 \subseteq L$.

It is well known that every variety is generated by its subdirectly irreducible algebras. By Lemma 4.1 any subdirectly irreducible algebra in $V(\mathrm{Gl})$ is either negative or of the form $\mathbf{A}^\Lambda$ for a suitable negative algebra $\mathbf{A}$. All negative algebras of $V(L)$ belong to $V(L_0 \cap \mathrm{Neg})$. All algebras of the form $\mathbf{A}^\Lambda$ belong to $\Lambda(L) = \Lambda(L_0)$, and so they also belong to $V(L_0 \cap \mathrm{Neg})$. Therefore $V(L) \subseteq V(L_0 \cap \mathrm{Neg})$ and $L_0 \cap \mathrm{Neg} \subseteq L$.

$\Leftarrow$: Let $L_0 \cap \mathrm{Neg} \subseteq L \subseteq L_0$. Then $\Lambda(L_0 \cap \mathrm{Neg}) \supseteq \Lambda(L) \supseteq \Lambda(L_0)$. We note that $L_0 \cap \mathrm{Neg}$ can be axiomatized by the formulas $\{\perp \vee A \mid A \in L_0\}$. As a consequence, any subdirectly irreducible algebra of $V(L_0 \cap \mathrm{Neg})$, which is not negative, belongs to $\Lambda(L_0)$. Thus $\Lambda(L_0 \cap \mathrm{Neg}) \subseteq \Lambda(L_0)$. It follows that $\Lambda(L) = \Lambda(L_0)$. $\square$

We now consider the extensions of Gl of a special kind. An axiomatization for these logics of the forms $L \uparrow \mathrm{Cl}$ and $L \Uparrow \mathrm{Cl}$, where L is a negative logic, is given in Proposition 4.7. The logic $\mathrm{Gl} = \mathrm{Neg} \uparrow \mathrm{Cl}$ is characterized by all algebras of the form $\mathbf{A}^\Lambda$, where $\mathbf{A}$ is a negative algebra (see Proposition 4.6). By Theorems 5.1 and 5.2 of [12], we immediately have

Proposition 7.2. *For every negative logic L the following are equivalent:*

(a) *$L \uparrow \mathrm{Cl}$ has* CIP*;*

(b) *$L \Uparrow \mathrm{Cl}$ has* CIP*;*

(c) *L has* CIP.

In [11, Theorem 5.5] all negative logics with CIP are found:

$$\mathrm{Neg},\ \mathrm{NC} = \mathrm{Neg} + (p \to q) \vee (q \to p),\ \mathrm{NE} = \mathrm{Neg} + p \vee (p \to q),\ \mathrm{For} = \mathrm{Neg} + p.$$

It is proved that CIP in a negative logic L is equivalent to the amalgamability of the variety $V(L)$ and the amalgamability of the class of finitely indecomposable algebras of $V(L)$. From Theorem 6.2 we immediately obtain

Proposition 7.3. *Let L be one of the following* Gl*-logics:* $\mathrm{Cl}, (\mathrm{NE} \uparrow \mathrm{Cl}), (\mathrm{NC} \uparrow \mathrm{Cl}), (\mathrm{Neg} \uparrow \mathrm{Cl}), (\mathrm{NE} \Uparrow \mathrm{Cl}), (\mathrm{NC} \Uparrow \mathrm{Cl})$, *and* $(\mathrm{Neg} \Uparrow \mathrm{Cl})$. *Then L has* CIP*, and the classes $V(L)$ and $\Lambda(L)$ are amalgamable.*

On the contrary, if a negative logic L does not possess CIP then the variety $V(L)$ and the class of finitely indecomposable algebras of $V(L)$ are nonamalgamable. Therefore, the classes $\Lambda(L \uparrow \mathrm{Cl})$ and $\Lambda(L \Uparrow \mathrm{Cl})$ are nonamalgamable, and the logics $L \uparrow \mathrm{Cl}$ and $L \Uparrow \mathrm{Cl}$ do not possess WIP. So we obtain

Proposition 7.4. *Let L be a negative logic. Then the following are equivalent:*
(1) $(L \uparrow \mathrm{Cl})$ *has* WIP*;*
(2) $(L \Uparrow \mathrm{Cl})$ *has* WIP*;*
(3) $(L \uparrow \mathrm{Cl})$ *has* CIP*;*
(4) $(L \Uparrow \mathrm{Cl})$ *has* CIP*;*
(5) L *has* CIP*;*
(6) L *is in the list* Neg, NC, NE, *and* For.

As a consequence, WIP is nontrivial in the propositional extensions of the minimal logic. Note that both sets of J-logics with WIP and of J-logics without WIP have the cardinality of the continuum. The former set contains all superintuitionistic logics; i.e., a family of the cardinality of the continuum. The latter set, by Proposition 7.4, is at least of the same cardinality as the set of negative logics different from Neg, NC, NE, and For, and the set of negative logics also has the cardinality of the continuum.

It is clear that not all the extensions of Gl can be presented as $(L \uparrow \mathrm{Cl})$ or $(L \Uparrow \mathrm{Cl})$. For example, there is no such representation for the logic $\mathrm{Neg} \cap (\mathrm{NE} \Uparrow \mathrm{Cl})$. If L is a negative logic without CIP then the logic $L \cap (\mathrm{NE} \Uparrow \mathrm{Cl})$ has WIP by Theorem 3.1 but we can prove that it does not possess CIP.

References

1. *Robinson A.*, "A result on consistency and its application to the theory of definition," Indag. Math., **18**, No. 1, 47–58 (1956).
2. *Barwise J. and Feferman S.,* eds., Model-Theoretic Logics, Springer-Verlag, New York (1985).
3. *Craig W.*, "Three uses of Herbrand–Gentzen theorem in relating model theory," J. Symbolic Logic, **22**, No. 3, 269–285 (1957).
4. *Gabbay D. M. and Maksimova L.,* Interpolation and Definability. Modal and Intuitionistic Logics, Oxford Univ. Press, Oxford (2005) (Oxford Logic Guides, 46; Oxford Sci. Publ.).
5. *Schütte K.,* "Der Interpolationssatz der intuitionistischen Prädikatenlogik," Math. Ann., Bd 148, 192–200 (1962).
6. *Gabbay D. M.*, Semantical Investigations in Heyting's Intuitionistic Logic, D. Reidel Publ. Co., Dordrecht (1981).
7. *Johansson I.*, "Der Minimalkalkül, ein reduzierter intuitionistische Formalismus," Compos. Math., **4**, 119–136 (1937).
8. *Segerberg K.*, "Propositional logics related to Heyting's and Johansson's," Theoria, **34**, No. 1, 26–61 (1968).
9. *Maksimova L.*, "Interpolation and joint consistency," in: We Will Show Them! Essays in Honour of Dov Gabbay. V. 2, S. Artemov, H. Barringer, A. d'Avila Garcez, L. Lamb and J. Woods, eds., King's College Publ., London, 2005, pp. 293–305.
10. *Maksimova L. L.*, "Craig's theorem in superintuitionistic logics and amalgamable varieties of pseudoboolean algebras," Algebra i Logika, **16**, No. 6, 643–681 (1977).
11. *Maksimova L. L.*, "Implicit definability in positive logics," Algebra and Logic, **42**, No. 1, 37–53 (2003).
12. *Maksimova L. L.*, "Interpolation and definability in extensions of the minimal logic," Algebra and Logic, **44**, No. 6, 407–421 (2005).
13. *Odintsov S. P.*, Constructive Negations and Paraconsistency, Springer-Verlag, Dordrecht (2008).
14. *Rasiowa H. and Sikorski R.*, The Mathematics of Metamathematics, Panstwowe Wydawnitstwo Naukowe, Warszawa (1963).
15. *Maksimova L. L.*, "Intuitionistic logic and implicit definability," Ann. Pure Appl. Logic, **105**, No. 1–3, 83–102 (2000).
16. *Maksimova L. L.*, "Weak form of interpolation in equational logic," Algebra and Logic, **47**, No. 1, 56–64 (2008).
17. *Maltsev A. I.*, Algebraic Systems [in Russian], Nauka, Moscow (1970).
18. *Odintsov S. P.*, "Logic of classical refutability and class of extensions of minimal logic," Logic Log. Philos., **9**, 91–107 (2001).

L. L. Maksimova
Sobolev Institute of Mathematics and Novosibirsk State University
Novosibirsk, Russia
E-mail address: lmaksi@math.nsc.ru