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# QUASIELLIPTIC OPERATORS AND SOBOLEV TYPE EQUATIONS

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**Abstract:** We consider a class of matrix quasielliptic operators on the $n$-dimensional space. For these operators, we establish the isomorphism properties in some special scales of weighted Sobolev spaces. Basing on these properties, we prove the unique solvability of the initial value problem for a class of Sobolev type equations.

**Keywords:** quasielliptic operator, weighted Sobolev space, isomorphism, Sobolev type equations

## § 1. Introduction

In this article we consider a class of matrix quasielliptic operators,

$$\mathscr{L}(D_x) = (l_{j,r}(D_x)) \tag{1.1}$$

on the whole space $\mathbb{R}^n$. For these operators, we establish the isomorphism properties in some special scales of weighted Sobolev spaces $W^l_{p,\sigma}$ and apply these properties to proving the solvability of the initial value problem for the class of Sobolev type equations

$$\begin{gathered}\mathscr{L}(D_x)D_t^m U + \sum_{k=0}^{m-1} \mathscr{L}_{m-k}(D_x) D_t^k U = F(t,x),\\ D_t^k U|_{t=0} = \varphi_{k+1}(x), \quad k = 0,\dots,m-1.\end{gathered} \tag{1.2}$$

The theorems on the isomorphism properties of differential operators have numerous applications in the theory of partial differential equations. However, in many cases their statements are far from obvious. For instance, the differential operator

$$\Delta - \varepsilon I : W_p^2(\mathbb{R}^n) \to L_p(\mathbb{R}^n), \quad 1 < p < \infty,$$

for $\varepsilon > 0$ establishes an isomorphism, but this is false for the Laplace operator

$$\Delta : W_p^2(\mathbb{R}^n) \to L_p(\mathbb{R}^n), \quad 1 < p < \infty.$$

The situation is similar for the polyharmonic operator

$$\Delta^m : W_p^{2m}(\mathbb{R}^n) \to L_p(\mathbb{R}^n), \quad 1 < p < \infty.$$

This operator is not an isomorphism for any $m$. In particular, using the results of [1, Chapter 12], we can show that the solvability of the equations

$$\Delta^m u = f(x), \quad x \in \mathbb{R}^n, \ n > 2m,$$

in the Sobolev space $W_p^{2m}(\mathbb{R}^n)$ with $p \le \frac{n}{n-2m}$ requires that the right-hand side $f(x)$ satisfy the orthogonality conditions of the form

$$\int\limits_{\mathbb{R}^n} f(x) x^\alpha dx = 0, \quad |\alpha| \le s.$$

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Note that the first isomorphism theorems for the Laplace operator $\Delta$ on $\mathbb{R}^n$ are proved in [2, 3], the theorems on the isomorphism properties of homogeneous elliptic operators appeared in the literature in the 1970s and 1980s (for instance, see [4–9]). The first isomorphism theorem for homogeneous quasielliptic operators on $\mathbb{R}^n$ was proved by the author in [10]; further research into the properties of these operators is presented in [11]. Some isomorphism theorems for a more general class of matrix quasielliptic operators on $\mathbb{R}^n$ are collected in [12–14]. This article continues the research in [10, 12–14].

## § 2. Statements of the Main Results

Let us impose some conditions on the class of matrix differential operators (1.1).

Condition 1. Assume that the matrix $\nu \times \nu$-differential operator $\mathscr{L}(D_x)$ has constant coefficients and its symbol $\mathscr{L}(i\xi) = (l_{j,r}(i\xi))$ satisfies the following conditions.

Condition 2. There exist vectors $\alpha = (\alpha_1, \dots, \alpha_n)$, $t = (t_1, \dots, t_\nu)$, $t_r > 0$, $t_r/\alpha_j \in \mathbb{N}$, such that for every $c > 0$ we have $l_{j,r}(c^\alpha i\xi) = c^{t_r} l_{j,r}(i\xi)$, $j, r = 1, \dots, \nu$.

Condition 3. $\det \mathscr{L}(i\xi) = 0$ for $\xi \in \mathbb{R}^n$ if and only if $\xi = 0$.

The matrix operators satisfying these conditions belong to the class of *quasielliptic operators* introduced by Volevich [15].

Let us give some examples of the operators satisfying Conditions 1–3.

Example 1. The *elliptic operators in the sense of Petrovskiĭ*: $\alpha_1 = \dots = \alpha_n = \frac{1}{m}$, $m \in \mathbb{N}$. Observe that the operators of this type with $t_1 = \dots = t_\nu = 1$ are called *homogeneous elliptic* operators. This class contains, for instance, the Navier operator of elasticity:

$$\mathscr{L}(D_x) = \begin{pmatrix} \mu\Delta + (\lambda+\mu)D^2_{x_1} & (\lambda+\mu)D^2_{x_1x_2} & (\lambda+\mu)D^2_{x_1x_3} \\ (\lambda+\mu)D^2_{x_1x_2} & \mu\Delta + (\lambda+\mu)D^2_{x_2} & (\lambda+\mu)D^2_{x_2x_3} \\ (\lambda+\mu)D^2_{x_1x_3} & (\lambda+\mu)D^2_{x_2x_3} & \mu\Delta + (\lambda+\mu)D^2_{x_3} \end{pmatrix} \tag{2.1}$$

where $\lambda$ and $\mu$ are the Lamé constants, and $\Delta$ is the Laplace operator with respect to $x$. Here $\det \mathscr{L}(i\xi) = -\mu^2(\lambda + 2\mu)|\xi|^6$ and $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{2}$.

Example 2. The *parabolic operators in the sense of Petrovskiĭ*:

$$\mathscr{L}(D_x) = \begin{pmatrix} D^{t_1}_{x_n} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & D^{t_\nu}_{x_n} \end{pmatrix} - \mathscr{L}^0(D_{x'}, D_{x_n}), \quad x = (x', x_n). \tag{2.2}$$

The homogeneity vector has the form $\alpha = \left(\frac{1}{2b}, \dots, \frac{1}{2b}, 1\right)$, $b \in \mathbb{N}$.

Example 3. The *parabolic operators in the sense of Èĭdel′man* of the form (2.2). In this case the homogeneity vector $\alpha$ is equal to $\left(\frac{1}{2b_1}, \dots, \frac{1}{2b_{n-1}}, 1\right)$, $b_j \in \mathbb{N}$, and the elements of the matrix $\mathscr{L}^0(i\xi', i\xi_n) = (l^0_{j,r}(i\xi))$ satisfy

$$l^0_{j,r}(c^\alpha i\xi) = c^{t_r} l^0_{j,r}(i\xi), \quad c > 0,\ j, r = 1, \dots, \nu,$$

while the imaginary parts of the roots of the equations $\det \mathscr{L}(is, i\lambda) = 0$, $s \in \mathbb{R}^{n-1}$, satisfy

$$\operatorname{Im} \lambda_j(s) \ge \delta\left(s_1^{2b_1} + \dots + s_{n-1}^{2b_{n-1}}\right), \quad \delta > 0.$$

Example 4. The *homogeneous quasielliptic operators*: $t_1 = \dots = t_\nu = 1$. This class contains, for instance, the operator *of parabolic type with "opposite time directions"* [16]:

$$\mathscr{L}(D_x) = \begin{pmatrix} D_{x_n} - \Delta' & 0 \\ 0 & D_{x_n} + \Delta' \end{pmatrix},$$

where $\Delta'$ is the Laplace operator with respect to $x'$. In this case $\alpha = \left(\frac{1}{2}, \dots, \frac{1}{2}, 1\right)$ and $t_1 = t_2 = 1$.

It is obvious that the class of homogeneous quasielliptic operators contains the scalar quasielliptic operators

$$\mathscr{L}(D_x) = \sum_{\beta\alpha=1} a_\beta D_x^\beta, \quad \mathscr{L}(i\xi) \neq 0, \quad \xi \in \mathbb{R}^n \backslash \{0\}.$$

By analogy with [10] we will establish the isomorphism properties of (1.1) using the special weighted Sobolev spaces from [17]:

$$W_{p,\sigma}^l(\mathbb{R}^n), \quad l = (k/\alpha_1, \dots, k/\alpha_n), \quad k/\alpha_i \in \mathbb{N},\ 1 < p < \infty,\ \sigma > 0.$$

Definition. We will say that a locally summable function $u(x)$ *belongs to the weighted Sobolev space* $W_{p,\sigma}^l(\mathbb{R}^n)$ if $u(x)$ has the generalized derivatives $D_x^\nu u(x)$ in $\mathbb{R}^n$ for $\nu\alpha \leq k$, and

$$\big\|(1 + \langle x\rangle)^{-\sigma(k-\nu\alpha)} D_x^\nu u(x), L_p(\mathbb{R}^n)\big\| < \infty, \quad \langle x\rangle^2 = \sum_{i=1}^n x_i^{2/\alpha_i}.$$

The norm on $W_{p,\sigma}^l(\mathbb{R}^n)$ is defined as

$$\big\|u(x), W_{p,\sigma}^l(\mathbb{R}^n)\big\| = \sum_{0\leq\nu\alpha\leq k} \big\|(1 + \langle x\rangle)^{-\sigma(k-\nu\alpha)} D_x^\nu u(x), L_p(\mathbb{R}^n)\big\|. \tag{2.3}$$

We will consider the general case where the components $k/\alpha_i$ of the smoothness vector $l$ need not be the same. Observe that in the isotropic case, when $k/\alpha_1 = \dots = k/\alpha_n = \bar{l}$, the norm (2.3) is equivalent to the norm

$$\sum_{0\leq|\beta|\leq\bar{l}} \big\|(1 + |x|)^{-\sigma(\bar{l}-|\beta|)} D_x^\beta u(x), L_p(\mathbb{R}^n)\big\|. \tag{2.4}$$

Observe that $C_0^\infty(\mathbb{R}^n)$ is dense in $W_{p,\sigma}^l(\mathbb{R}^n)$ for $\sigma \leq 1$ (see [17]).

Let us give two well-known examples of weighted Sobolev spaces, which in the isotropic case $k/\alpha_i = \bar{l}$ coincide with those defined above with $\sigma = 1$.

Example 5. The *Kudryavtsev space* $W_{p,\square}^{\bar{l}}(\mathbb{R}^n)$, where $\square = \{x \in \mathbb{R}^n : |x_j| < 1,\ j = 1, \dots, n\}$, whose norm is defined as

$$\big\|u(x), W_{p,\square}^{\bar{l}}(\mathbb{R}^n)\big\| = \int_\square |u(x)|\, dx + \sum_{|\beta|=\bar{l}} \big\|D_x^\beta u(x), L_p(\mathbb{R}^n)\big\|.$$

It is shown in [18] that in the case $p > n$ this norm is equivalent to (2.4) with $\sigma = 1$. Consequently, in the case that $k/\alpha_1 = \dots = k/\alpha_n = \bar{l}$, $p > n$, and $\sigma = 1$, the spaces $W_{p,\square}^{\bar{l}}(\mathbb{R}^n)$ and $W_{p,\sigma}^l(\mathbb{R}^n)$ coincide.

Example 6. The *Nirenberg–Walker–Cantor space* $M_{\bar{l},m}^p(\mathbb{R}^n)$, $1 < p < \infty$, whose norm is defined [19, 20] as

$$\big\|u(x), M_{\bar{l},m}^p(\mathbb{R}^n)\big\| = \sum_{0\leq|\beta|\leq\bar{l}} \big\|(1 + |x|)^{m+|\beta|} D_x^\beta u(x), L_p(\mathbb{R}^n)\big\|.$$

Consequently, in the case that $k/\alpha_1 = \dots = k/\alpha_n = \bar{l}$, $m = -\bar{l}$, $\sigma = 1$, $M_{\bar{l},m}^p(\mathbb{R}^n)$ and $W_{p,\sigma}^l(\mathbb{R}^n)$ coincide.

We say that $U(x) = (U^1(x), \dots, U^\nu(x))^T$ *belongs to*

$$\mathbf{W}_{p,\sigma}^{\mathbf{l}}(\mathbb{R}^n) = \prod_{r=1}^\nu W_{p,\sigma}^{l^r}(\mathbb{R}^n), \quad l^r = (t_r/\alpha_1, \dots, t_r/\alpha_n),\ 1 < p < \infty,\ \sigma \geq 0,$$

if every component $U^r(x)$ belongs to $W_{p,\sigma}^{l^r}(\mathbb{R}^n)$ and write

$$\big\|U(x), \mathbf{W}_{p,\sigma}^{\mathbf{l}}(\mathbb{R}^n)\big\| = \sum_{r=1}^\nu \big\|U^r(x), W_{p,\sigma}^{l^r}(\mathbb{R}^n)\big\|.$$

We will also write

$$\mathbf{L}_p(\mathbb{R}^n) = \prod_{r=1}^{\nu} L_p(\mathbb{R}^n)$$

and put

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad t_{\max} = \max\{t_1, \dots, t_\nu\}.$$

Let us state the isomorphism theorem.

**Theorem 1.** *Suppose that* $|\alpha|/p > t_{\max}$. *Then*

$$\mathscr{L}(D_x) : \mathbf{W}^{\mathbf{l}}_{p,\sigma}(\mathbb{R}^n) \to \mathbf{L}_p(\mathbb{R}^n), \quad 1 < p < \infty, \ \sigma = 1,$$

*is an isomorphism.*

We give an example of application of Theorem 1.

Consider the Cauchy problem (1.2) for a system of equations that are not solved with respect to the highest time derivative:

$$\mathscr{L}(D_x) D_t^m U + \sum_{k=0}^{m-1} \mathscr{L}_{m-k}(D_x) D_t^k U = F(t,x). \tag{2.5}$$

Nowadays the equations of the form (2.5) are called *Sobolev type equations* because it was precisely Sobolev's articles (see [21, Vol. I]) that pioneered the study of equations of this type. Note also that in the literature the *Sobolev equation*

$$\Delta D_t^2 u + D_{x_n}^2 u = f(t,x)$$

is among the most popular equations that are not solved with respect to the highest derivative.

We will assume that the matrix $\nu \times \nu$-differential operators $\mathscr{L}_k(D_x)$, $k = 1, \dots, m$, have constant coefficients and satisfy Condition 2; i.e., their symbols $\mathscr{L}_k(i\xi) = \big(l^k_{j,r}(i\xi)\big)$, as well as the symbol of the operator $\mathscr{L}(D_x)$, are quasihomogeneous.

Let us state the theorem on the unconditional solvability of the Cauchy problem (1.2), assuming for simplicity that the initial conditions are zero.

**Theorem 2.** *Suppose that* $|\alpha|/p > t_{\max}$, $\varphi_k(x) \equiv 0$, $k = 1, \dots, m$. *Then for every* $F(t,x) \in C([0,T]; \mathbf{L}_p(\mathbb{R}^n))$ *the problem* (1.2) *has a unique solution* $U(t,x) \in C^m\big([0,T]; \mathbf{W}^{\mathbf{l}}_{p,1}(\mathbb{R}^n)\big)$, *and*

$$\big\| U(t,x), C^m\big([0,T]; \mathbf{W}^{\mathbf{l}}_{p,1}(\mathbb{R}^n)\big) \big\| \le c(T) \| F(t,x), C([0,T]; \mathbf{L}_p(\mathbb{R}^n)) \| \tag{2.6}$$

*with some constant* $c(T)$ *independent of* $F(t,x)$.

## § 3. Proofs of the Main Results

In proving Theorem 1 we will follow the scheme of [10, 12]. Thus, we briefly sketch the proof and dwell on the substantial differences.

Conditions 1 and 2 imply that $\mathscr{L}(D_x)$ maps $\mathbf{W}^{\mathbf{l}}_{p,\sigma}(\mathbb{R}^n)$ into $\mathbf{L}_p(\mathbb{R}^n)$ and is a bounded operator.

Indeed, given $U(x) \in \mathbf{W}^{\mathbf{l}}_{p,\sigma}(\mathbb{R}^n)$, we can write the $j$th component of $\mathscr{L}(D_x)U(x)$ as

$$(\mathscr{L}(D_x)U(x))_j = \sum_{r=1}^{\nu} l_{j,r}(D_x) U^r(x), \quad l_{j,r}(D_x) = \sum_{\beta\alpha = t_r} a_{j,r,\beta} D_x^\beta.$$

Hence, $l_{j,r}(D_x)U^r(x) \in L_p(\mathbb{R}^n)$, and so $\mathscr{L}(D_x)U(x) \in \mathbf{L}_p(\mathbb{R}^n)$; since the coefficients $a_{j,r,\beta}$ are constant, we also have

$$\| \mathscr{L}(D_x)U(x), \mathbf{L}_p(\mathbb{R}^n) \| \le c \big\| U(x), \mathbf{W}^{\mathbf{l}}_{p,\sigma}(\mathbb{R}^n) \big\|,$$

where $c > 0$ is some constant independent of $U(x)$.

Consequently, in order to prove Theorem 1 we must establish that for $|\alpha|/p > t_{\max}$ the system of differential equations

$$\mathscr{L}(D_x)U(x) = F(x), \quad x \in \mathbb{R}^n, \tag{3.1}$$

has a unique solution $U(x) \in \mathbf{W}^{\mathbf{l}}_{p,1}(\mathbb{R}^n)$ for every right-hand side $F(x) \in \mathbf{L}_p(\mathbb{R}^n)$, and we have the estimate

$$\big\|U(x), \mathbf{W}^{\mathbf{l}}_{p,1}(\mathbb{R}^n)\big\| \le C\|F(x), \mathbf{L}_p(\mathbb{R}^n)\| \tag{3.2}$$

with some constant $C > 0$ independent of $F(x)$.

Let us mention the crucial steps in proving the solvability of (3.1) and deriving the estimate (3.2).

In order to prove the solvability of (3.1) we use the method for constructing approximate solutions which is detailed in [22]. This method bases on the integral representation for summable functions due to Uspenskiĭ [23] (also see [22, Chapter 1]):

$$\varphi(x) = \lim_{h\to 0}(2\pi)^{-n}\int\limits_h^{h^{-1}} v^{-|\hat{\alpha}|-1}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\exp\left(i\frac{x-y}{v^{\hat{\alpha}}}\xi\right)G(\xi)\varphi(y)\,d\xi dy dv, \tag{3.3}$$

where

$$G(\xi) = 2m\langle\xi\rangle^{2m}\exp(-\langle\xi\rangle^{2m}), \quad \langle\xi\rangle^2 = \sum_{i=1}^n \xi_i^{2/\hat{\alpha}_i}, \quad m, 1/\hat{\alpha}_i \in \mathbb{N}.$$

Initially, we assume that the components $F^j(x)$ of $F(x) \in \mathbf{L}_p(\mathbb{R}^n)$ are compactly supported. Consider the family of integral operators $P_{k,h}$, $k = 1,\dots,\nu$, $0 < h < 1$, defined as

$$P_{k,h}F(x) = (2\pi)^{-n}\int\limits_h^{h^{-1}} v^{-|\alpha|/t_k}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\exp\left(i\frac{x-y}{v^{\alpha/t_k}}\xi\right)G_k(\xi)\left(\sum_{r=1}^{\nu} l^{k,r}(\xi)F^r(y)\right)d\xi dy dv, \tag{3.4}$$

where $l^{k,r}(\xi)$ are the elements of the inverse matrix $(\mathscr{L}(i\xi))^{-1}$, and

$$G_k(\xi) = 2m\langle\xi\rangle_k^{2m}\exp\big(-\langle\xi\rangle_k^{2m}\big), \quad \langle\xi\rangle_k^2 = \sum_{i=1}^n \xi_i^{2t_k/\alpha_i}, \quad m \in \mathbb{N}. \tag{3.5}$$

It follows from (3.4) and Conditions 1–3 that the functions $P_{k,h}F(x)$ are infinitely differentiable and

$$\sum_{k=1}^{\nu} l_{j,k}(D_x)P_{k,h}F(x) = F_h^j(x),$$

where

$$F_h^j(x) = (2\pi)^{-n}\int\limits_h^{h^{-1}} v^{-|\alpha|/t_j-1}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\exp\left(i\frac{x-y}{v^{\alpha/t_j}}\xi\right)G_j(\xi)F^j(y)\,d\xi dy dv.$$

Using (3.3) with $\hat{\alpha}_i = \alpha_i/t_j$, we have

$$\big\|F_h^j(x) - F^j(x), L_p(\mathbb{R}^n)\big\| \to 0 \quad \text{as } h \to 0.$$

Consequently, we can consider $U_h(x)$ with the components $U_h^k(x) = P_{k,h}F(x)$, $k = 1,\dots,\nu$, as an approximate solution to (3.1). Henceforth we will write $U_h(x)$ as

$$U_h(x) = P_hF(x) = (P_{1,h}F(x),\dots,P_{\nu,h}F(x))^T. \tag{3.6}$$

It is obvious that we can find a positive integer $m_1$ such that for $m \geq m_1$ the functions $P_{k,h}F(x)$ in (3.5) will be summable to every power $p \geq 1$ (see [12]). Henceforth we will assume in (3.5) that $m \geq m_1$.

In the next section we will prove that in the case $|\alpha|/p > t_{\max}$ the estimate

$$\left\|U_h(x), \mathbf{W}^{\mathbf{l}}_{p,1}(\mathbb{R}^n)\right\| \leq C\|F(x), \mathbf{L}_p(\mathbb{R}^n)\| \tag{3.7}$$

holds with some constant $C > 0$ independent of $F(x)$ and $h$ and also establish that

$$\left\|U_{h_1}(x) - U_{h_2}(x), \mathbf{W}^{\mathbf{l}}_{p,1}(\mathbb{R}^n)\right\| \to 0 \quad \text{as } h_1, h_2 \to 0. \tag{3.8}$$

Let us show how to use this to prove Theorem 1.

Since the space $\mathbf{W}^{\mathbf{l}}_{p,1}(\mathbb{R}^n)$ is complete, it follows from (3.7) and (3.8) that there exists a continuous linear operator

$$P : \mathbf{L}_p(\mathbb{R}^n) \to \mathbf{W}^{\mathbf{l}}_{p,\sigma}(\mathbb{R}^n), \quad 1 < p < \infty,\ \sigma = 1,$$

defined on the compactly supported vector functions $F(x)$ as $PF(x) = \lim_{h\to 0} P_h F(x)$, and $U(x) = PF(x) \in \mathbf{W}^{\mathbf{l}}_{p,1}(\mathbb{R}^n)$ will be a solution to (3.1). Since the set of compactly supported vector functions is dense in $\mathbf{L}_p(\mathbb{R}^n)$, by a well-known theorem $P$ has a unique continuous extension to the whole space $\mathbf{L}_p(\mathbb{R}^n)$ with the same norm. We use the same notation $P$ for the extension.

The inequality (3.7) implies that the linear operators

$$P_h : \mathbf{L}_p(\mathbb{R}^n) \to \mathbf{W}^{\mathbf{l}}_{p,\sigma}(\mathbb{R}^n), \quad 1 < p < \infty,\ \sigma = 1,$$

are continuous and the norms $\{\|P_h\|\}$ are uniformly bounded: $\|P_h\| \leq C$. Consequently, by the Banach–Steinhaus theorem we have

$$\left\|P_h F(x) - PF(x), \mathbf{W}^{\mathbf{l}}_{p,1}(\mathbb{R}^n)\right\| \to 0 \quad \text{as } h \to 0$$

for all $F(x) \in \mathbf{L}_p(\mathbb{R}^n)$.

The above arguments imply the existence of a solution $U(x) \in \mathbf{W}^{\mathbf{l}}_{p,1}(\mathbb{R}^n)$ to (3.1) for every right-hand side $F(x) \in \mathbf{L}_p(\mathbb{R}^n)$; and, moreover, we have (3.2).

The uniqueness of a solution to (3.1) in the space under consideration is proved by analogy with [10].

Thus, the quasielliptic operator

$$\mathscr{L}(D_x) : \mathbf{W}^{\mathbf{l}}_{p,1}(\mathbb{R}^n) \to \mathbf{L}_p(\mathbb{R}^n), \quad 1 < p < \infty,$$

is continuous; if $|\alpha|/p > t_{\max}$ then the domain of $\mathscr{L}(D_x)$ coincides with the whole space $\mathbf{L}_p(\mathbb{R}^n)$; the kernel $\operatorname{Ker}\mathscr{L}(D_x)$ is zero; and the bounded operator $P : \mathbf{L}_p(\mathbb{R}^n) \to \mathbf{W}^{\mathbf{l}}_{p,1}(\mathbb{R}^n)$ is the inverse of $\mathscr{L}(D_x)$.

Consequently, in order to complete the proof of Theorem 1 we must obtain (3.7) and establish (3.8). This question is discussed in the next section.

The claim of Theorem 2 is a simple corollary to that isomorphism theorem for (1.1).

Indeed, by Theorem 1 and the condition on the symbols of $\mathscr{L}_k(D_x)$ the operators

$$(\mathscr{L}(D_x))^{-1}\mathscr{L}_{m-k}(D_x) : \mathbf{W}^{\mathbf{l}}_{p,1}(\mathbb{R}^n) \to \mathbf{W}^{\mathbf{l}}_{p,1}(\mathbb{R}^n), \quad 1 < p < \infty,\ k = 0, \dots, m-1,$$

for $|\alpha|/p > t_{\max}$ are continuous. Consequently, for $\varphi_k(x) \equiv 0$, $k = 1, \dots, m$, (1.2) is equivalent to the Cauchy problem for the differential equation

$$D_t^m U + \sum_{k=0}^{m-1} (\mathscr{L}(D_x))^{-1}\mathscr{L}_{m-k}(D_x) D_t^k U = (\mathscr{L}(D_x))^{-1} F(t,x),$$

$$D_t^k U|_{t=0} = 0, \quad k = 0, \dots, m-1,$$

with bounded operator coefficients.

Since the right-hand side $(\mathscr{L}(D_x))^{-1}F(t,x)$ belongs to $C\big([0,T]; \mathbf{W}^{\mathbf{l}}_{p,1}(\mathbb{R}^n)\big)$, the general theory of Cauchy problems implies the existence and uniqueness of a solution $U(t,x) \in C^m\big([0,T]; \mathbf{W}^{\mathbf{l}}_{p,1}(\mathbb{R}^n)\big)$ to the problem, as well as (2.6).

## § 4. The Estimate for Approximate Solutions to (3.1)

We split the proof of (3.7) and (3.8) into a series of lemmas.

Firstly, we give an estimate of the highest derivatives of the components of the vector function (3.6).

**Lemma 4.1.** *Let* $\beta = (\beta_1, \dots, \beta_n)$ *and* $\beta\alpha = t_k$. *Then*

$$\left\| D_x^\beta U_h^k(x), L_p(\mathbb{R}^n) \right\| \leq C_\beta \| F(x), \mathbf{L}_p(\mathbb{R}^n) \| \tag{4.1}$$

*with some constant* $C_\beta > 0$ *independent of* $F(x)$ *and* $h$*; moreover,*

$$\left\| D_x^\beta U_{h_1}^k(x) - D_x^\beta U_{h_2}^k(x), L_p(\mathbb{R}^n) \right\| \to 0 \quad \text{as } h_1, h_2 \to 0. \tag{4.2}$$

Proof. It is obvious that it suffices to obtain (4.1) for $F(x) \in \mathbf{C}_0^\infty(\mathbb{R}^n)$. By (3.4)

$$D_x^\beta U_h^k(x) = (2\pi)^{-n} \int\limits_h^{h^{-1}} v^{-|\alpha|/t_k - 1} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \exp\left( i \frac{x-y}{v^{\alpha/t_k}} \xi \right) G_k(\xi)(i\xi)^\beta \left( \sum_{r=1}^{\nu} l^{k,r}(\xi) F^r(y) \right) d\xi dy dv.$$

Using the properties of the Fourier transform, we can rewrite this formula as

$$D_x^\beta U_h^k(x) = (2\pi)^{-n} \int\limits_h^{h^{-1}} v^{-1} \int\limits_{\mathbb{R}^n} \left( \int\limits_{\mathbb{R}^n} \exp(i(x-y)s) G_k(s v^{\alpha/t_k})\, ds \right) F_{k,\beta}(y)\, dy dv, \tag{4.3}$$

where

$$F_{k,\beta}(y) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} \exp(iy\xi)(i\xi)^\beta \left( \sum_{r=1}^{\nu} l^{k,r}(\xi) \widehat{F}^r(\xi) \right) d\xi.$$

The multi-index $\beta$ satisfies $\beta\alpha = t_k$; therefore, by Conditions 2 and 3 the function

$$\mu_{k,\beta}(\xi) = (i\xi)^\beta \sum_{r=1}^{\nu} l^{k,r}(\xi)$$

satisfies the hypotheses of the multiplier theorem [24]. Then the theorem yields the inequality

$$\| F_{k,\beta}(x), L_p(\mathbb{R}^n) \| \leq c_\beta \| F(x), \mathbf{L}_p(\mathbb{R}^n) \|$$

with some constant $c_\beta$ independent of $F(x)$. Consequently, using the properties of the integral representation (3.3) (see [22, Chapter 1]), we derive (4.1) from (4.3).

The proof of (4.2) goes similarly. The proof of the lemma is complete.

In order to obtain the weighted estimates for the derivatives

$$D_x^\beta U_h^k(x), \quad 0 \leq \beta\alpha < t_k, \tag{4.4}$$

we will need some estimates for the integrals

$$\mathscr{K}_{\beta,h}^{k,r}(x) = (2\pi)^{-n} \int\limits_h^{h^{-1}} v^{-|\alpha|/t_k - \beta\alpha/t_k} \int\limits_{\mathbb{R}^n} \exp\left( i \frac{x\xi}{v^{\alpha/t_k}} \right) G_k(\xi)(i\xi)^\beta l^{k,r}(\xi) d\xi dv, \quad 0 < h < 1. \tag{4.5}$$

**Lemma 4.2.** *Suppose that* $|\alpha| + \beta\alpha > t_k$. *Then there exists* $m_2$ *such that if* $m \geq m_2$ *in the definition* (3.5) *of* $G_k(\xi)$ *then the estimate*

$$\langle x \rangle^{|\alpha| + \beta\alpha - t_k} \left| \mathscr{K}_{\beta,h}^{k,r}(x) \right| \leq c, \quad x \in \mathbb{R}^n, \tag{4.6}$$

*holds with some constant* $c > 0$ *independent of* $h$.

A proof of (4.6) is easy to give by analogy with that of Lemma 3.3 in [12].

Henceforth we will assume in (3.5) that $m \geq \max\{m_1, m_2\}$. Let us estimate the derivatives (4.4).

**Lemma 4.3.** *Let $\beta = (\beta_1, \dots, \beta_n)$, $\beta\alpha < t_k$, and $|\alpha|/p > t_k$. Then the inequality*

$$\big\|\langle x\rangle^{-(t_k-\beta\alpha)} D_x^\beta U_h^k(x), L_p(\mathbb{R}^n)\big\| \le c\|F(x), \mathbf{L}_p(\mathbb{R}^n)\| \tag{4.7}$$

*holds with some constant $c > 0$ independent of $F(x)$ and $h$; moreover,*

$$\big\|\langle x\rangle^{-(t_k-\beta\alpha)}\big(D_x^\beta U_{h_1}^k(x) - D_x^\beta U_{h_2}^k(x)\big), L_p(\mathbb{R}^n)\big\| \to 0 \quad \text{as } h_1, h_2 \to 0. \tag{4.8}$$

Proof. It is obvious that it suffices to obtain (4.7) for $F(x) \in \mathbf{C}_0^\infty(\mathbb{R}^n)$. Taking (3.4), (3.5), and (4.5) into account, we write the derivative $D_x^\beta U_h^k(x)$ as

$$D_x^\beta U_h^k(x) = \sum_{r=1}^{\nu} \int_{\mathbb{R}^n} \mathscr{K}_{\beta,h}^{k,r}(x-y) F^r(y)\, dy.$$

Using Lemma 4.2, we obtain the estimate

$$\big\|\langle x\rangle^{-(t_k-\beta\alpha)} D_x^\beta U_h^k(x), L_p(\mathbb{R}^n)\big\|$$
$$\le c_1 \sum_{r=1}^{\nu} \bigg\|\langle x\rangle^{-(t_k-\beta\alpha)} \int_{\mathbb{R}^n} \langle x-y\rangle^{-(|\alpha|+\beta\alpha-t_k)} |F^r(y)|\, dy, L_p(\mathbb{R}^n)\bigg\|.$$

By hypotheses, $|\alpha|/p > t_k$ and $t_k - \beta\alpha > 0$; therefore, $|\alpha| + \beta\alpha - t_k > 0$. Hence, we will have

$$\big\|\langle x\rangle^{-(t_k-\beta\alpha)} D_x^\beta U_h^k(x),\ L_p(\mathbb{R}^n)\big\|$$
$$\le c_2 \sum_{r=1}^{\nu} \bigg\| \int_{\mathbb{R}^n} \prod_{i=1}^{n} |x_i|^{(\beta\alpha-t_k)/|\alpha|} |x_i - y_i|^{(t_k-\beta\alpha)/|\alpha|-1} |F^r(y)|\, dy, L_p(\mathbb{R}^n)\bigg\|.$$

Consequently, applying the Hardy–Littlewood inequality [25], we obtain (4.7).

A proof of (4.8) goes by analogy with that of (4.7).

The proof of the lemma is complete.

Lemmas 4.1 and 4.3 immediately yield (3.7) and (3.8) for the vector function (3.6) which is an approximate solution to (3.1). This completes the proof of Theorem 1.

## § 5. Examples

Let us give as corollaries to Theorem 1 the statements on the isomorphism properties of the differential operators

$$\mathscr{L}(D_x) : \mathbf{W}_{p,1}^{\mathbf{l}}(\mathbb{R}^n) = \prod_{r=1}^{\nu} W_{p,1}^{l^r}(\mathbb{R}^n) \to \mathbf{L}_p(\mathbb{R}^n), \quad l^r = (t_r/\alpha_1, \dots, t_r/\alpha_n),\ p > 1, \tag{5.1}$$

considered in Examples 1–4.

**1.** Let $\mathscr{L}(D_x)$ be an elliptic matrix differential operator in the sense of Petrovskiĭ, and as in Example 1, let the homogeneity vector have the form $\alpha = \big(\frac{1}{m}, \dots, \frac{1}{m}\big)$. Then $l^r = (t_r m, \dots, t_r m)$, $|\alpha| = \frac{n}{m}$, and according to Theorem 1 the operator (5.1) establishes an isomorphism provided that $n > mpt_{\max}$. For the homogeneous elliptic operators ($t_1 = \dots = t_\nu = 1$) this holds for $n > mp$. In particular, the Navier operator (2.1) establishes an isomorphism

$$\mathscr{L}(D_x) : \mathbf{W}_{p,1}^{\mathbf{2}}(\mathbb{R}^3) \to \mathbf{L}_p(\mathbb{R}^3) \quad \text{for } 1 < p < 3/2.$$

**2.** Let $\mathscr{L}(D_x)$ be a parabolic operator in the sense of Petrovskiĭ, and as in Example 2, let the homogeneity vector have the form $\alpha = \big(\frac{1}{2b}, \dots, \frac{1}{2b}, 1\big)$. Then

$$l^r = t_r(2b, \dots, 2b, 1), \quad |\alpha| = \frac{n-1}{2b} + 1.$$

By Theorem 1, (5.1) establishes an isomorphism provided that $n + 2b - 1 > 2bpt_{\max}$.

**3.** Let $\mathscr{L}(D_x)$ be a parabolic operator in the sense of Èĭdel′man, and as in Example 3, let the homogeneity vector have the form $\alpha = \left(\frac{1}{2b_1}, \dots, \frac{1}{2b_{n-1}}, 1\right)$. Then

$$l^r = t_r(2b_1, \dots, 2b_{n-1}, 1), \quad |\alpha| = 1 + \sum_{i=1}^{n-1} \frac{1}{2b_i}.$$

According to Theorem 1, (5.1) is an isomorphism provided that

$$1 + \sum_{i=1}^{n-1} \frac{1}{2b_i} > pt_{\max}.$$

**4.** Let $\mathscr{L}(D_x)$ be a homogeneous quasielliptic operator ($t_1 = \dots = t_\nu = 1$). Then by Theorem 1 (5.1) is an isomorphism provided that $|\alpha| > p$. In particular, the operator of parabolic type with "opposite time directions"

$$\mathscr{L}(D_x) = \begin{pmatrix} D_{x_n} - \Delta' & 0 \\ 0 & D_{x_n} + \Delta' \end{pmatrix} : W_{p,1}^{\mathbf{2},1}(\mathbb{R}^n) \times W_{p,1}^{\mathbf{2},1}(\mathbb{R}^n) \to L_p(\mathbb{R}^n) \times L_p(\mathbb{R}^n)$$

is an isomorphism for $1 < p < \frac{n+1}{2}$.

## § 6. Generalizations

Following [14] and the scheme of this article, we can study the isomorphism properties of quasielliptic operators in a larger class. More precisely, we can replace Condition 2 on the symbol of the operator $\mathscr{L}(D_x)$ with the following condition:

Condition $2^0$. There exist vectors

$$\alpha = (\alpha_1, \dots, \alpha_n), \quad (t_1, \dots, t_\nu), \quad t_k > 0, \quad (s_1, \dots, s_\nu), \quad s_k \le 0, \quad \max\{s_k\} = 0,$$

such that for every $c > 0$ we have

$$\mathscr{L}(c^\alpha i\xi) = \begin{pmatrix} c^{s_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & c^{s_\nu} \end{pmatrix} \mathscr{L}(i\xi) \begin{pmatrix} c^{t_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & c^{t_\nu} \end{pmatrix}, \quad \xi \in \mathbb{R}^n.$$

The class of differential operators satisfying Conditions 1, $2^0$ and 3 includes, in particular, the operators elliptic in the sense of Douglis–Nirenberg.

Using the isomorphism theorems for this class of operators, we can prove a theorem on the unconditional solvability of the initial value problem for a larger class of Sobolev type equations. For instance, the article [14] contains a proof, using the isomorphism theorem [12] for some certain operators elliptic in the sense of Douglis–Nirenberg, of the well-posedness theorem for the initial value problem for the class of first order systems

$$A_0 D_t u + \sum_{j=1}^{n} A_j D_{x_j} u + Bu = F(t, x) \tag{6.1}$$

with degenerate matrix $A_0$. This class contains, in particular, the *Sobolev system* [26]

$$v_t + [\vec{\omega}, v] + \nabla p = f(t, x), \quad \operatorname{div} v = g(t, x), \ x \in \mathbb{R}^3. \tag{6.2}$$

Recall that in the case that $A_0 = A_0^* > 0$, $A_k = A_k^*$, $k = 1, \dots, n$, the systems of the form (6.1) are called *hyperbolic in the sense of Friedrichs* [27], but (6.2) does not belong to this class since $\det A_0 = 0$.

Remark. It is interesting to note that the two fundamental articles [26] and [27] on the theory of systems of the form (6.1) were published in the same year.

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