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ARITHMETICAL D-DEGREES

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Abstract: Description is given of the isomorphism types of the principal ideals of the join semilattice of m-degrees which are generated by arithmetical sets. A result by Lachlan of 1972 on computably enumerable m-degrees is extended to the arbitrary levels of the arithmetical hierarchy. As a corollary, a characterization is given of the local isomorphism types of the Rogers semilattices of numberings of finite families, and the nontrivial Rogers semilattices of numberings which can be computed at the different levels of the arithmetical hierarchy are proved to be nonisomorphic provided that the difference between levels is more than 1.

Keywords: arithmetical hierarchy, m-reducibility, distributive join semilattice, Lachlan semilattice, numbering, Rogers semilattice

In 1972, Lachlan [1] described the semilattices that are isomorphic to the principal ideals of the join semilattice of computably enumerable m-degrees. As shown later in [2], a semilattice satisfies Lachlan's description if and only if it is a distributive join semilattice with top and bottom which has Σ^0_3-representation. By these results, the following is true: A join semilattice is isomorphic to a principal ideal of the semilattice of computably enumerable m-degrees if and only if it is a bounded distributive semilattice admitting Σ^0_3-representation.

The main result of this paper is a generalization of the last statement which allows us to extend it to arithmetical m-degrees. It is proved that for every natural number n, a join semilattice is isomorphic to a principal ideal of the semilattice of m-degrees of Σ^0_{n+1}-sets if and only if this semilattice is bounded distributive and admits Σ^0_{n+3}-representation. Moreover, all principal ideals of the join semilattice of m-degrees, generated by Δ^0_{n+2}-sets appear to be bounded distributive semilattices having Σ^0_{n+3}-representation. What is more, to every bounded semilattice with Σ^0_{n+3}-representation, there is either a coimmune or computable Σ^0_{n+1}-set generating an isomorphic principal ideal of the join semilattice of m-degrees. The last statement strengthens a result of [3]; together with the results of [4], it allows us to expand the class of semilattices which are principal ideals or segments in the Roger semilattices of arithmetical numberings. This implies, in particular, a strengthening of the results on the difference of the isomorphism types of the Roger semilattices of the arithmetical numberings of the various levels of the arithmetical hierarchies presented in [5, 6].

§ 1. Principal Ideals of the Semilattice of Arithmetical m-Degrees

All basic concepts of computation theory can be found in [7]; those of lattice theory can be found in [8]; and those of enumeration theory, in [9]. We assume the reader familiar with them. In the introduction of [9], we can also find some useful facts on distributive semilattices.

To denote the value of a numbering ν at x, we traditionally write νx instead of $\nu(x)$, thus omitting parentheses. Given a partial function f, by δf we denote the domain of f and by ρf, the range of f.

Given a quasiordered set $\mathscr{A} = \langle A, \leq \rangle$, the associated partially ordered set will be denoted by $\widetilde{\mathscr{A}} = \langle \widetilde{A}, \leq \rangle$ (thus employing the same notation for the quasiorder and the associated partial order); we denote the coset $\widetilde{A}$ containing $x \in A$ by $[x]_{\mathscr{A}}$ (or simply by $[x]$ if $\mathscr{A}$ is clear from the context).

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A quasiordered set $\mathscr{A}$ is called a *prelattice* (*join presemilattice* or *meet presemilattice*), if $\widetilde{\mathscr{A}}$ is a lattice (join semilattice or meet semilattice). A prelattice (join presemilattice) is called *distributive* if the associated lattice (join semilattice) is distributive. In what follows, we simply call these join semilattices (join presemilattices) semilattices (presemilattices), since we will not consider any meet semilattices. Given a prelattice (presemilattice) $\mathscr{A}$, write $\mathscr{A} = \langle A, \leq; u, v\rangle$ ($\mathscr{A} = \langle A, \leq; u\rangle$) in the case that u and v are binary operations on $\mathscr{A}$ representing the join and meet on $\widetilde{\mathscr{A}}$ (i.e., $[u(x,y)] = \sup\{[x],[y]\}$ and $[v(x,y)] = \inf\{[x],[y]\}$ for all $x, y \in A$).

If $\mathscr{L}$ is a semilattice with bottom and top then we denote these by $\bot_{\mathscr{L}}$ and by $\top_{\mathscr{L}}$ respectively.

Let $\mathscr{L} = \langle L, \leq^{\mathscr{L}}; \vee^{\mathscr{L}}\rangle$ be a semilattice with ground set L, let ν be a numbering of L, and let $n \in \mathbb{N}$. We say that ν is a Σ^0_n*-representation* of a semilattice $\mathscr{L}$ if the following hold:

(1) the binary relation "$\nu x \leq^{\mathscr{L}} \nu y$" on $\mathbb{N}$ belongs to the class Σ^0_n of the arithmetical hierarchy;

(2) there is a computable function $u : \mathbb{N}^2 \to \mathbb{N}$ such that $\nu u(x,y) = \nu x \vee^{\mathscr{L}} \nu y$ for all $x, y \in \mathbb{N}$.

We say that ν is an *n-Lachlan representation* of a semilattice $\mathscr{L}$, if there exists a sequence $\{\mathscr{D}_i = \langle D_i, \leq_i\rangle\}_{i\in\mathbb{N}}$ of finite distributive prelattices with the following properties:

(1) $D_0 \subseteq D_1 \subseteq \dots$ is a strongly commutable sequence of finite subsets of the set of natural numbers such that $\bigcup_{i\in\mathbb{N}} D_i = \mathbb{N}$;

(2) $0, 1 \in D_i$ for all i and $0 \leq_i x \leq_i 1$ for all $x \in D_i$;

(3) for $x, y \in D_i$, $x \leq_i y$ implies $x \leq_{i+1} y$; the naturally defined maps from $\widetilde{\mathscr{D}}_i$ to $\widetilde{\mathscr{D}}_{i+1}$ preserve joins;

(4) the ternary relation "$x \leq_i y$" belongs to the class Π^0_{n+2} of the arithmetical hierarchy;

(5) there are sequences $\{u_i : D_i^2 \to D_i\}_{i\in\mathbb{N}}$ uniformly computable in i and $\{v_i : D_i^2 \to D_i\}_{i\in\mathbb{N}}$ such that $\mathscr{D}_i = \langle D_i, \leq_i; u_i, v_i\rangle$;

(6) $\nu x \leq^{\mathscr{L}} \nu y \Leftrightarrow (\exists i \in \mathbb{N})(x, y \in D_i \,\&\, x \leq_i y)$ for all $x, y \in \mathbb{N}$.

It follows from the definitions that an n-Lachlan representation is a Σ^0_{n+3}-representation. It is easy to show that if a semilattice $\mathscr{L}$ has an n-Lachlan representation for some $n \in \mathbb{N}$, then $\mathscr{L}$ has the bottom $\bot_{\mathscr{L}} = \nu 0$ and the top $\top_{\mathscr{L}} = \nu 1$ and is distributive since $\mathscr{L}$ is isomorphic to the direct limit $\varinjlim_{i\in\mathbb{N}} \widetilde{\mathscr{D}}_i$.

In [2], the following is proved:

If $\mathscr{L}$ is a distributive semilattice with top and bottom possessing a Σ^0_{n+3}-representation ν for some $n \in \mathbb{N}$ then there exists an n-Lachlan representation μ of $\mathscr{L}$ such that $\nu \leq \mu$.

For $n = 0$, this statement is the content of Theorem 1 of [2]. Remark 1 of [2] noticed that the result remains valid for an arbitrary $n \in \mathbb{N}$; only one small change to be made in the proof.

When considering m-degrees, we ignore the m-degrees of $\varnothing$ and $\mathbb{N}$ thus assuming that in the semilattice $\mathscr{L}^m = \langle L^m, \leq^{\mathscr{L}^m}; \vee^{\mathscr{L}^m}\rangle$ of all m-degrees, there is a bottom $\bot_{\mathscr{L}^m}$ consisting of computable sets. For every $U \subseteq \mathbb{N}$ different from $\varnothing$ and $\mathbb{N}$, we denote by $\deg_m(U)$ the m-degree of U; and by $\mathscr{L}^m_U = \langle L^m_U, \leq^{\mathscr{L}^m_U}; \vee^{\mathscr{L}^m_U}\rangle$, the principal ideal in $\mathscr{L}^m$ generated by $\deg_m(U)$.

If $X \subseteq \mathbb{N}$ and ε is an equivalence on $\mathbb{N}$ then $[X]_\varepsilon$ denotes $\{y \in \mathbb{N} : (\exists x \in X)(\langle x, y\rangle \in \varepsilon)\}$. If X and ε are computably enumerable then so is $[X]_\varepsilon$. We say that ε *agrees with* X if $X = [X]_\varepsilon$.

Following [1], we introduce the Ψ-operator. Given $U \subseteq \mathbb{N}$ and a computably enumerable $X \subseteq \mathbb{N}$, put:

(1) if $X \cap U \neq \varnothing$ and $X \nsubseteq U$ then $\Psi(U \mid X) = \deg_m(f^{-1}(U))$, where f is a computable total function such that $\rho f = X$ (it should be clear that $\Psi(U \mid X)$ does not depend on the choice of f);

(2) if $X \cap U = \varnothing$ or $X \subseteq U$ then $\Psi(U \mid X) = \bot_{\mathscr{L}^m}$.

Without any proof we mention the following basic properties of the Ψ-operator:

(1) for $a \in L^m$, $a \leq^{\mathscr{L}^m} \deg_m(U) \Leftrightarrow$ there is a computably enumerable set X such that $a = \Psi(U \mid X)$;

(2) $\Psi(U \mid X_1 \cup X_2) = \Psi(U \mid X_1) \vee^{\mathscr{L}^m} \Psi(U \mid X_2)$;

(3) if $U_1 =^* U_2$ and $X_1 =^* X_2$ then $\Psi(U_1 \mid X_1) = \Psi(U_2 \mid X_2)$ (hereinafter, $=^*$ stands for equality modulo finite sets);

(4) if a computably enumerable equivalence ε agrees with U then $\Psi(U \mid X) = \Psi(U \mid [X]_\varepsilon)$;

(5) if $X_2 \cap U \neq \varnothing$ and $X_2 \not\subseteq U$ then $\Psi(U \mid X_1) \leq^{\mathscr{L}^m} \Psi(U \mid X_2)$ if and only if there is a computable partial function θ such that $X_1 \subseteq \delta\theta$, $\theta(X_1) \subseteq X_2$, and $(\forall x \in X_1)(x \in U \leftrightarrow \theta(x) \in U)$.

Let $\mathscr{E}$ denote the lattice of all computably enumerable sets and let $\mathscr{E}^*$ denote the factor lattice of $\mathscr{E}$ modulo finite sets. It is easy from Properties (1)–(3) of the Ψ-operator that for every fixed set $U \neq \varnothing, \mathbb{N}$ the map $\lambda X\,\Psi(U \mid X)$ is an epimorphism from $\mathscr{E}^*$ (considered as a join semilattice) onto the semilattice $\mathscr{L}^m_U$.

Let $\{\theta_i\}_{i\in\mathbb{N}}$ be a universal computable sequence of all unary computable partial functions. Given $i \in \mathbb{N}$, let W_i denote the computably enumerable set with index i (that is, $W_i = \delta\theta_i$) and let $\{W_i^t\}_{t,i\in\mathbb{N}}$ be a double strongly computable sequence of finite sets such that $W_i^0 \subseteq W_i^1 \subseteq \dots$ and $W_i = \bigcup_{t\in\mathbb{N}} W_i^t$ for all $i \in \mathbb{N}$.

Theorem 1. *Let $n \in \mathbb{N}$ and let $U \subseteq \mathbb{N}$ be an arbitrary Δ^0_{n+2}-set different from $\varnothing$ and $\mathbb{N}$. Then $\mathscr{L}^m_U$ is a distributive join semilattice with top and bottom having Σ^0_{n+3}-representation.*

PROOF. Existence of the bottom $\bot_{\mathscr{L}^m_U} = \bot_{\mathscr{L}^m}$ and the top $\top_{\mathscr{L}^m_U} = \deg_m(U)$ in $\mathscr{L}^m_U$ is obvious. Distributivity of $\mathscr{L}^m_U$ is immediate from the distributivity of $\mathscr{L}^m$ which is a well-known fact in computability theory.

Let $\nu x = \Psi(U \mid W_x)$ for all $x \in \mathbb{N}$. Then by Property (1) of the Ψ-operator, ν is a numbering of the set L^m_U. Suppose that $m \in U$ and $k \in \mathbb{N} \setminus U$. By (3), $\nu y = \Psi(U \mid W_y \cup \{m,k\})$ for all $y \in \mathbb{N}$. By (5)

$$\nu x \leq^{\mathscr{L}^m_U} \nu y \Leftrightarrow (\exists i \in \mathbb{N})(W_x \subseteq \delta\theta_i \,\&\, \theta_i(W_x)$$
$$\subseteq W_y \cup \{m,k\} \,\&(\forall z \in W_x)(x \in U \leftrightarrow \theta_i(x) \in U)).$$

Using the Tarski–Kuratowski algorithm, we infer that the relation “$\nu x \leq^{\mathscr{L}^m_U} \nu y$” belongs to the class Σ^0_{n+3}. Finally, there is a computable function u such that $W_{u(x,y)} = W_x \cup W_y$ for all $x, y \in \mathbb{N}$; by (2) $\nu u(x,y) = \nu x \vee^{\mathscr{L}^m_U} \nu y$ for all $x, y \in \mathbb{N}$. Therefore, ν is a Σ^0_{n+3}-representation of $\mathscr{L}^m_U$. $\square$

The following also holds:

Theorem 2. *Let $n \in \mathbb{N}$ and let $\mathscr{L}$ be a distributive join semilattice with bottom and top having Σ^0_{n+3}-representation. Then there exists a coimmune computable Σ^0_{n+1}-set U such that $\mathscr{L} \cong \mathscr{L}^m_U$.*

A proof of this theorem is given in the next section.

From Theorems 1 and 2 a characterization is immediate of the isomorphism types of the principal ideals of the semilattice of m-degrees generated by arithmetical sets.

Corollary 1. *Given $n \in \mathbb{N}$ and a semilattice $\mathscr{L}$, the following are equivalent:*

(1) *$\mathscr{L}$ is a distributive join semilattice with bottom and top having Σ^0_{n+3}-representation;*

(2) *$\mathscr{L}$ is isomorphic to a principal ideal of the semilattice of m-degrees which is generated either by a coimmune or computable Σ^0_{n+1}-set;*

(3) *$\mathscr{L}$ is isomorphic to a principal ideal of the semilattice of m-degrees which is generated either by a coimmune or computable Π^0_{n+1}-set;*

(4) *$\mathscr{L}$ is isomorphic to a principal ideal of the semilattice of m-degrees which is generated by a Δ^0_{n+2}-set.*

PROOF. The implications (4) $\Rightarrow$ (1) $\Rightarrow$ (2) are due to Theorems 1 and 2. The implication (2) $\Rightarrow$ (4) follows from the inclusion $\Sigma^0_{n+1} \subseteq \Delta^0_{n+2}$. Finally, (2) $\Leftrightarrow$ (3) is justified by the fact that the map sending each subset of $\mathbb{N}$ to its complement defines an automorphism of $\mathscr{L}^m$. $\square$

§ 2. Proof of Theorem 2

Let $n \in \mathbb{N}$ and let ν' be a Σ^0_{n+3}-representation of a distributive semilattice $\mathscr{L} = \langle L, \leq^{\mathscr{L}}; \vee^{\mathscr{L}} \rangle$ with bottom and top. According to a statement mentioned in the previous section and proved in [2], there is a numbering $\nu \geq \nu'$ of L that is an n-Lachlan representation of $\mathscr{L}$.

Theorem 1 from [3] is equivalent to the following:

Each semilattice having 0*-Lachlan representation is isomorphic to the semilattice* $\mathscr{L}_U^m$ *for a computable or hypersimple set* U*.*

Thus, for $n = 0$ we may assume Theorem 2 to be proved. In what follows, we suppose that $n > 0$.

We fix a sequence $\{\mathscr{D}_i = \langle D_i, \leq_i\rangle\}_{i\in\mathbb{N}}$ of finite distributive presemilattices which possesses all six properties from the definition of an n-Lachlan representation.

We employ the technique of working with towers which was invented by Lachlan in [1] and further developed by other people in [10, 11]. However, frames and towers will be built not for the sequence of presemilattices $\{\mathscr{D}_i\}_{i\in\mathbb{N}}$ itself as it was done in the above-mentioned papers, but for the approximating spectra of the sequence as it was performed in the paper [12] by Yu. L. Ershov. In other words, we will make a "mixture" of the Lachlan and Ershov techniques, borrowing the essential parts of both of them. (Ershov did work not with towers but with equivalent representations.) We give necessary definitions.

A *spectrum* is a finite sequence of binary relations $\mathfrak{S} = \left\langle \leq_0^{\mathfrak{S}}, \dots, \leq_k^{\mathfrak{S}}\right\rangle$ with the following properties

(1) for every $i \leq k$, the relation $\leq_i^{\mathfrak{S}}$ is a quasiorder on D_i such that $\mathscr{D}_i^{\mathfrak{S}} = \left\langle D_i, \leq_i^{\mathfrak{S}}\right\rangle$ is a finite distributive prelattice with the bottom 0 and the top 1.

(2) for arbitrary $i < k$ and $x, y \in D_i$, $x \leq_i^{\mathfrak{S}} y$ implies $x \leq_{i+1}^{\mathfrak{S}} y$; moreover, the naturally defined map from $\widetilde{\mathscr{D}}_i^{\mathfrak{S}}$ to $\widetilde{\mathscr{D}}_{i+1}^{\mathfrak{S}}$ preserves joins.

The number k in the above definition is called the *length* of $\mathfrak{S}$. The length of $\mathfrak{S}$ is denoted by $\mathrm{lh}(\mathfrak{S})$. We say that a spectrum $\mathfrak{S}$ is a *beginning* of a spectrum $\mathfrak{S}'$ and we write $\mathfrak{S} \preccurlyeq \mathfrak{S}'$ provided that $\mathrm{lh}(\mathfrak{S}) \leq \mathrm{lh}(\mathfrak{S}')$ and $\leq_i^{\mathfrak{S}}$ coincides with $\leq_i^{\mathfrak{S}'}$ for any $i \leq \mathrm{lh}(\mathfrak{S})$. If $\mathfrak{S}$ is a spectrum and $m \leq \mathrm{lh}(\mathfrak{S})$ then we denote the unique spectrum $\mathfrak{S}'$ of length m for which $\mathfrak{S}' \preccurlyeq \mathfrak{S}$ by $\mathfrak{S} \upharpoonright m$.

A spectrum $\mathfrak{S}$ is called *good* provided that for every $i \leq k$, the relation $\leq_i^{\mathfrak{S}}$ coincides with $\leq_i$. For every number k, there is exactly one good spectrum of length k; we denote this spectrum by $\mathfrak{S}^k$.

Let $\mathfrak{S}$ be an arbitrary spectrum of length k and let $A \subseteq D_i$ for $i \leq k$. We say that A is an *atom of spectrum* $\mathfrak{S}$ *of level* i if $\inf\{[x] : x \in A\} \not\leq_i^{\mathfrak{S}} \sup\{[x] : x \notin A\}$ in $\widetilde{\mathscr{D}}_i^{\mathfrak{S}}$. In other words, the atoms of level i are exactly the principal upper order cones in $\mathscr{D}_i^{\mathfrak{S}}$ which are generated by the elements different from $[0]$ and which are join irreducible.

We observe the following two properties of atoms (as before, here $\mathfrak{S}$ denotes an arbitrary spectrum of length k; the proofs of all properties can be found in [1, 10, 11]):

(1) for $i \leq k$, the lattice $\widetilde{\mathscr{D}}_i^{\mathfrak{S}}$ is isomorphic to a sublattice of the powerset lattice of the set of atoms of $\mathfrak{S}$ of level i; the corresponding isomorphism is defined by $[x]_{\mathscr{D}_i^{\mathfrak{S}}} \mapsto \{A \text{ is an atom of } \mathfrak{S} \text{ of level } i : x \in A\}$;

(2) for every $i < k$ and every atom A of $\mathfrak{S}$ of level $i+1$, there is a unique set $\{A_1, \dots, A_m\}$ consisting of the atoms of $\mathfrak{S}$ of level i none of which includes another one such that $A \cap D_i = A_1 \cup \dots \cup A_m$.

For $k \in \mathbb{N}$, a *tree of height* k is a finite tree where every maximal branch has length $k+1$ (we assume a tree to grow "downward"). In other words, a *tree of height* k is a finite poset $\mathscr{M} = \langle M, \leq_{\mathscr{M}}\rangle$ with top such that for every $a \in M$ the set $\{x \in M : a \leq_{\mathscr{M}} x\}$ is totally ordered, and its every maximal totally ordered subposet has $k+1$ elements. The top of $\mathscr{M}$ is called a *root* of this tree and is denoted by $r_{\mathscr{M}}$. For an element $a \in M$, the *level* of a is the length of a maximal chain in the set of elements below a. The level of a is denoted by $h_{\mathscr{M}}(a)$. It is easy to see that the height of a tree is the same as the level of its root; moreover, if $\mathscr{M}$ is a tree of height k then for all $a \in M$ we have $h_{\mathscr{M}}(a) = k + 1 - |\{x \in M : a \leq_{\mathscr{M}} x\}|$.

A *frame of height* k is an arbitrary triple $\mathscr{F} = \langle \mathfrak{S}, \mathscr{M}, c\rangle$ such that

(1) $\mathfrak{S}$ is a spectrum of length k;

(2) $\mathscr{M} = \langle M, \leq_{\mathscr{M}}\rangle$ is a tree of height k;

(3) c is a map from M into $\mathscr{P}(D_k)$ such that, for all $a \in M$, the set $c(a)$ is an atom of $\mathfrak{S}$ of level $h_{\mathscr{M}}(a)$;

(4) if for an element $a \in M$ of a nonzero level $\mathrm{Sc}(a)$ denotes the set of successors of a (that is, the set of all elements of level $h_{\mathscr{M}}(a) - 1$ which are below a) then $c(a) \cap D_{h_{\mathscr{M}}(a)-1} = \bigcup\{c(b) : b \in \mathrm{Sc}(a)\}$.

We say that frames $\langle \mathfrak{S}, \mathscr{M}, c\rangle$ and $\langle \mathfrak{S}', \mathscr{M}', c'\rangle$ are *isomorphic* if $\mathfrak{S} = \mathfrak{S}'$ and there is an isomorphism f of $\mathscr{M}$ onto $\mathscr{M}'$ such that $c = c' \circ f$. In what follows, we identify isomorphic frames.

If $\mathscr{F} = \langle \mathfrak{S}, \mathscr{M}, c\rangle$ is a frame then $\mathfrak{S}$ is called the *spectrum* of $\mathscr{F}$ and $c(r_{\mathscr{M}})$ is called an *atom* of $\mathscr{F}$. It follows from the second property of atoms that if $\mathfrak{S}$ is a spectrum of length k and A is an atom of $\mathfrak{S}$ of level k then there exists a unique frame of height k with this particular spectrum and this particular atom (modulo an isomorphism; we will not to emphasize the latter anymore).

Let $\mathscr{F}_1 = \langle \mathfrak{S}_1, \mathscr{M}_1, c_1\rangle$ and $\mathscr{F}_2 = \langle \mathfrak{S}_2, \mathscr{M}_2, c_2\rangle$ be frames, let $a \in M_1$, $b \in M_2$, and $k = h_{\mathscr{M}_1}(a) = h_{\mathscr{M}_2}(b)$, and let $\mathfrak{S}_1 \restriction k = \mathfrak{S}_2 \restriction k$. Using the second property of atoms and induction on k, it is easy to show that the following are equivalent:

(1) $c_1(a) \subseteq c_2(b)$;

(2) there exists a level preserving monotone map f from $\{x \in M_1 : x \leq_{\mathscr{M}_1} a\}$ into $\{x \in M_2 : x \leq_{\mathscr{M}_2} b\}$ such that $c_1(x) \subseteq c_2(f(x))$ for any $x \leq_{\mathscr{M}_1} a$.

We denote the set of all maps f possessing all properties of (2) by $\Phi(\mathscr{F}_1, \mathscr{F}_2, a, b)$.

A *tower of height* k is a quadruple $\mathscr{T} = \langle \mathfrak{S}, \mathscr{M}, c, \varphi\rangle$ such that

(1) $\langle \mathfrak{S}, \mathscr{M}, c\rangle$ is a frame of height k;

(2) φ is a map assigning to each element of M a nonempty finite subset of the set of natural numbers;

(3) if a and $\mathrm{Sc}(a)$ denote the same as in (4) in the definition of frame then $\varphi(a) = \bigcup\{\varphi(b) : b \in \mathrm{Sc}(a)\}$ and $\varphi(b_1) \cap \varphi(b_2) = \varnothing$ for all different $b_1, b_2 \in \mathrm{Sc}(a)$.

Given a tower $\mathscr{T} = \langle \mathfrak{S}, \mathscr{M}, c, \varphi\rangle$, the frame $\langle \mathfrak{S}, \mathscr{M}, c\rangle$ is called the *frame of the tower* and $\varphi(r_{\mathscr{M}})$ is called its *base*; we denote the latter by $\mathrm{base}(\mathscr{T})$. It should be clear that given a frame and a finite subset of $\mathbb{N}$ large enough, we can build a tower on this frame taking the given finite set as a base.

Suppose that $\mathscr{T}_1 = \langle \mathfrak{S}_1, \mathscr{M}_1, c_1, \varphi_1\rangle$ and $\mathscr{T}_2 = \langle \mathfrak{S}_2, \mathscr{M}_2, c_2, \varphi_2\rangle$ are towers with frames $\mathscr{F}_1$ and $\mathscr{F}_2$ respectively. Let $a \in M_1$ and $b \in M_2$ be elements of the same level k, let $\mathfrak{S}_1 \restriction k = \mathfrak{S}_2 \restriction k$, $c_1(a) \subseteq c_2(b)$, and let $f \in \Phi(\mathscr{F}_1, \mathscr{F}_2, a, b)$. We define a map φ as follows: Given $y \in M_2$, put $\varphi(y) = \varphi_2(y) \cup \bigcup\{\varphi_1(x) : x \leq_{\mathscr{M}_1} a,\ f(x) \leq_{\mathscr{M}_2} y\}$. It is easy to check that $\mathscr{T} = \langle \mathfrak{S}_2, \mathscr{M}_2, c_2, \varphi\rangle$ is a tower whose frame coincides with the frame of $\mathscr{T}_2$, whose base equals the union of the base of $\mathscr{T}_2$ and $\varphi_1(a)$, and the following hold:

(1) for every $x \in M_1$ of level $\leq k$, either $x \not\leq_{\mathscr{M}_1} a$ and $\varphi_1(x) \cap \mathrm{base}(\mathscr{T}) = \varnothing$ or $x \leq_{\mathscr{M}_1} a$ and $\varphi_1(x) \subseteq \varphi(f(x))$;

(2) $\varphi_2(y) \subseteq \varphi(y)$ for all $y \in M_2$.

We say that the tower $\mathscr{T}$ is obtained by *modifying* $\mathscr{T}_2$ using $\mathscr{T}_1$ and f. The number k is called the *level of modification*. In what follows, if we use a step-by-step construction, and a tower is modified at some step; then, abusing terminology, we will often refer to the initial and resulting towers as to the same one. This will not lead us to any confusion.

We now describe the process of building towers. Before each step of our construction, finitely many towers will be built with bases subsets of a certain initial segment of the poset of natural numbers. This segment consists of the numbers that have appeared in the bases of the towers already built, and the numbers 0 and 1. The numbers belonging to this segment are called *used* and the remaining are called *nonused*. Each time when we build a new tower, a rather larger initial segment of nonused numbers will be used as the base of the tower. The numbers 0 and 1 are assumed to be used even before we start our construction; that is, before we build the very first tower. Along the way of building new towers, we will also modify and destroy some old towers.

We recall that we consider all frames to within isomorphism. In particular, for each number k, there is only finitely many different frames of height k. Given a frame $\mathscr{F}$, let $\mathscr{T}_1^{\mathscr{F},t}, \dots, \mathscr{T}_{s(\mathscr{F},t)}^{\mathscr{F},t}$ be all towers with the frame $\mathscr{F}$ already existing before step t and listed in the course of their building. We assume that the set of all pairs of the form $\langle \mathscr{F}, m\rangle$, where $\mathscr{F}$ is a frame and m is a number, is effectively ordered by ω (the first infinite cardinal). We may also assume that if $m_1 < m_2$ then $\langle \mathscr{F}, m_1\rangle < \langle \mathscr{F}, m_2\rangle$ for all frames $\mathscr{F}$. Let l be a computable function of big span; i.e., a computable surjection from $\mathbb{N}$ onto $\mathbb{N}$ such that the preimage of every number is infinite. We describe step t of the construction.

Step t. This step consists of the two stages:

STAGE I. For $i = l(t)$ we check whether there are towers $\mathscr{T} = \mathscr{T}_{m_1}^{\mathscr{F},t} = \langle \mathfrak{S}_1, \mathscr{M}_1, c_1, \varphi_1 \rangle$ and $\mathscr{S} = \mathscr{T}_{m_2}^{\mathscr{G},t} = \langle \mathfrak{S}_2, \mathscr{M}_2, c_2, \varphi_2 \rangle$ with frames $\mathscr{F}$, $\mathscr{G}$ respectively, and elements $a \in M_1$, $b \in M_2$ such that

(1) both towers have height at least i;
(2) $\mathfrak{S}_1 \restriction i = \mathfrak{S}_2 \restriction i$;
(3) $h_{\mathscr{M}_1}(a) = h_{\mathscr{M}_2}(b) = i$;
(4) $c_1(a) \subseteq c_2(b)$;
(5) $\mathscr{S}$ was built before $\mathscr{T}$, and $\langle \mathscr{G}, m_2 \rangle < \langle \mathscr{F}, m_1 \rangle$;
(6) $\varphi_2(b) \cap W_i^t = \varnothing$ and $\varphi_1(a) \cap W_i^t \neq \varnothing$.

If there are no tower and no element with required properties then we pass to the next stage. Otherwise, we choose the oldest $\mathscr{S}$, and for this $\mathscr{S}$ we (effectively) choose a tower $\mathscr{T}$, elements a, b, and a function $f \in \Phi(\mathscr{F}, \mathscr{G}, a, b)$. We modify $\mathscr{S}$ using $\mathscr{T}$ and f, destroy $\mathscr{T}$, and pass to the next stage.

STAGE II. We find a frame $\mathscr{F}$ and an integer $m > 0$ with $\langle \mathscr{F}, m \rangle$ least possible such that the number of the already existing towers with frame $\mathscr{F}$ is less than m, and we build a new tower with frame $\mathscr{F}$ taking as a base a large enough initial segment of nonused numbers. Then we pass to the next step.

Description of the construction is finished. It is clear that the construction is effective. We will investigate its properties.

Each built tower either is destroyed at some point or is never destroyed but modified at some steps. In the latter case we call a tower *permanent*. Every permanent tower can be modified only finitely many times. Indeed, if for a tower $\mathscr{T} = \langle \mathfrak{S}, \mathscr{M}, c, \varphi \rangle$, the number of elements a belonging to M and such that $\varphi(a) \cap W_{h_{\mathscr{M}}(a)}^t \neq \varnothing$ does not decrease when t increases, it increases each time when the tower is modified. We say that a permanent tower is *final* at step t, if it exists and is not modified at steps $\geq t$.

Lemma 1. *For each frame $\mathscr{F}$ the following is true: $\lim_{t\to\infty} s(\mathscr{F}, t) = \infty$, and, for every $m > 0$, $\mathscr{T}_m^{\mathscr{F},t}$ is the same final tower at almost all steps t.*

PROOF. It suffices to show that for every frame $\mathscr{F}$ and every $m > 0$, the towers $\mathscr{T}_m^{\mathscr{F},t}$ exist and coincide at almost all steps t.

Suppose not. Let $\langle \mathscr{F}, m \rangle$ be a least pair for which the statement fails. Let t_0 be a step large enough so that for all $\langle \mathscr{F}', m' \rangle < \langle \mathscr{F}, m \rangle$ with $m' > 0$ the towers $\mathscr{T}_{m'}^{\mathscr{F}',t}$ exist, coincide, and are final at all steps $t \geq t_0$. If before some step $t \geq t_0$ the tower $\mathscr{T}_m^{\mathscr{F},t}$ is not defined then $s(\mathscr{F}, t) = m - 1$, and it follows from the description of Stages I and II that none of the towers with frame $\mathscr{F}$ will be destroyed and a tower with frame $\mathscr{F}$ will be built at step t, so that the value of $\mathscr{T}_m^{\mathscr{F},t}$ is defined for all $t > t_0$. By the choice of the pair $\langle \mathscr{F}, m \rangle$ of step t_0 and by the description of Stage I, the tower $\mathscr{T}_m^{\mathscr{F},t_0+1}$ is not destroyed; thus, it becomes final after some finitely many modifications. $\square$

Given a frame $\mathscr{F}$ and $m \in \mathbb{N}$, we denote the tower $\mathscr{T}_m^{\mathscr{F},t}$ (in its final variant) which is the same for all t large enough by $\mathscr{T}_m^{\mathscr{F}}$. For $x > 1$, the following is true: x appears in the base of some tower at some step and a few steps, it either disappears from all bases or appears in the base of a final tower. The first part of the statement is obvious, as at Stage II of each step, a tower is built. After a number has appeared in the base of a tower, once it disappears, it will never appear there again. Finally, if a number x is in the base of one tower before some step t and it is also in the base of another tower after step t then the second tower was built before the first. It follows that the number of these "transitions" is bounded for each x.

Each final tower equals $\mathscr{T}_m^{\mathscr{F}}$ for some frame $\mathscr{F}$ and some integer $m > 0$. We denote the set of all numbers that do not appear in the bases of the final towers by D. It is clear that D is computably enumerable and $0, 1 \in D$. Since $n > 0$; given $x \notin D$, we can compute the frame $\mathscr{F}$ with oracle $\mathbf{0}^{(n)}$ and the number m such that $x \in \operatorname{base}(\mathscr{T}_m^{\mathscr{F}})$ effectively.

Let $\mathscr{F} = \langle \mathfrak{S}, \mathscr{M}, c \rangle$ be a frame of height k and let $A = c(r_{\mathscr{M}})$ be an atom of $\mathscr{F}$. We say that $\mathscr{F}$ is a *dense* frame, if there are infinitely many final towers $\mathscr{T} = \langle \mathfrak{S}', \mathscr{M}', c', \varphi' \rangle$ of height $\geq k$ such that $\mathfrak{S} \preccurlyeq \mathfrak{S}'$ and $A = c'(a)$ and $\varphi'(a) \cap W_k \neq \varnothing$ hold for some element $a \in M'$ of level k. We say that this frame is *saturated* if for all final towers $\mathscr{T} = \langle \mathfrak{S}', \mathscr{M}', c', \varphi' \rangle$ with $\mathfrak{S} \preccurlyeq \mathfrak{S}'$ and all $a \in M'$ of level k such that $A \subseteq c'(a)$ one has $\varphi'(a) \cap W_k \neq \varnothing$.

Lemma 2. *A frame is dense if and only if it is saturated.*

Proof. Sufficiency is obvious. Indeed, for each frame, there are infinitely many towers built on it so that density is immediate from saturation.

We prove necessity. Suppose that $\mathscr{F} = \langle \mathfrak{S}, \mathscr{M}, c \rangle$ is a dense frame of height k. Let $\mathscr{T} = \langle \mathfrak{S}', \mathscr{M}', c', \varphi' \rangle$ be a final tower such that $\mathfrak{S} \preccurlyeq \mathfrak{S}'$ and $a \in M'$ is an element of level k for which $c(r_{\mathscr{M}}) = A \subseteq c'(a)$. We have $\mathscr{T} = \mathscr{T}^{\mathscr{G}}_m$ for some frame $\mathscr{G}$ and some $m > 0$. Let t_0 be large enough so that for all $\langle \mathscr{G}', m' \rangle \leq \langle \mathscr{G}, m \rangle$, one has $\mathscr{T}^{\mathscr{G}', t_0}_{m'} = \mathscr{T}^{\mathscr{G}'}_{m'}$, and all these towers have become final by step t_0. By density, there are a frame $\mathscr{G}''$ and $m'' > 0$ such that $\langle \mathscr{G}'', m'' \rangle > \langle \mathscr{G}, m \rangle$, and for the final tower $\mathscr{T}^{\mathscr{G}''}_{m''} = \langle \mathfrak{S}'', \mathscr{M}'', c'', \varphi'' \rangle$ built after $\mathscr{T}^{\mathscr{G}}_m$, there is an element $b \in M''$ of level k such that $\mathfrak{S} \preccurlyeq \mathfrak{S}''$, $c''(b) = A$, and $\varphi''(b) \cap W_k \neq \varnothing$. Let $t_1 \geq t_0$ be large enough so that the tower $\mathscr{T}^{\mathscr{G}'', t_1}_{m''} = \mathscr{T}^{\mathscr{G}''}_{m''}$ has become final by step t_1 and $\varphi''(b) \cap W^{t_1}_k \neq \varnothing$. Finally, let $t \geq t_1$ be such that $k = l(t)$. By the choice of t, none of the towers $\mathscr{T}^{\mathscr{G}', t}_{m'}$ for $\langle \mathscr{G}', m' \rangle \leq \langle \mathscr{G}, m \rangle$ will be modified at Stage I of this step; therefore, $\varphi'(a) \cap W^t_k \neq \varnothing$. $\square$

We consider a series of equivalences and sets. Let i be an arbitrary natural number, let $\mathfrak{S}$ be a spectrum of length i, and let $d \in D_i$. Put

$$\begin{aligned}
\varepsilon =& \{\langle x, y \rangle : x, y \in D\} \cup \{\langle x, y \rangle : x, y \in \mathrm{base}(\mathscr{T}) \text{ for some final tower } \mathscr{T}\},\\
\varepsilon_i =& \{\langle x, y \rangle : x, y \in D \vee x = y\} \cup \{\langle x, y \rangle : \text{there are a final tower } \langle \mathfrak{S}', \mathscr{M}, c, \varphi \rangle\\
& \text{of height } \geq i \text{ and an element } a \in M \text{ of level } i \text{ such that } x, y \in \varphi(a)\},\\
R^{\mathfrak{S},0}_{d,i} =& \bigcup \{x : \text{there is a final tower } \langle \mathfrak{S}, \mathscr{M}, c, \varphi \rangle \text{ such that } d \in c(r_{\mathscr{M}}) \text{ and } x \in \varphi(r_{\mathscr{M}})\},\\
R^{\mathfrak{S},1}_{d,i} =& D \cup \bigcup \{x : \text{there are a final tower } \langle \mathfrak{S}', \mathscr{M}, c, \varphi \rangle \text{ of height } > i \text{ and}\\
& \text{an element } a \in M \text{ of level } i \text{ such that } \mathfrak{S} \preccurlyeq \mathfrak{S}',\ d \in c(a), \text{ and } x \in \varphi(a)\}.
\end{aligned}$$

It is clear that $\varepsilon_i \subseteq \varepsilon$ for all $i \in \mathbb{N}$. We notice also that those sets and equivalences are computable with oracle $\mathbf{0}^{(n)}$ uniformly in all parameters.

Lemma 3. *Given $i \in \mathbb{N}$, a spectrum $\mathfrak{S}$ of length i, and $d \in D_i$, the relation ε_i as well as the sets $R^{\mathfrak{S},0}_{d,i}$ and $R^{\mathfrak{S},1}_{d,i}$ is computably enumerable.*

Proof. Let $\mathfrak{F}^1_i$ be the set of all saturated frames of height $\leq i$ and let $\mathfrak{F}^2_i$ be the set of all frames of height $\leq i$ which are not dense. By Lemma 2, any frame of height $\leq i$ appears in exactly one of those sets.

For a frame $\mathscr{F} = \langle \mathfrak{S}', \mathscr{M}', c' \rangle \in \mathfrak{F}^2_i$ of height $k \leq i$, there is a final tower $\mathscr{T}_{\mathscr{F}} = \langle \mathfrak{S}', \mathscr{M}', c', \varphi' \rangle$ built on this frame such that $\varphi'(r_{\mathscr{M}'}) \cap W_k = \varnothing$. Let t_0 be large enough so that by step t_0, the towers $\mathscr{T}_{\mathscr{F}}$ have been already built for all $\mathscr{F} \in \mathfrak{F}^2_i$. Let $t_1 \geq t_0$ be such that by step t_1 all towers built by step t_0 became destroyed or final.

We say that for a tower $\mathscr{T} = \langle \mathfrak{S}'', \mathscr{M}'', c'', \varphi'' \rangle$ and for $t \in \mathbb{N}$, the condition $C(\mathscr{T}, t)$ holds, if for every $k \leq i$, $a \in M''$ of level k, and a frame $\mathscr{F}$ with spectrum $\mathfrak{S}'' \restriction k$ and atom $c''(a)$, we have $\mathscr{F} \in \mathfrak{F}^1_i \Rightarrow \varphi''(a) \cap W^t_k \neq \varnothing$. It follows from the definition of saturation that this condition is satisfied for all final towers and all t large enough.

Let D^t denote the set of numbers appeared in D by step t. We consider the following sets:

$$\begin{aligned}
\varepsilon^t_i =& \{\langle x, y \rangle : x, y \in D^t \vee x = y\} \cup \{\langle x, y \rangle : \text{by step } t, \text{ there are}\\
& \text{a tower } \langle \mathfrak{S}', \mathscr{M}, c, \varphi \rangle \text{ of height } \geq i \text{ and an element } a \in M \text{ of level } i\\
& \text{such that } x, y \in \varphi(a) \text{ and for all towers } \mathscr{T} \text{ built before, one has } C(\mathscr{T}, t)\},\\
R^{\mathfrak{S},0,t}_{d,i} =& \bigcup \{x : \text{by step } t \text{ there is a tower } \langle \mathfrak{S}, \mathscr{M}, c, \varphi \rangle \text{ such that } d \in c(r_{\mathscr{M}}),\ x \in \varphi(r_{\mathscr{M}})
\end{aligned}$$

and for all towers $\mathscr{T}$, built before, one has $C(\mathscr{T},t)\}$,

$$R^{\mathfrak{S},1,t}_{d,i} = D^t \cup \bigcup\{x : \text{by step } t \text{ there are a tower } \langle \mathfrak{S}', \mathscr{M}, c, \varphi\rangle \text{ of height } > i$$
$$\text{and an element } a \in M \text{ of level } i \text{ such that } \mathfrak{S} \preccurlyeq \mathfrak{S}',\ d \in c(a),$$
$$x \in \varphi(a) \text{ and for all towers } \mathscr{T}, \text{ built before one has } C(\mathscr{T},t)\}.$$

It is clear that the sets introduced are computable uniformly in t. Since the condition C holds for all final towers and all t large enough, for all $x,y \in \mathbb{N}$ we have

$$\langle x,y\rangle \in \varepsilon_i \Leftrightarrow \langle x,y\rangle \in \varepsilon_i^t \text{ for almost all } t,$$

$$x \in R^{\mathfrak{S},0}_{d,i} \Leftrightarrow x \in R^{\mathfrak{S},0,t}_{d,i} \text{ for almost all } t,$$

$$x \in R^{\mathfrak{S},1}_{d,i} \Leftrightarrow x \in R^{\mathfrak{S},1,t}_{d,i} \text{ for almost all } t.$$

To complete the proof, it suffices to show that $\varepsilon_i^t \subseteq \varepsilon_i^{t+1}$, $R^{\mathfrak{S},0,t}_{d,i} \subseteq R^{\mathfrak{S},0,t+1}_{d,i}$, and $R^{\mathfrak{S},1,t}_{d,i} \subseteq R^{\mathfrak{S},1,t+1}_{d,i}$ for all $t \geq t_1$.

Let $t \geq t_1$ and $\langle x,y\rangle \in \varepsilon_i^t$. If $x,y \in D^t$ or $x=y$ then $\langle x,y\rangle \in \varepsilon_i^{t+1}$. Otherwise, before step t, there are a tower $\mathscr{T} = \langle \mathfrak{S}', \mathscr{M}, c, \varphi\rangle$ of height $\geq i$ and an element $a \in M$ of level i such that $x,y \in \varphi(a)$, and $C(\mathscr{T}',t)$ holds for all towers $\mathscr{T}'$ built before $\mathscr{T}$. If at step t the tower $\mathscr{T}$ is not destroyed then $\langle x,y\rangle \in \varepsilon_i^{t+1}$. Suppose that $\mathscr{T}$ is destroyed at this step. Then there is modification of some tower $\mathscr{S} = \langle \mathfrak{S}'', \mathscr{M}', c', \varphi'\rangle$ built before $\mathscr{T}$. Let k be the level of this modification and let $b' \in M'$, $a' \in M$ be such that $\mathscr{S}$ is modified using $f \in \Phi(\mathscr{F}, \mathscr{G}, a', b')$, where $\mathscr{F}$ and $\mathscr{G}$ are frames of the towers $\mathscr{T}$ and $\mathscr{S}$ respectively. Let $\mathscr{H}$ be the frame of height k built on the spectrum $\mathscr{S}'' \restriction k$ and atom $\varphi'(b')$. The frame $\mathscr{H}$ cannot belong to $\mathfrak{F}_i^1$, as condition C holds for $\mathscr{S}$. The frame $\mathscr{H}$ also cannot belong to $\mathfrak{F}_i^2$ by the choice of t_1, since otherwise the tower $\mathscr{T}_{\mathscr{H}}$ built before $\mathscr{S}$ would be modified instead of $\mathscr{S}$. The only possibility remains: $k > i$, that is, our modification is of level greater than i. But then either $a \leq_{\mathscr{M}} a'$ and $x,y \in \varphi'(f(a))$ or $a \not\leq_{\mathscr{M}} a'$ and $x,y \in D^{t+1}$. In both cases $\langle x,y\rangle \in \varepsilon_i^{t+1}$.

The implications $x \in R^{\mathfrak{S},0,t}_{d,i} \Rightarrow x \in R^{\mathfrak{S},0,t+1}_{d,i}$ and $x \in R^{\mathfrak{S},1,t}_{d,i} \Rightarrow x \in R^{\mathfrak{S},1,t+1}_{d,i}$ can be proved in the same vein for all $x \in \mathbb{N}$ and $t \geq t_1$. As in the previous case, the main idea is as follows: If x has appeared in $R^{\mathfrak{S},0,t}_{d,i}$ or $R^{\mathfrak{S},1,t}_{d,i}$ because it has appeared in the base of a tower then at step t this tower can be destroyed only as a result of modification of a level greater than i. □

We pass now to the definition of a coimmune set U belonging to the class Σ^0_{n+1}. We need the following

Lemma 4. *There is a sequence $\{V_{\mathfrak{S}} : \mathfrak{S}$ is a spectrum$\}$ of subsets of $\mathbb{N}$ enumerable uniformly in $\mathfrak{S}$ with oracle $\mathbf{0}^{(n)}$ such that*

(1) *$V_{\mathfrak{S}}$ is an initial segment of $\mathbb{N}$ for all $\mathfrak{S}$;*
(2) *$\mathfrak{S} \preccurlyeq \mathfrak{S}'$ implies $V_{\mathfrak{S}} \subseteq V_{\mathfrak{S}'}$;*
(3) *the spectrum $\mathfrak{S}$ is good $\Rightarrow V_{\mathfrak{S}}$ is finite;*
(4) *the spectrum $\mathfrak{S}$ is bad $\Rightarrow V_{\mathfrak{S}'} = \mathbb{N}$ for almost all $\mathfrak{S}' \succcurlyeq \mathfrak{S}$.*

Proof. For every $i \in \mathbb{N}$ and $P \subseteq D_i^2$, let $R(P,i)$ denote the condition $(\forall\langle x,y\rangle \in P)(x \leq_i y)$ and let $Q(P,i)$ denote the condition $(\forall\langle x,y\rangle \notin P)(x \not\leq_i y)$. Then R is a Π^0_{n+2}-condition and Q is a Σ^0_{n+2}-condition on P and i.

Since $R \in \Pi^0_{n+2}$, there is a $\mathbf{0}^{(n)}$-computable function $h(P,i,t)$ such that $h(P,i,0) \leq h(P,i,1) \leq \dots$ and $\lim_{t\to\infty} h(P,i,t) = \infty \Leftrightarrow R(P,i)$. Since $Q \in \Sigma^0_{n+2}$, there is a Σ^0_{n+1}-predicate T such that for all $i \in \mathbb{N}$ and $P \subseteq D_i^2$ the set $\{t : T(P,i,t)\}$ is an initial segment of natural numbers equal to $\mathbb{N}$ if and only if $Q(P,i)$ fails.

Let $\mathfrak{S} = \big\langle \leq_0^{\mathfrak{S}}, \dots, \leq_k^{\mathfrak{S}}\big\rangle$ be a spectrum of length k. We put

$$V_{\mathfrak{S}} = \big\{t : (\exists i \leq k) T\big(\leq_i^{\mathfrak{S}}, i, t\big)\big\} \cup \big\{t : (\exists i \leq k)\big(h\big(\leq_i^{\mathfrak{S}}, i, t\big) \leq k\big)\big\}.$$

It is easy to check that all required properties are satisfied. □

We fix a $\mathbf{0}^{(n)}$-computable function $p(\mathfrak{S},t)$ such that $p(\mathfrak{S},0) \le p(\mathfrak{S},1) \le \dots$ and $V_{\mathfrak{S}} = \{x : \exists t(x < p(\mathfrak{S},t))\}$. For each frame $\mathscr{F} = \langle \mathfrak{S}, \mathscr{M}, c\rangle$, we put $p(\mathscr{F},t)$ to be equal to $p(\mathfrak{S},t)$. As in § 1, let $\{\theta_i\}_{i\in\mathbb{N}}$ be a universal computable sequence of all partial computable functions. We describe a process of numbering of U. Its distinctive feature is as follows: The base of any final tower either does not contain numbers from U at all or is enumerated in U completely at some step. In the process, some towers will be marked by $[\mathscr{F}]$, where $\mathscr{F}$ is a frame. Some marks will be left out of our consideration; those marks will never appear again.

Step 0. We enumerate in U all numbers from D but zero. We pass to the next step.

Step $t+1$. This step consists of the two stages:

Stage I. We enumerate in U all numbers from the $\mathrm{base}\big(\mathscr{T}^{\mathscr{F}}_m\big)$ for all pairs $\langle \mathscr{F}, m\rangle$ such that $\mathscr{F}$ is a frame of height $\le t$ and $m < p(\mathscr{F},t)$. Then for any $i \le t$, if W^t_i does not contain a number that was already enumerated in U and does contain a number from the base of a final tower of height $> i$, then we enumerate in U all numbers from the base of this tower. After that, we delete all marks from the towers whose bases were enumerated in U. Finally, we choose a least pair $\langle \mathscr{F}, m\rangle$ such that the tower $\mathscr{T}^{\mathscr{F}}_m$ was not marked yet and its base was not enumerated in U. We enumerate the $\mathrm{base}\big(\mathscr{T}^{\mathscr{F}}_m\big)$ in U and pass to the next stage.

Stage II. We find a frame $\mathscr{F}$ with the value of the pair $\langle \mathscr{F}, p(\mathscr{F},t)\rangle$ least possible such that the mark $[\mathscr{F}]$ does not appeared yet and was not deleted yet. Let m be a least number such that the base of $\mathscr{T}^{\mathscr{F}}_m$ was not enumerated in U and i is the height of $\mathscr{F}$. We check whether there are numbers $x \in \mathrm{base}\big(\mathscr{T}^{\mathscr{F}}_m\big) \cap \delta\theta_i$ such that $\theta_i(x)$ does not belong to the base of a tower with the frame $\mathscr{F}$. If there are no such numbers then we exclude $[\mathscr{F}]$ from our consideration and pass to the next step. Otherwise, we choose such a number x. If $\theta_i(x) = 0$ then we enumerate the base of $\mathscr{T}^{\mathscr{F}}_m$ in U, exclude the mark $[\mathscr{F}]$ from our consideration and pass to the next step. If $\theta_i(x)$ was already enumerated in U then we mark $\mathscr{T}^{\mathscr{F}}_m$ with $[\mathscr{F}]$ and pass to the next step. If $\theta_i(x) \ne 0$ was not enumerated in U yet then this means that $\theta_i(x)$ is in the base of some final tower $\mathscr{T}^{\mathscr{G}}_k$. If the tower $\mathscr{T}^{\mathscr{G}}_k$ has no mark then we enumerate the $\mathrm{base}\big(\mathscr{T}^{\mathscr{G}}_k\big)$ in U, mark the tower $\mathscr{T}^{\mathscr{F}}_m$ by $[\mathscr{F}]$, and pass to the next step. Finally, if $\mathscr{T}^{\mathscr{G}}_k$ already has a different mark then we have the two cases:

(1) if $\langle \mathscr{F}, p(\mathscr{F},t)\rangle < \langle \mathscr{G}, p(\mathscr{G},t)\rangle$ then we mark $\mathscr{T}^{\mathscr{F}}_m$ by $[\mathscr{F}]$, delete all marks from the tower $\mathscr{T}^{\mathscr{G}}_k$, enumerate its base in U, and pass to the next step;

(2) if $\langle \mathscr{F}, p(\mathscr{F},t)\rangle > \langle \mathscr{G}, p(\mathscr{G},t)\rangle$ then we mark $\mathscr{T}^{\mathscr{G}}_k$ by $[\mathscr{F}]$, enumerate the base of $\mathscr{T}^{\mathscr{F}}_m$ in U, and pass to the next step.

The description of our construction is now complete. It is clear that the construction is effective with oracle $\mathbf{0}^{(n)}$; therefore, $U \subseteq \Sigma^0_{n+1}$.

Before each step, each mark is assigned to at most one tower. For each frame $\mathscr{F}$, before each step, at most one tower with this frame is marked; moreover, one of its marks should be $[\mathscr{F}]$. A tower can be marked only if its base was not enumerated in U yet; once it was enumerated, all marks are deleted from that tower. What is more, according to the last actions of the first stage, sooner or later either each tower will get a mark or the base of this tower will be enumerated in U. This means that for every frame either the bases of all towers with this frame are enumerated in U or there is a unique tower for which this is not so.

By construction, $D \setminus \{0\} \subseteq U$ and either the base of every final tower is contained in U or it fails to contain numbers from U at all. Thus the equivalence ε and all equivalences ε_i for $i \in \mathbb{N}$ agree with $U \cup \{0\}$.

We say that a spectrum $\mathfrak{S}$ is *quasigood* if $V_{\mathfrak{S}}$ is finite. It is easy to see that the quasigood spectra form an initial segment in the tree of all spectra, and the only infinite branch of this segment consists of good spectra. It is not hard to notice that, due to Stage I, the bases of all towers whose spectrum is not quasigood appear in U.

Given $i \in \mathbb{N}$, a spectrum $\mathfrak{S}$ of length i, and $d \in D_i$, put $R^{\mathfrak{S}}_{d,i} = R^{\mathfrak{S},0}_{d,i} \cup R^{\mathfrak{S},1}_{d,i}$. The set $R^{\mathfrak{S},0}_{d,i}$ may contain only finitely many elements not belonging to U, so that $\Psi\big(U \mid R^{\mathfrak{S},0}_{d,i}\big) = \perp_{\mathscr{L}^m}$. It is not hard to notice that $\left[R_{d,i}^{\mathfrak{S},1}\right]_{\varepsilon_{i+1}} = R_{d,i+1}^{\mathfrak{S}_1} \cup \dots \cup R_{d,i+1}^{\mathfrak{S}_k}$, where $\mathfrak{S}_1, \dots, \mathfrak{S}_k$ are all spectra of length $i+1$ which extend $\mathfrak{S}$. Since ε_{i+1} is an enumerable equivalence agreeing with $U \cup \{0\}$, we have

$$\begin{aligned}\Psi\big(U \mid R_{d,i}^{\mathfrak{S}}\big) &= \Psi\big(U \mid R_{d,i}^{\mathfrak{S},0}\big) \vee^{\mathscr{L}^m} \Psi\big(U \mid R_{d,i}^{\mathfrak{S},1}\big) = \bot_{\mathscr{L}^m} \vee^{\mathscr{L}^m} \Psi\big(U \cup \{0\} \mid \left[R_{d,i}^{\mathfrak{S},0}\right]_{\varepsilon_{i+1}}\big) \\ &= \Psi\big(U \mid R_{d,i+1}^{\mathfrak{S}_1} \cup \dots \cup R_{d,i+1}^{\mathfrak{S}_k}\big) = \Psi\big(U \mid R_{d,i+1}^{\mathfrak{S}_1}\big) \vee^{\mathscr{L}^m} \dots \vee^{\mathscr{L}^m} \Psi\big(U \mid R_{d,i+1}^{\mathfrak{S}_k}\big).\end{aligned}$$

Iterating this transformation finitely many times, we see that $\Psi\big(U \mid R_{d,i}^{\mathfrak{S}}\big) = \Psi\big(U \mid R_{d,i+m}^{\mathfrak{S}_1}\big) \vee^{\mathscr{L}^m} \dots \vee^{\mathscr{L}^m} \Psi\big(U \mid R_{d,i+m}^{\mathfrak{S}_l}\big)$, where m is an arbitrary natural number and $\mathfrak{S}_1, \dots, \mathfrak{S}_l$ are all spectra of length $i+m$ which extend $\mathfrak{S}$.

If a spectrum $\mathfrak{S}$ is not quasigood then $R_{d,i}^{\mathfrak{S}} \subseteq U \cup \{0\}$ and $\Psi\big(U \mid R_{d,i}^{\mathfrak{S}}\big) = \bot_{\mathscr{L}^m}$. Since almost all spectra extending a bad spectrum are not quasigood, we infer from the previous two equalities that $\Psi\big(U \mid R_{d,i}^{\mathfrak{S}}\big) = \bot_{\mathscr{L}^m}$ for every bad spectrum $\mathfrak{S}$.

Finally, if $\mathfrak{S} = \mathfrak{S}^i$ is a good spectrum, then among all spectra of length $i+1$ extending $\mathfrak{S}$, one spectrum is equal to $\mathfrak{S}^{i+1}$ and all of the remaining are bad, so that $\Psi\big(U \mid R_{d,i}^{\mathfrak{S}^i}\big) = \Psi\big(U \mid R_{d,i+1}^{\mathfrak{S}^{i+1}}\big)$.

We define a map λ from $\mathscr{L}$ to $\mathscr{L}_U^m$ by $\lambda(x) = \Psi\big(U \mid R_{d,i}^{\mathfrak{S}^i}\big)$, where $x \in L$, $d \in \mathbb{N}$ are such that $x = \nu(d)$ and i is chosen with the property $d \in D_i$. This map is correctly defined, presenting a semilattice homomorphism. Indeed, it was shown above that $\Psi\big(U \mid R_{d,i}^{\mathfrak{S}^i}\big)$ does not depend on the choice of i. If $\nu(d_1) = \nu(d_2)$ then we may choose i so that $d_1 \equiv_i d_2$; thus each atom of level i in every good spectrum contains d_1 if and only if it contains d_2 and $R_{d_1,i}^{\mathfrak{S}^i} = R_{d_2,i}^{\mathfrak{S}^i}$. We see that the value of $\lambda(x)$ does not depend on the choice of d with $x = \nu(d)$. Finally, if $x_1 \vee^{\mathscr{L}} x_2 = x_3$, $x_1 = \nu(d_1)$, $x_2 = \nu(d_2)$, $x_3 = \nu(d_3)$, and $d_1, d_2, d_3 \in D_i$; then, in the lattice $\widetilde{\mathscr{D}}_i$, the element $[d_3]$ is the join of $[d_1]$ and $[d_2]$, each atom of level i in every good spectrum contains d_3 if and only if it contains d_1 or d_2, $R_{d_3,i}^{\mathfrak{S}^i} = R_{d_1,i}^{\mathfrak{S}^i} \cup R_{d_2,i}^{\mathfrak{S}^i}$ and $\lambda(x_3) = \lambda(x_1) \vee^{\mathscr{L}^m} \lambda(x_2)$.

Lemma 5. *λ is an embedding.*

Proof. We show first that for every frame $\mathscr{F}$ with quasigood spectrum the mark $[\mathscr{F}]$ either is left out from our consideration or becomes permanent (i.e., it is assigned to some tower and never deleted from that tower) at some step.

It is clear that the spectrum of $\mathscr{F}$ is quasigood if and only if $p(\mathscr{F}) = \lim_{t\to\infty} p(\mathscr{F}, t) < \infty$. We prove our statement by induction on $\langle \mathscr{F}, p(\mathscr{F})\rangle$.

Let $\mathscr{F}$ be a frame with a quasigood spectrum of height i, let $\mathfrak{F} = \{\mathscr{F}' : p(\mathscr{F}) < \infty$, and $\langle \mathscr{F}', p(\mathscr{F}')\rangle < \langle \mathscr{F}, p(\mathscr{F})\rangle\}$. Suppose that for all $\mathscr{F}' \in \mathfrak{F}$ our claim about marks is true. Let t_0 be large enough so that by step $t_0 + 1$, all marks $[\mathscr{F}']$ for $\mathscr{F}' \in \mathfrak{F}$ either were excluded from consideration or have become permanent. Let $t_1 \geq t_0$ be large enough so that $p(\mathscr{F}') = p(\mathscr{F}', t_1)$ for all $\mathscr{F}' \in \mathfrak{F} \cup \{\mathscr{F}\}$ and $\langle \mathscr{F}', p(\mathscr{F}', t_1)\rangle > \langle \mathscr{F}, p(\mathscr{F})\rangle$ for all $\mathscr{F}' \notin \mathfrak{F} \cup \{\mathscr{F}\}$. Let i' be equal to the maximal height of a frame from $\mathfrak{F} \cup \{\mathscr{F}\}$. Let $t_2 \geq t_1$ be such that either $j < i'$, or $W_j \cap U = \varnothing$, or $W_j^{t_2}$ contains the elements already enumerated in U by step $t_2 + 1$.

Suppose that our statement about $[\mathscr{F}]$ is false. Then this mark is never left out from consideration and there exist infinitely many numbers $t \geq t_2$ such that after Stage I of step $t+1$ this mark is not assigned to any tower. Let t_3 be one of these t's. From the description of Stage II and our choice of t_1, it follows that at step $t_3 + 1$ this mark is either left out of consideration (which is impossible) or assigned to one of the towers $\mathscr{T}_m^{\mathscr{G}}$ for some $\mathscr{G} \in \mathfrak{F} \cup \{\mathscr{F}\}$. Let $t > t_3$ be such that at step $t+1$ this mark is again deleted. By the choice of t_2, this cannot happen at the first stage of step $t+1$. However, this cannot happen either at Stage II of this step; since, otherwise, the mark $[\mathscr{G}]$ would come into play for $\mathscr{G} \in \mathfrak{F}$, which is impossible by the choice of t_0.

We now pass to the proof of the main claim of our lemma. We should show that for all $a, b \in \mathbb{N}$, if $\nu(a) \nleq^{\mathscr{L}} \nu(b)$ then $\lambda(\nu(a)) \nleq^{\mathscr{L}^m} \lambda(\nu(b))$. Suppose that the assumption is true while the conclusion is not. Let i be such that $a, b \in D_i$. Since $\lambda(\nu(a)) = \Psi(U \mid R^{\mathfrak{S}^i}_{a,i})$ and $\lambda(\nu(b)) = \Psi(U \mid R^{\mathfrak{S}^i}_{b,i})$, either there is j such that $\delta\theta_j = R^{\mathfrak{S}^i}_{a,i}$, $\rho\theta_j = R^{\mathfrak{S}^i}_{b,i}$, and $x \in U \Leftrightarrow \theta_j(x) \in U$ for all $x \in \delta\theta_j$, or $R^{\mathfrak{S}^i}_{b,i} \subseteq U$, or $R^{\mathfrak{S}^i}_{b,i} \cap U = \varnothing$.

The inclusion $R^{\mathfrak{S}^i}_{b,i} \subseteq U$ is impossible since $0 \in D \subseteq R^{\mathfrak{S}^i}_{b,i}$ and $0 \notin U$. The equality $R^{\mathfrak{S}^i}_{b,i} \cap U = \varnothing$ is impossible either, since $1 \in D \setminus \{0\} \subseteq R^{\mathfrak{S}^i}_{b,i} \cap U$. We consider the remaining case. Since every partial computable function has infinitely many numbers, we may assume that $j \geq i$. Since $\lambda(\nu(a)) \not\leq^{\mathscr{L}_m} \lambda(\nu(b))$, there is a frame $\mathscr{F}$ of height j with spectrum $\mathfrak{S}^j$ such that a belongs and b does not belong to the atom of this frame. The base of every tower with frame $\mathscr{F}$ contains numbers from $R^{\mathfrak{S}^i}_{a,i}$ and does not contain numbers from $R^{\mathfrak{S}^i}_{b,i}$, so that the mark $[\mathscr{F}]$ can be excluded from our consideration only in the case when in the base of a final tower with frame $\mathscr{F}$, there is $x \in U$ such that $\theta_i(x) = 0$. However, this is impossible; thus, this mark appears at some step and never disappears. But then for some x one of the elements of $\{x, \theta_i(x)\}$ will be enumerated in U at this step and the other will appear in the base of a tower on which a permanent mark will be put, and will never be enumerated in U; a contradiction. $\square$

Lemma 6. *λ is an isomorphism from $\mathscr{L}$ onto $\mathscr{L}^m_U$.*

Proof. Since λ is an embedding, it suffices to show that every element from $\mathscr{L}^m_U$ belongs to the image of λ.

Let $u \in L^m_U$. Then $u = \Psi(U \mid W_i)$ for some $i \in \mathbb{N}$. Let $\mathscr{F}_1, \dots, \mathscr{F}_k$ be all dense frames with spectrum $\mathfrak{S}^i$ and let $A_1, \dots, A_k$ be their atoms. Let d be an element of D_i belonging to all atoms $A_1, \dots, A_k$ and not belonging to any other atom of the spectrum $\mathfrak{S}^i$ of level i. Such an element exists by Lemma 2 since for every atom A of level i of $\mathfrak{S}^i$ such that $A \supseteq A_m$ for some $m \in [1, k]$, the frame built on this atom with spectrum $\mathfrak{S}^i$ is saturated, whence $A \in \{A_1, \dots, A_k\}$.

We show that $\left[R^{\mathfrak{S}^i}_{1,i} \cap W_i\right]_{\varepsilon_i} \cup D =^* R^{\mathfrak{S}^i}_{d,i}$. Let $x \in R^{\mathfrak{S}^i}_{d,i}$. Then either $x \in D$ or $x \in \varphi(a)$ for some final tower $\mathscr{T} = \langle \mathscr{S}, \mathscr{M}, c, \varphi \rangle$ and some $a \in M$ of level i such that $\mathfrak{S}^i \preccurlyeq \mathfrak{S}$ and $d \in \varphi(a)$. The latter means that the atom $\varphi(a)$ belongs to $\{A_1, \dots, A_k\}$. Hence the frame with spectrum $\mathfrak{S}^i$ built on the atom $\varphi(a)$ is saturated and $\varphi(a)$ contains an element y belonging to W_i. The atom $\varphi(a)$ also contains 1, whence $y \in R^{\mathfrak{S}^i}_{1,i} \cap W_i$. The fact that $x, y \in \varphi(a)$ means that $\langle x, y \rangle \in \varepsilon_i$. Therefore, $x \in \left[R^{\mathfrak{S}^i}_{1,i} \cap W_i\right]_{\varepsilon_i}$.

Conversely, suppose that $x \in \left[R^{\mathfrak{S}^i}_{1,i} \cap W_i\right]_{\varepsilon_i} \cup D$. If $x \in D$ then $x \in R^{\mathfrak{S}^i}_{d,i}$. If $x \notin D$ then there are a final tower $\mathscr{T} = \langle \mathscr{S}, \mathscr{M}, c, \varphi \rangle$ with $\mathfrak{S}^i \preccurlyeq \mathfrak{S}$ and an element $a \in M$ of level i such that $x \in \varphi(a)$ and $\varphi(a) \cap W_i \neq \varnothing$. If $\varphi(a) \in \{A_1, \dots, A_k\}$ then $d \in \varphi(a)$ and $x \in R^{\mathfrak{S}^i}_{d,i}$. If $d \notin \varphi(a)$ then the frame with spectrum $\mathfrak{S}^i$ built on $\varphi(a)$ is not dense and there are only finitely many possibilities for x.

We have $\Psi(U \mid R^{\mathfrak{S}^i}_{d,i}) = \Psi(U \mid \left[R^{\mathfrak{S}^i}_{1,i} \cap W_i\right]_{\varepsilon_i} \cup D) = \Psi(U \cup \{0\} \mid \left[R^{\mathfrak{S}^i}_{1,i} \cap W_i\right]_{\varepsilon_i}) \vee^{\mathscr{L}^m} \Psi(U \mid D) = \Psi(U \cup \{0\} \mid R^{\mathfrak{S}^i}_{1,i} \cap W_i) \vee^{\mathscr{L}^m} \bot_{\mathscr{L}^m} = \Psi(U \mid R^{\mathfrak{S}^i}_{1,i} \cap W_i)$. Let X be the union of all Y such that Y equals either $R^{\mathfrak{S},0}_{1,m}$ for a spectrum $\mathfrak{S}$ of length $m < i$ or $R^{\mathfrak{S}}_{1,i}$ for a bad spectrum of length i. For all Y composing X, $\Psi(U \mid Y) = \bot_{\mathscr{L}^m}$, whence $\Psi(U \mid X) = \bot_{\mathscr{L}^m}$. We derive from $\mathbb{N} = X \cup R^{\mathfrak{S}^i}_{1,i}$ that

$$\begin{aligned} u = \Psi(U \mid W_i) &= \Psi\big(U \mid (X \cap W_i) \cup (R^{\mathfrak{S}^i}_{1,i} \cap W_i)\big) \\ &= \bot_{\mathscr{L}^m} \vee^{\mathscr{L}^m} \Psi(U \mid R^{\mathfrak{S}^i}_{1,i} \cap W_i) = \Psi(U \mid R^{\mathfrak{S}^i}_{d,i}) = \lambda(\nu(d)). \quad \square \end{aligned}$$

To complete the proof of the theorem, it remains to asserting that U is computable or coimmune. It is, indeed, the case since every infinite computably enumerable set contains elements from U. Indeed, if W_i is infinite then it contains either nonzero elements from D, or numbers from the bases of the final towers of height $> i$, or infinitely many numbers belonging to the bases of the final towers of height $\leq i$. In the first case, $W_i \cap U \neq \varnothing$ since $D \subseteq U \cup \{0\}$; in the second case, $W_i \cap U \neq \varnothing$ because of Stage I of enumerating U; and in the third case, $W_i \cap U \neq \varnothing$ since only finitely many final towers of height $\leq i$ do not contain in their bases the numbers not from U.

Theorem 2 is now proved.

§ 3. Applications to Enumeration Theory

The results of the previous sections allow us to solve a series of open problems and to strengthen some results in enumeration theory.

Let $n \in \mathbb{N}$, let $\mathscr{F}$ be a nonempty family of sets belonging to the class Σ^0_{n+1} of the arithmetical hierarchy, and let ν be a numbering of $\mathscr{F}$. Then ν is called Σ^0_{n+1}-*computable* if the set of the pairs $\{\langle x, y\rangle : x \in \nu y\}$ belongs to Σ^0_{n+1}. The Σ^0_{n+1}-computable numberings form an ideal in the semilattice of all numberings of $\mathscr{F}$. This ideal is called the *Rogers semilattice* of $\mathscr{F}$ and is denoted by $\mathscr{R}^0_{n+1}(\mathscr{F})$.

For $n = 0$, the notion of Σ^0_{n+1}-computable numbering coincides with the classical definition of computable numbering. Study of these numberings was carried out since the end of the sixties of the twentieth century. Many papers on this topic has appeared; among those, we should mention the monograph [9]. For $n > 0$, the research on Σ^0_{n+1}-computable numberings has started in the second half of the last decade of the twentieth century. Among all papers devoted to those numberings, we can mention [3–6, 13, 14] as well as a number of other papers which did not appear in the bibliography list of the current paper.

One of the main objectives of enumeration theory in studying the *arithmetical numberings* (that is, Σ^0_{n+1}-computable for some n) is the study of the algebraic properties of the Rogers semilattices; namely, the description of their isomorphism types in general as well as the description of the isomorphism types of the principal ideals and intervals of these semilattices (the so-called *local description*), the description of the elements possessing certain properties (minimal, irreducible, accessible elements, etc.). Some results have been already obtained in this direction; see the papers cited above.

To strengthen some of those results and to obtain new ones, we extend the operator Ψ to numberings (in the same way as it was done in [3, 4, 11, 12, 14] and in different terms in some other papers). Let ν be a numbering of an arbitrary set S and let X be a nonempty computably enumerable set. We let $\Psi(\nu \mid X)$ to be equal to the coset of the set of numberings of $\nu(X)$ containing the numbering $\nu \circ f$, where f is a computable total function with range X. It is not hard to check that this definition does not depend on the choice of f and for this notation the properties analogous to those of the Ψ-operator for sets hold. Namely:

(1) for an arbitrary numbering μ, $\mu \le \nu \Leftrightarrow$ there is a computably enumerable set X such that $[\mu] = \Psi(\nu \mid X)$;

(2) $\Psi(\nu \mid X_1 \cup X_2) = \Psi(\nu \mid X_1) \vee \Psi(\nu \mid X_2)$;

(3) $\Psi(\nu \mid X_1) \le \Psi(\nu \mid X_2)$ if and only if $\nu(X_1) \subseteq \nu(X_2)$ and there is a partial computable function θ such that $X_1 \subseteq \delta\theta$, $\theta(X_1) \subseteq X_2$, and $(\forall x \in X_1)(\nu x = \nu\theta(x))$.

Lemma 7. *Let $a, b \in \mathscr{R}^0_{n+1}(\mathscr{F})$ for some $n, \mathscr{F}$ and $a \le b$. Then the interval $[a, b]$ in $\mathscr{R}^0_{n+1}(\mathscr{F})$ is a bounded distributive semilattice having Σ^0_{n+3}-representation.*

Proof. Boundedness (that is, the existence of some bottom and top) is obvious. Distributivity follows from the distributivity of $\mathscr{R}^0_{n+1}(\mathscr{F})$.

Let ν be a numbering such that $b = [\nu]$. We have $a = \Psi(\nu \mid X)$ for some computably enumerable set X. Let δ be a numbering such that $\delta e = \Psi(\nu \mid X \cup W_e)$. From the properties of the Ψ-operator, it is clear that δ is a numbering of the interval $[a, b]$ and for every computable function u such that $W_{u(d,e)} = W_d \cup W_e$ we have $\delta u(d, e) = \delta d \vee \delta e$. To prove that δ is a Σ^0_{n+3}-representation, it remains to show that the relation "$\delta d \le \delta e$" belongs to the class Σ^0_{n+3} of the arithmetical hierarchy. Note that $\delta d \le \delta e \Leftrightarrow (\exists i \in \mathbb{N})(X \cup W_d \subseteq \delta\theta_i \,\&\, \theta_i(X \cup W_d) \subseteq W_e \,\&(\forall z \in W_d)(\nu z = \nu\theta_i(z))$. Since ν is Σ^0_{n+1}-computable, the relation "$\nu z_1 = \nu z_2$" belongs to Π^0_{n+2}, and the Tarski–Kuratowski algorithm applies to give the desired conclusion. □

Corollary 2. *If the semilattice $\mathscr{R}^0_{n+1}(\mathscr{F})$ has a bottom then each of its principal ideals is a bounded distributive lattice having Σ^0_{n+3}-representation.*

Corollary 3. *If $\mathscr{F}$ is a nonempty finite family of Σ^0_{n+1}-sets then any principal ideal of the semilattice $\mathscr{R}^0_{n+1}(\mathscr{F})$ is a bounded distributive lattice having Σ^0_{n+3}-representation.*

We say that a semilattice is *trivial* if it has only one element. It is clear that if a family $\mathscr{F}$ of Σ^0_{n+1}-sets is one-element then $\mathscr{R}^0_{n+1}(\mathscr{F})$ is trivial. For $n > 0$, this sufficient condition is also necessary: it is known that if $n > 0$ and a family $\mathscr{F}$ having more than one element has a Σ^0_{n+1}-computable numbering, then $\mathscr{R}^0_{n+1}(\mathscr{F})$ is infinite [4, 14]. For $n = 0$, the situation is more complicated. For finite $\mathscr{F} \subseteq \mathscr{E}$, the semilattice $\mathscr{R}^0_1(\mathscr{F})$ is trivial if and only if all elements of $\mathscr{F}$ are not subsets of each other. Moreover, the examples are known of an infinite $\mathscr{F}$ containing a pair of sets, one subset of the other, and for which $\mathscr{R}^0_1(\mathscr{F})$ is trivial.

The *local isomorphism type* of a semilattice is the collection of the isomorphism types of all principal ideals of this semilattice. We say that two semilattices are *locally isomorphic* if their local isomorphism types are the same. If semilattices are isomorphic then they are also locally isomorphic; the converse is obviously false.

Description of local isomorphism types of the semilattices $\mathscr{R}^0_1(\mathscr{F})$ for finite $\mathscr{F}$ is a well-known fact in enumeration theory. There are only two of these types. One of those consists of a single trivial semilattice and it appears for the trivial $\mathscr{R}^0_1(\mathscr{F})$; that is, when $\mathscr{F}$ does not contain a pair of sets such that one of them is a subset of the other. If $\mathscr{F}$ does contain a pair of this sort then the principal ideals of $\mathscr{R}^0_1(\mathscr{F})$ are exactly the Lachlan semilattices; that is, the semilattices that have 0-Lachlan representation; in other words, all bounded distributive semilattices having Σ^0_3-representation.

Below, we will generalize this result for an arbitrary n.

Lemma 8. *For every $n \in \mathbb{N}$, there is a bounded distributive semilattice having Σ^0_{n+1}-representation but no Σ^0_n-representation.*

Proof. In [15], it is claimed (with a reference to a paper of Feiner) that there is a Boolean algebra having Σ^0_1-representation but no computable representation. This algebra cannot have a computable representation either as a join semilattice, since in the same book it is proved that a Boolean algebra is computable as an algebra if and only if it is computable as a poset. Therefore, for $n = 0$, our lemma is proved. Relativizing the claim of the lemma to $\mathbf{0}^{(n)}$, we get its validity for an arbitrary n. $\square$

Lemma 9. *Let $\mathscr{R}^0_{n+2}(\mathscr{F})$ be a nontrivial semilattice and let $\mathscr{L}$ be an arbitrary bounded distributive semilattice having Σ^0_{n+3}-representation. Then there is an ideal in $\mathscr{R}^0_{n+2}(\mathscr{F})$ isomorphic to*

(1) *$\mathscr{L}$ if $\mathscr{F}$ is finite;*

(2) *$\mathscr{L}$ without a bottom if $\mathscr{F}$ is infinite.*

Proof. In [4], the following result is proved:

If $\mathscr{R}^0_{n+2}(\mathscr{F})$ is a nontrivial semilattice and a set U is $\mathbf{0}^{(n+1)}$-computable and immune, then there is an ideal in $\mathscr{R}^0_{n+2}(\mathscr{F})$ isomorphic to $\mathscr{L}^m_U$ in case $\mathscr{F}$ is finite and to $\mathscr{L}^m_U$ without a bottom in case $\mathscr{F}$ is infinite.

Let $\mathscr{R}^0_{n+2}(\mathscr{F})$ be a nontrivial semilattice and let $\mathscr{L}$ be a bounded distributive semilattice having Σ^0_{n+3}-representation. By Corollary 1, there is an immune or computable Π^0_{n+1}-set U such that $\mathscr{L} \cong \mathscr{L}^m_U$. If U is computable then $\mathscr{L}$ is trivial and the claim of the lemma is obvious. If U is immune then it suffices to notice that all sets from Π^0_{n+1} are $\mathbf{0}^{(n+1)}$-computable and to refer to the result cited above. $\square$

Corollary 4. *If $\mathscr{R}^0_{n+2}(\mathscr{F})$ is a nontrivial semilattice and $\mathscr{L}$ is an arbitrary bounded distributive semilattice having Σ^0_{n+3}-representation, then there is an ideal in $\mathscr{R}^0_{n+2}(\mathscr{F})$ isomorphic to $\mathscr{L}$.*

Proof. If $\mathscr{F}$ is finite then we refer to Lemma 9. If $\mathscr{F}$ is infinite then we can consider instead of $\mathscr{L}$ the semilattice $\mathscr{L}$ with a new bottom added and use Lemma 9 again. $\square$

Theorem 3. *Let $\mathscr{F}$ be a finite collection of Σ^0_{n+2}-sets. Then one of the following three cases occurs:*

(1) *$\mathscr{F}$ is one-element and $\mathscr{R}^0_{n+2}(\mathscr{F})$ is trivial;*

(2) *$\mathscr{F}$ is not one-element, any of its elements does not contain any other element as a subset, and all principal ideals in $\mathscr{R}^0_{n+2}(\mathscr{F})$ are exactly the bounded distributive semilattices having Σ^0_{n+3}-representation;*

(3) $\mathscr{F}$ *contains a pair of different sets, one of which is a subset of the other, and all principal ideals in* $\mathscr{R}^0_{n+2}(\mathscr{F})$ *are exactly the bounded distributive semilattices having* Σ^0_{n+4}*-representation.*

PROOF. If $\mathscr{F}$ is one-element then we are in the first case.

Let $\mathscr{F}$ contain more than one element and let $\mathscr{F}$ do not contain a pair of different sets, one of which is a subset of the other. By Lemma 9, any bounded distributive semilattice having Σ^0_{n+3}-representation is isomorphic to a principal ideal of $\mathscr{R}^0_{n+2}(\mathscr{F})$. We show that there is no other ideal. Let b be an arbitrary element of $\mathscr{R}^0_{n+2}(\mathscr{F})$ and let a be a bottom of this semilattice The ideal generated by b is the interval $[a, b]$. We proceed in the same way as in Lemma 7: we introduce ν, X, δ, etc. The only thing to be verified is that the relation "$\nu z_1 = \nu z_2$" belongs to the class Π^0_{n+2} of the arithmetical hierarchy.

Since all elements of $\mathscr{F}$ are pairwise noncomparable with respect to inclusion, there is a finite collection $\mathscr{S}$ of finite sets such that, for every $F \in \mathscr{F}$, there is $S \in \mathscr{S}$ with $S \subseteq F$; and, conversely, for every $S \in \mathscr{S}$ there is a unique $F \in \mathscr{F}$ with $S \subseteq F$. Hence,

$$\nu z_1 = \nu z_2 \Leftrightarrow (\forall S_1, S_2 \in \mathscr{S})(S_1 \subseteq \nu z_1 \,\&\, S_2 \subseteq \nu z_2 \to S_1 = S_2).$$

Applying the Tarski–Kuratowski algorithm to the right-hand side, we immediately get the desired conclusion.

We consider the last case. Let $F_1, F_2 \in \mathscr{F}$ be such that $F_1 \subset F_2$. By Corollary 3, each principal ideal in $\mathscr{R}^0_{n+2}(\mathscr{F})$ is a bounded distributive semilattice having Σ^0_{n+4}-representation. We have to show that the converse is also true; that is, each bounded distributive semilattice having Σ^0_{n+4}-representation is isomorphic to a principal ideal in U. Let $\mathscr{L}$ be a semilattice of this kind. By Corollary 1, there is $U \in \Sigma^0_{n+2}$ such that $\mathscr{L} \cong \mathscr{L}^m_U$. Let ν be a decidable numbering of $\mathscr{F}$ and let μ be a numbering of $\{F_1, F_2\}$ such that $\mu x = F_2 \Leftrightarrow x \in U$. It is easy to see that $\nu \oplus \mu$ is a Σ^0_{n+2}-computable numbering of $\mathscr{F}$ and the principal ideal in $\mathscr{R}^0_{n+2}(\mathscr{F})$ generated by $[\nu \oplus \mu]$ is isomorphic to $\mathscr{L}^m_U$. $\square$

Thus all local isomorphism types are described of the semilattices $\mathscr{R}^0_{n+1}(\mathscr{F})$ for finite $\mathscr{F}$. As Lemma 8 and Theorem 3 show, there are three of these types for $n > 0$.

Moreover, Theorem 3 implies that, for all $n \in \mathbb{N}$, there are finite collections $\mathscr{F}$ and $\mathscr{G}$ such that the semilattices $\mathscr{R}^0_{n+1}(\mathscr{F})$ and $\mathscr{R}^0_{n+2}(\mathscr{G})$ are nontrivial and locally isomorphic. If the "level difference" is more than 1; then, as the following theorem shows, the situation is totally different for all rather than only finite collections.

Theorem 4. *Let* $n, m \in \mathbb{N}$ *be such that* $n+2 \le m$. *If for some collections* $\mathscr{F}$ *and* $\mathscr{G}$ *their semilattices* $\mathscr{R}^0_{n+1}(\mathscr{F})$ *and* $\mathscr{R}^0_{m+1}(\mathscr{G})$ *are locally isomorphic then the latter are trivial.*

PROOF. By Lemma 8, there is a bounded distributive semilattice having Σ^0_{n+4}-representation and not having Σ^0_{n+3}-representation. Let $\mathscr{R}^0_{m+1}(\mathscr{G})$ be nontrivial. By Corollary 4, there is a principal ideal in $\mathscr{R}^0_{m+1}(\mathscr{G})$ which is isomorphic to $\mathscr{L}$. However, by Lemma 7 the semilattice $\mathscr{R}^0_{n+1}(\mathscr{F})$ cannot contain ideals isomorphic to $\mathscr{L}$, since any ideal with bottom is an interval. Thus, if the semilattices are locally isomorphic then $\mathscr{R}^0_{m+1}(\mathscr{G})$ is trivial; whence $\mathscr{R}^0_{n+1}(\mathscr{F})$ is also trivial. $\square$

Corollary 5. *If* $n + 2 \le m$ *and the semilattices* $\mathscr{R}^0_{n+1}(\mathscr{F})$ *and* $\mathscr{R}^0_{m+1}(\mathscr{G})$ *are both nontrivial then they are not isomorphic.*

It is proved in [5] that for all $\mathscr{F}$ and $m \ge n + 4$ there is $\mathscr{G}$ such that $\mathscr{R}^0_{n+1}(\mathscr{F})$ and $\mathscr{R}^0_{m+1}(\mathscr{G})$ are not isomorphic. Then it was proved in [6] that if $m \ge n + 3$ then $\mathscr{R}^0_{n+1}(\mathscr{F})$ and $\mathscr{R}^0_{m+1}(\mathscr{G})$ are either not isomorphic or trivial. Corollary 5 gives a stronger result: comparing with [6], the "level difference" is reduced from 3 to 2.

The question remains open whether this "level difference" can be further reduced from 2 to 1; that is, whether the nontrivial semilattices $\mathscr{R}^0_{n+1}(\mathscr{F})$ and $\mathscr{R}^0_{n+2}(\mathscr{G})$ should be nonisomorphic. It is only known that for finite collections and for $n = 0$, this question has a positive answer. As it was noticed after Theorem 3, even if all these semilattices are not isomorphic, this is impossible to justify on investigating only their local isomorphism types.

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