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DETERMINING THE PARAMETERS OF A STRATIFIED PIECEWISE CONSTANT MEDIUM FOR THE UNKNOWN SHAPE OF AN IMPULSE SOURCE

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Abstract: For a hyperbolic wave equation with some parameter λ, we consider the problem of finding the piecewise constant wave propagation speed and a series of parameters in the conjugation condition. Moreover, the shape is assumed unknown of the impulse point source that excites the oscillation process. We prove that, under certain assumptions on the structure of the medium, its sought parameters are determined uniquely from the displacements of points of the boundary given for two different values of λ. We give an algorithm for solving the problem.

Keywords: wave equation, inverse problem, solution algorithm

§ 1. Introduction and the Main Results

Suppose that $y_0, y_1, y_2, \dots, y_k, \dots$, $0 = y_0 < y_1 < y_2 < \dots < y_k < \dots$, is an ordered finite or infinite sequence of points of the half-axis $\{y \geq 0\}$ which divides the latter into finitely or infinitely many intervals (y_{k-1}, y_k), while $\{c_k, k = 1, 2, \dots\}$ and $\{a_k, k = 1, 2, \dots\}$ are two sequences of positive numbers. Suppose that $(c_k, a_k) \neq (c_{k+1}, a_{k+1})$ for all $k = 1, 2, \dots$. We assume also that the point at infinity is the unique limit point in the case of an infinite sequence $\{y_k\}$. In the domain $G = \mathbb{R}_+ \times \mathbb{R}$, $\mathbb{R}_+ = \{y \in \mathbb{R} \mid y > 0\}$, consider the wave propagation process described by the differential equation

$$u_{tt} - c_k^2 u_{yy} + \lambda^2 c_k^2 u = 0, \quad (y,t) \in (y_{k-1}, y_k) \times \mathbb{R}, \quad k = 1, 2, \dots, \tag{1.1}$$

the initial and boundary conditions

$$u|_{t<0} \equiv 0, \quad u_y|_{y=0} = 0, \tag{1.2}$$

the conjugation conditions

$$u|_{y=y_k+0} = u|_{y=y_k-0}, \quad a_{k+1} u_y|_{y=y_k+0} = a_k u_y|_{y=y_k-0}, \quad k = 1, 2, \dots, \tag{1.3}$$

and the condition of excitation of waves by a point impulse source concentrated at some $y^* \in (y_0, y_1)$:

$$u|_{y=y^*+0} = u|_{y=y^*-0}, \quad u_y|_{y=y^*+0} - u_y|_{y=y^*-0} = f(t) \quad (f(t) \equiv 0,\ t < 0). \tag{1.4}$$

Here $u = u(\lambda, y, t)$ and λ is some parameter of the problem. This parameter arises usually in the wave propagation problems in stratified media after application of the Fourier transform to the wave equation in all spatial variables other than y.

For the given c_k, a_k, and $f(t)$ problem (1.1)–(1.4) is well-posed and determines the function $u(\lambda, y, t)$ with compact support for every finite t. In applications, of interest is the problem of finding the structure of the medium (in our case the constants y_k, c_k, and a_k, $k = 1, 2, \dots$) from the displacements of the points of the medium which are measured on the boundary of the domain

$$u|_{y=0} = F(\lambda, t), \quad t \in (0, T), \tag{1.5}$$

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for a finite time interval and different values of λ. Actually, we speak of constructing the two piecewise constant functions $c = c(y)$ and $a = a(y)$ that coincide with the respective numbers c_k and a_k on their constancy intervals (y_{k-1}, y_k). Since the speed of propagation is finite, having information of the form (1.5), we can try to determine these functions only on some finite interval whose length depends on T and is monotone increasing with the growth of T. In the case when the wave propagation speed $c = c(y)$ is a continuous function of y (in this case we do not need the conjugation equations), a similar problem for a known function $f(t)$ was considered in a series of articles (for example, see [1–6]) and in the case similar to (1.1)–(1.5) but with some more complicated equations that describe propagation of waves in an elastic stratified plate, in [7]. In these articles a series of uniqueness and stability theorems were established for the corresponding inverse problems.

Here we consider the problem of finding y_k, c_k, and a_k, $k = 1, 2, \dots$, from information (1.5) on an unknown function $f(t)$. Of course, in this case we should make some assumptions about the structure of $f(t)$ (we return to this issue below). Note that in the case of a continuously nonhomogeneous stratified medium a similar statement of the problem appeared first in M. L. Gerver's monograph [8]. However, the differential equation of string oscillations had no parameter λ (this equation corresponds to the case of the normal incidence of an acoustic or elastic wave onto the boundary of the half-space, while (1.1) describes a more general case of oblique incidence). This circumstance became an obstacle for obtaining a uniqueness theorem and an algorithm for solution of the inverse problem suitable for applications. Some results connected with stability of a solution to the inverse problem and the method for its solution in the statement with parameter λ were obtained in the author's article [9]. In this paper, we establish a uniqueness theorem for (1.1)–(1.5) and construct an algorithm for effective solution of the problem. Moreover, as in [9], we use (1.5) given for two different values of λ.

We make some transformations of the initial problem in order to make it more convenient for computations. First of all, introduce the new variable z and the sequence of points z_k, $k = 1, 2, \dots$, by the formula

$$z = \begin{cases} \frac{y}{c_1}, & y \in (y_0, y_1),\ z_1 = \frac{y_1}{c_1}, \\ z_k + \frac{y - y_k}{c_{k+1}}, & y \in (y_k, y_{k+1}),\ z_{k+1} = z_k + \frac{y_{k+1} - y_k}{c_{k+1}},\ k = 1, 2, \dots. \end{cases} \tag{1.6}$$

Additionally, we denote $z_0 = 0$ and $z^* = y^*/c_1$. It is obvious that $z^* \in (z_0, z_1)$. The physical meaning of the new variable is the traveling time of a signal along the straight line from y_0 to y. Let $y = y(z)$ be the inverse of the function defined by (1.6). Introduce the new function $v(\lambda, z, t) = u(\lambda, y(z), t)$. This function is a solution to the problem

$$v_{tt} - v_{zz} + \lambda^2 c_k^2 v = 0, \quad (z, t) \in (z_{k-1}, z_k) \times \mathbb{R}, \quad k = 1, 2, \dots, \tag{1.7}$$

$$v|_{t<0} \equiv 0, \quad v_z|_{z=0} = 0, \tag{1.8}$$

$$v|_{z=z_k+0} = v|_{z=z_k-0}, \quad v_z|_{z=z_k+0} = d_{k,k+1} v_z|_{z=z_k-0}, \quad k = 1, 2, \dots, \tag{1.9}$$

$$v|_{z=z^*+0} = v|_{z=z^*-0}, \quad v_z|_{z=z^*+0} - v_z|_{z=z^*-0} = g(t), \quad g(t) = c_1 f(t), \tag{1.10}$$

in which $d_{k,k+1} = c_{k+1} a_k / (c_k a_{k+1})$ characterizes the value of the jump of the normal derivative of v on the boundary $z = z_k$ which is the interface between the kth layer, $z_{k-1} < z < z_k$, and the $(k+1)$th layer. Note that the coefficient $d_{k,k+1}$ can equal 1 even under the above condition $(c_k, a_k) \neq (c_{k+1}, a_{k+1})$. In this case it is necessary that $c_k \neq c_{k+1}$. To demonstrate by way of contradiction; assuming that $c_k = c_{k+1}$, from the condition $d_{k,k+1} = 1$ we find that $a_k = a_{k+1}$ and hence $(c_k, a_k) = (c_{k+1}, a_{k+1})$. This contradicts the initial assumption. Thus, it is possible that the function v remains continuous upon the passage through the boundary $z = z_k$ together with its normal derivative, the wave propagation speed has a finite jump but on this boundary. As we see from (1.7), in this situation the second derivative v_{zz} has a finite jump; therefore, $z = z_k$ is the boundary of weak discontinuity of v.

We can write the information of the solution to (1.7)–(1.10) used for solution of the inverse problem as

$$v|_{z=0} = F(\lambda, t), \quad t \in (0, T). \tag{1.11}$$

Below, studying the inverse problem, we suppose that $g(t)$ has the structure

$$g(t) = b\delta(t) + \hat{g}(t)\theta_0(t), \tag{1.12}$$

where $b \neq 0$, $\theta_0(t)$ is the Heaviside function: $\theta_0(t) = 1$ for $t \geq 0$ and $\theta_0(t) = 0$ for $t < 0$, and $\hat{g}(t) \in \mathbf{C}^1[0, T - z^*)$, $T > 0$.

Theorem 1.1. *Suppose that* (1.12) *holds with* $b \neq 0$ *and* $\hat{g}(t) \in \mathbf{C}^1[0, T - z^*)$. *Assume that* $T > 0$ *is chosen so that* $F(\lambda, t) \not\equiv 0$ *for* $t \in (0, T)$. *Then the prescription of* $F(\lambda, t)$ *on* $t \in (0, T)$ *for two values* λ_1 *and* λ_2 *such that* $\lambda_1^2 \neq \lambda_2^2$ *uniquely determines* b, z^*, *and the set of numbers* z_k, c_k, c_{k+1}, *and* $d_{k,k+1}$ *corresponding to those values of* k *for which* $z_k < (T + z^*)/2$. *Moreover,* $\hat{g}(t)$ *is determined uniquely for all* $t \in [0, T - z^*)$.

This theorem is proven in § 3 and provides a constructive algorithm for finding all parameters of the medium. The algorithm is based on construction of a special function $v^*(\lambda, z, t)$ that contains all discontinuities of the solution v to problem (1.7)–(1.10) as well as the discontinuities of its first and second derivatives with respect to the variable t. This function is a collection of traveling waves excited by the point source at $(z^*, 0)$ and multiply by reflected in the boundaries $z = z_k$, $k = 0, 1, 2, \dots$. It is defined by the formula

$$v^*(\lambda, z, t) = \theta_0(t) \sum_{s \geq 1} [\alpha_i(z)\theta_0(t + \chi_i z - \tau_i)$$
$$+\beta_i(\lambda, z)\theta_1(t + \chi_i z - \tau_i) + \gamma_i(\lambda, z)\theta_2(t + \chi_i z - \tau_i)], \tag{1.13}$$

where $i = (i_1, i_2, \dots, i_s)$ is the multi-index describing the "history" of the traveling wave and comprising the numbers of the boundaries on which this wave was reflected or refracted, the parameter χ_i characterizes the direction of its propagation and takes the values $+1$ or -1 depending on the values of the multi-index, τ_i is some number defined by a recurrent formula, and $\theta_k(t) = t^k\theta_0(t)/k!$, $k = 1, 2$. The system of notation connected with the function $v^*(\lambda, z, t)$ and the construction of the latter are described in detail in § 2. Moreover, the function v^* is constructed so that the difference $v - v^* = w$ be a smooth function. Namely, we have the following theorem:

Theorem 1.2. *Suppose that* (1.12) *holds with* $b \neq 0$ *and* $\hat{g}(t) \in \mathbf{C}^1[0, T - z^*)$. *Then the function* $w(\lambda, z, t) = v(\lambda, z, t) - v^*(\lambda, z, t)$ *is continuous in* $D(T) = \{(z, t) \mid z \geq 0,\ 0 \leq t < T - z\}$ *together with* w_t *and* w_{tt}, *while* $w(\lambda, z, t) \equiv 0$ *for* $t \leq |z - z^*|$.

This theorem is proven in § 4. It follows from this theorem that $F(\lambda, t)$ is the trace of the sum of the two functions v^* and w for $z = 0$; moreover, the trace of the second function is twice continuously differentiable with respect to t, while the trace of the first is just a piecewise smooth quadratic function containing the finite discontinuities of the function itself and its first and second derivatives. On the one hand, these discontinuities coincide with the corresponding discontinuities of $F(\lambda, t)$ and consequently are known. On the other hand, the discontinuities of $v^*(\lambda, 0, t)$ are expressed in terms of the parameters of the medium. With these two facts in mind, we can determine the sought parameters, tracing the values of discontinuities of $F(\lambda_1, t)$ and $F(\lambda_2, t)$ and their first two derivatives.

Remark 1. Theorem 1.1 claims that from the given (1.5) we can find some finitely many parameters $z_1, \dots, z_n$, $c_1, \dots, c_{n+1}$, $d_{1,2}, \dots, d_{n,n+1}$ that characterize the stratified medium. Naturally, in this case we easily find the initial boundaries of the layers $y_1, \dots, y_n$ and the speed inside them. However, $a_1, \dots, a_n$ cannot be reconstructed uniquely. It was clear from the very beginning, since the conjugation conditions are determined uniquely only by a_k/a_{k+1}, $k = 1, \dots, n$. Obviously, all these relations are determined by $c_1, \dots, c_{n+1}$ and $d_{1,2}, \dots, d_{n,n+1}$.

Remark 2. We can replace the above assumption about the structure of the source with a more general assumption. Namely, we can suppose that (1.12) is valid with its left-hand side replaced with the derivative of $g(t)$ of some integer order $k \geq 0$. In this case differentiation of all relations (1.7)–(1.11) with respect to the variable t of order k reduces the problem to the one under consideration.

§ 2. Construction of v^*

Under the conditions of Theorem 1.1, we have the representation $\hat{g}(t) = g_0 + g_1 t + o(t)$ in which $g_0 = \hat{g}(0)$ and $g_1 = \hat{g}'(0)$. In this connection, consider the question of construction of the discontinuous component v^* of the solution to (1.7)–(1.10) which also contains the discontinuities of its first and second derivatives with respect to t, putting

$$g(t) = b\delta(t) + g_0\theta_0(t) + g_1\theta_1(t) + \bar{g}(t)\theta_1(t) \tag{2.1}$$

in (1.10). Here $\bar{g}(t) = o(t)$ as $t \to 0$. The way of construction of this function is well known (for example, see [10]). The sought function consists of the collection of traveling waves propagating along the characteristics of the differential equation and excited by the source singularities (such as $\delta(t)$, $\theta_0(t)$, and $\theta_1(t)$) localized at $(z^*, 0)$. These waves, propagating on the z, t-plane, meet the discontinuity boundaries $z = z_k$ of the medium parameters which leads to the appearance of reflected and refracted traveling waves whose computation agrees with the boundary condition and the conjugation conditions. We observe an intricate picture of the waves multiply reflected and refracted from boundaries. To describe it, we need to introduce an appropriate system of notation.

Consider the function v^* defined above by (1.13). We now explain in detail the meaning of each summand of this formula and the symbols involved. We start with the multi-index $i = (i_1, i_2, \dots, i_s)$. By definition, its components are the indices of boundaries, i.e., nonnegative integers. For $s = 1$ this multi-index becomes the usual index and takes only two values 0 or 1. With the index $i = 0$ we associate the wave propagating from the source to the left towards the boundary $z = z_0 = 0$, and with the index $i = 1$ we associate the wave traveling from the source to the right towards the boundary $z = z_1$. For $s = 2$ the multi-index i is equal to (i_1, i_2), where i_1 can take only the values 0 or 1 and the index i_2 cannot coincide with i_1 and differs from i_1 by one (if this is admissible, i.e., does not lead to the appearance of negative values). Thus, for $s = 2$ three possible values of the multi-index are: $i = (0, 1)$, $i = (1, 0)$, and $i = (1, 2)$. The first of them is associated with the wave reflected from the boundary $z = z_0$ and traveling towards the boundary $z = z_1$, the second, with the wave reflected from the boundary $z = z_1$ and traveling towards the boundary $z = z_0$, and third, with the wave refracted on the boundary $z = z_1$ and traveling towards the boundary $z = z_2$. The further definition is done by the recurrent scheme: the multi-index $i = (i_1, i_2, \dots, i_s)$ consists of a multi-index $i' = (i_1, i_2, \dots, i_{s-1})$ and a nonnegative number i_s that differs from the index i_{s-1} by one. This multi-index is associated with the wave that propagates first from the source towards the boundary $z = z_{i_1}$ then, after refraction or reflection on this boundary, towards the boundary $z = z_{i_2}$, and so on. By definition, this wave is defined only for $z \in [z_{k-1}, z_k]$, where $k = \max(i_{s-1}, i_s)$. Thus, $\alpha_i^j = \beta_i^j = \gamma_i^j \equiv 0$ outside $[z_{k-1}, z_k]$.

We now turn to explanation of the symbols χ_i and τ_i. If $i = 0$ then $\chi_i = 1$ and $\tau_i = z^*$; if $i = 1$ then we put $\chi_i = -1$ and $\tau_i = -z^*$. Then the characteristics of the differential equation $t + \chi_i z - \tau_i = 0$ corresponding to these values of χ_i and τ_i pass through $(z^*, 0)$ and correspond to the waves propagating from the source to the left or to the right. Now, for two-component multi-indices we put

$$\chi_i = \begin{cases} -1, & i = (0,1), \\ +1, & i = (1,0), \\ -1, & i = (1,2), \end{cases} \qquad \tau_i = \begin{cases} z^*, & i = (0,1), \\ 2z_1 - z^*, & i = (1,0), \\ -z^*, & i = (1,2). \end{cases}$$

In the case when the number of components is greater than two we put

$$\chi_i = \begin{cases} -1, & i_s = i_{s-1} + 1, \\ +1, & i_s = i_{s-1} - 1, \end{cases} \qquad \tau_i = \begin{cases} \tau_{i_1,\dots,i_{s-1}}, & i_s = i_{s-1} + 1 = i_{s-2} + 2, \\ \tau_{i_1,\dots,i_{s-1}}, & i_s = i_{s-1} - 1 = i_{s-2} - 2, \\ \tau_{i_1,\dots,i_{s-1}} - 2z_{i_{s-1}}, & i_s = i_{s-1} + 1 = i_{s-2}, \\ \tau_{i_1,\dots,i_{s-1}} + 2z_{i_{s-1}}, & i_s = i_{s-1} - 1 = i_{s-2}. \end{cases}$$

These definitions of χ_i and τ_i guarantee continuity of the polygonal line starting at the source and composed of the segments of characteristics of positive or negative slope. Each segment of the characteristic

$t+\chi_i z-\tau_i=0$ lying in the strip $z\in(z_{k-1},z_k)$, where $k=\max(i_{s-1},i_s)$, supports a discontinuity of a solution to (1.7)–(1.10). The expression $\alpha_i(z)\theta_0(t+\chi_i z-\tau_i)$ describes a finite jump of a solution upon the passage through the corresponding characteristic, while the expressions $\beta_i(\lambda,z)\theta_1(t+\chi_i z-\tau_i)$ and $\gamma_i(\lambda,z)\theta_2(t+\chi_i z-\tau_i)$ describe the respective finite jumps of the first and second derivatives of a solution in the variable t. To obtain formulas for calculation of the coefficients $\alpha_i(z)$, $\beta_i(\lambda,z)$, and $\gamma_i(\lambda,z)$, we have to insert $v=v^*+w$ in (1.7) and equate to zero the coefficients of $\delta(t+\chi_i z-\tau_i)$, $\theta_0(t+\chi_i z-\tau_i)$, and $\theta_1(t+\chi_i z-\tau_i)$. Eventually, we obtain the following differential equations for $\alpha_i(z)$, $\beta_i(\lambda,z)$, $\gamma_i(\lambda,z)$, and $w(\lambda,z,t)$:

$$\alpha_i'=0,\quad -2\beta_i'\chi_i-\alpha_i''+\lambda^2c_k^2\alpha_i=0,\quad -2\gamma_i'\chi_i-\beta_i''+\lambda^2c_k^2\beta_i=0, \tag{2.2}$$

$$w_{tt}-w_{zz}+\lambda^2c_k^2w=\varphi_k,\quad (z,t)\in(z_{k-1},z_k)\times\mathbb{R},\quad k=1,2,\dots. \tag{2.3}$$

In these equalities the symbol $'$ stands for differentiation with respect to the variable z, the number k agrees with the multi-index $i=(i_1,\dots,i_s)$ as follows: $k=\max(i_{s-1},i_s)$ if $s>1$ and $k=1$ if $s=1$. The function φ_k is calculated by the formula

$$\varphi_k(\lambda,z,t)=\sum_{s\geq 1}\left[\gamma_i''-\lambda^2c_k^2\gamma_i\right]\theta_2(t+\chi_i z-\tau_i)\quad (z,t)\in(z_{k-1},z_k)\times\mathbb{R},\quad k=1,2,\dots, \tag{2.4}$$

in which summation is carried out only over these multi-indices satisfying either $(i_{s-1},i_s)=(k-1,k)$ or $(i_{s-1},i_s)=(k,k-1)$. Note that, in the domain $D(T)$ where (2.3) will be considered, the function φ_k comprises only finitely many summands different from zero, since $\theta_2(t+\chi_i z-\tau_i)=0$ if $t+\chi_i z-\tau_i<0$. Hence, for $\chi_i=1$ in the sum defining φ_k we have to consider only those summands for which $\tau_i<T$ and for $\chi_i=-1$ we take only those summands for which $\tau_i+2z_{k-1}<T$. Equations (2.2) can be easily integrated. If $i=0$ or $i=1$ then the corresponding solutions have the form

$$\begin{gathered}\alpha_i(z)=\alpha_i(z^*),\quad \beta_i(\lambda,z)=\beta_i(z^*)+\frac{\chi_i\alpha_i}{2}\lambda^2c_1^2(z-z^*),\\ \gamma_i(z,\lambda)=\gamma_i(\lambda,z^*)+\frac{\chi_i}{2}\lambda^2c_1^2\beta_i(z^*)(z-z^*)+\frac{\alpha_i}{8}\lambda^4c_1^4(z-z^*)^2.\end{gathered} \tag{2.5}$$

In the general case $i=(i_1,\dots,i_s)$, $s\geq 2$, the solutions to (2.2) are calculated by the formulas

$$\begin{gathered}\alpha_i(z)=\alpha_i(z_{i_{s-1}}),\quad \beta_i(\lambda,z)=\beta_i(\lambda,z_{i_{s-1}})+\frac{\chi_i\alpha_i}{2}\lambda^2c_k^2(z-z_{i_{s-1}}),\\ \gamma_i(z,\lambda)=\gamma_i(\lambda,z_{i_{s-1}})+\frac{\chi_i}{2}\lambda^2c_k^2\beta_i(\lambda,z_{i_{s-1}})(z-z_{i_{s-1}})+\frac{\alpha_i}{8}\lambda^4c_k^4(z-z_{i_{s-1}})^2,\end{gathered} \tag{2.6}$$

in which $k=\max(i_{s-1},i_s)$. The constants in these formulas (for a fixed λ) are found from (1.8)–(1.10). Substituting v^*+w for v in (1.10) and equating the coefficients of the same singularities, we find the following relations for determining α_i, $\beta_i(z^*)$, and $\gamma_i(\lambda,z^*)$ as well as the conjugation conditions for w at $z=z^*$:

$$\begin{gathered}\alpha_1-\alpha_0=0,\quad \alpha_1+\alpha_0=-b,\\ \beta_1(z^*)-\beta_0(z^*)=0,\quad \beta_1(z^*)+\beta_0(z^*)=-g_0,\\ \gamma_1(\lambda,z^*)-\gamma_0(\lambda,z^*)=0,\quad \gamma_1(\lambda,z^*)+\gamma_0(\lambda,z^*)+\frac{1}{2}\lambda^2c_1^2(\alpha_1+\alpha_0)=-g_1,\end{gathered} \tag{2.7}$$

$$w|_{z=z^*+0}=w|_{z=z^*-0},\quad w_z|_{z=z^*+0}-w_z|_{z=z^*-0}=h(\lambda,t). \tag{2.8}$$

Here $h(\lambda,t)$ is defined by the formula

$$h(\lambda,t)=\bar{g}(t)\theta_1(t)+\frac{1}{2}\lambda^2c_1^2(\beta_0(z^*)+\beta_1(z^*))\theta_2(t). \tag{2.9}$$

From (2.7) we find that

$$\alpha_1=\alpha_0=-\frac{b}{2},\quad \beta_1(z^*)=\beta_0(z^*)=-\frac{g_0}{2},\quad \gamma_1(\lambda,z^*)=\gamma_0(\lambda,z^*)=-\frac{g_1}{2}+\frac{b}{4}\lambda^2c_1^2. \tag{2.10}$$

Similarly, substituting $v^* + w$ for v in (1.8) and equating the coefficients of the same singularities, we find the following recurrent relations for determining α_i, $\beta_i(\lambda, z_0)$, and $\gamma_i(\lambda, z_0)$ for those values of the multi-indices $i = (i_1, \dots, i_s)$ in which the last two indices are already defined by the equalities $i_{s-1} = 0$ and $i_s = 1$:

$$\begin{gathered}\alpha_{i_1,\dots,i_s} = \alpha_{i_1,\dots,i_{s-1}}, \quad \beta_{i_1,\dots,i_s}(\lambda, z_0) = \beta_{i_1,\dots,i_{s-1}}(\lambda, z_0), \\ \gamma_{i_1,\dots,i_s}(\lambda, z_0) = \gamma_{i_1,\dots,i_{s-1}}(\lambda, z_0).\end{gathered} \tag{2.11}$$

Moreover, we find the initial conditions for w and its boundary condition for $z = 0$. They have the form

$$w|_{t<0} \equiv 0, \quad w_z|_{z=0} = 0. \tag{2.12}$$

Acting similarly, from (1.9) we find the recurrent relations for determining α_i, $\beta_i(\lambda, z_{i_{s-1}})$, and $\gamma_i(\lambda, z_{i_{s-1}})$ for all multi-indices $i = (i_1, \dots, i_s)$. First, consider the case that differs from the general case presented below. It corresponds to the multi-indices with only two components. This case is connected with the waves starting from the source $(z^*, 0)$ refracted and reflected on the boundary $z = z_1$. Here the incident wave corresponds to the value $i = 1$, the reflected wave, to the multi-index $i = (1, 0)$, and the refracted wave, to the multi-index $i = (1, 2)$. The corresponding relations have the form

$$\begin{gathered}\alpha_{1,2} = \alpha_1 + \alpha_{1,0}, \quad \beta_{1,2}(\lambda, z_1) = \beta_1(\lambda, z_1) + \beta_{1,0}(\lambda, z_1), \\ \gamma_{1,2}(\lambda, z_1) = \gamma_1(\lambda, z_1) + \gamma_{1,0}(\lambda, z_1), \quad -\alpha_{1,2} = d_{1,2}(-\alpha_1 + \alpha_{1,0}), \\ -\beta_{1,2}(\lambda, z_1) = d_{1,2}(-\beta_1(\lambda, z_1) + \beta_{1,0}(\lambda, z_1)), \\ -\gamma_{1,2}(\lambda, z_1) - \frac{1}{2}\lambda^2 c_2^2 \alpha_{1,2} = d_{1,2}\Big[-\gamma_1(\lambda, z_1) + \gamma_{1,0}(\lambda, z_1) - \frac{1}{2}\lambda^2 c_1^2(\alpha_1 - \alpha_{1,0})\Big].\end{gathered} \tag{2.13}$$

Hence, we find that

$$\begin{gathered}\alpha_{1,0} = \eta_{1,2}\alpha_1, \quad \beta_{1,0}(\lambda, z_1) = \eta_{1,2}\beta_1(\lambda, z_1), \\ \gamma_{1,0}(\lambda, z_1) = \eta_{1,2}\gamma_1(\lambda, z_1) + \frac{\lambda^2}{4}\big(1 - \eta_{1,2}^2\big)\big(c_1^2 - c_2^2\big)\alpha_1, \\ \alpha_{1,2} = (1 + \eta_{1,2})\alpha_1, \quad \beta_{1,2}(\lambda, z_1) = (1 + \eta_{1,2})\beta_1(\lambda, z_1), \\ \gamma_{1,2}(\lambda, z_1) = (1 + \eta_{1,2})\gamma_1(\lambda, z_1) + \frac{\lambda^2}{4}\big(1 - \eta_{1,2}^2\big)\big(c_1^2 - c_2^2\big)\alpha_1.\end{gathered} \tag{2.14}$$

In these formulas $\eta_{1,2} = \frac{d_{1,2}-1}{d_{1,2}+1}$. The number $\eta_{1,2}$ is called the *reflection index* for the boundary $z = z_1$; and the number $1 + \eta_{1,2}$ is called the *transmission index*. It is obvious that $\eta_{1,2} \in (-1, 1)$.

In the general case the wave can come to the boundary $z = z_k$ either from the left, like in the above case, or from the right travelling from the layer (z_k, z_{k+1}). If the two but last values of the multi-index $i = (i_1 \dots, i_s)$ are equal to $i_{s-2} = k - 1$ and $i_{s-1} = k$ and the last equals $i_s = k - 1$ then this multi-index describes the wave reflected from the boundary $z = z_k$ into the layer (z_{k-1}, z_k); if $i_s = k + 1$ then this is the wave transmitted to the layer (z_k, z_{k+1}). Moreover, the formulas for calculation of α_i, $\beta_i(\lambda, z_{i_{s-1}})$, and $\gamma_i(\lambda, z_{i_{s-1}})$ are quite similar to (2.14) and have the form

$$\begin{gathered}\alpha_{i_1,\dots,i_s} = \eta_{k,k+1}\alpha_{i_1,\dots,i_{s-1}}, \quad \beta_{i_1,\dots,i_s}(\lambda, z_{i_{s-1}}) = \eta_{k,k+1}\beta_{i_1,\dots,i_{s-1}}(\lambda, z_{i_{s-1}}), \\ \gamma_{i_1,\dots,i_s}(\lambda, z_{i_{s-1}}) = \eta_{k,k+1}\gamma_{i_1,\dots,i_{s-1}}(\lambda, z_{i_{s-1}}) \\ + \frac{\lambda^2}{4}\big(1 - \eta_{k,k+1}^2\big)\big(c_k^2 - c_{k+1}^2\big)\alpha_{i_1,\dots,i_{s-1}}, \text{ if } i_{s-2} = k - 1,\ i_{s-1} = k,\ i_s = k - 1,\end{gathered} \tag{2.15}$$

$$\begin{gathered}\alpha_{i_1,\dots,i_s} = (1 + \eta_{k,k+1})\alpha_{i_1,\dots,i_{s-1}}, \quad \beta_{i_1,\dots,i_s}(\lambda, z_{i_{s-1}}) = (1 + \eta_{k,k+1})\beta_{i_1,\dots,i_{s-1}}(\lambda, z_{i_{s-1}}), \\ \gamma_{i_1,\dots,i_s}(\lambda, z_{i_{s-1}}) = (1 + \eta_{k,k+1})\gamma_{i_1,\dots,i_{s-1}}(\lambda, z_{i_{s-1}}) \\ + \frac{\lambda^2}{4}\big(1 - \eta_{k,k+1}^2\big)\big(c_k^2 - c_{k+1}^2\big)\alpha_{i_1,\dots,i_{s-1}}, \text{ if } i_{s-2} = k - 1,\ i_{s-1} = k,\ i_s = k + 1.\end{gathered} \tag{2.16}$$

In these formulas $\eta_{k,k+1} = \frac{d_{k,k+1}-1}{d_{k,k+1}+1}$ is the reflection index for the boundary $z = z_k$ (towards the layer (z_{k-1}, z_k)).

If two but last values of the multi-index $i = (i_1 \dots, i_s)$ are equal to $i_{s-2} = k+1$ and $i_{s-1} = k$ then the multi-index describes the wave reflected from the boundary $z = z_k$ if $i_s = k+1$ and the wave transmitted to the layer (z_{k-1}, z_k) if $i_s = k-1$. To use the available formulas, it suffices to rewrite the conjugation conditions interchanging the layers k and $k+1$; namely,

$$v|_{z=z_k-0} = v|_{z=z_k+0}, \quad v_z|_{z=z_k-0} = d_{k+1,k} v_z|_{z=z_k+0}, \quad k = 1, 2, \dots. \tag{2.17}$$

Here $d_{k+1,k} = 1/d_{k,k+1}$. In this case the formulas for calculation of α_i, $\beta_i(\lambda, z_{i_{s-1}})$, and $\gamma_i(\lambda, z_{i_{s-1}})$ take the form

$$\alpha_{i_1,\dots,i_s} = \eta_{k+1,k}\alpha_{i_1,\dots,i_{s-1}}, \quad \beta_{i_1,\dots,i_s}(\lambda, z_{i_{s-1}}) = \eta_{k+1,k}\beta_{i_1,\dots,i_{s-1}}(\lambda, z_{i_{s-1}}),$$

$$\gamma_{i_1,\dots,i_s}(\lambda, z_{i_{s-1}}) = \eta_{k+1,k}\gamma_{i_1,\dots,i_{s-1}}(\lambda, z_{i_{s-1}})$$
$$+\frac{\lambda^2}{4}\big(1-\eta_{k+1,k}^2\big)\big(c_{k+1}^2 - c_k^2\big)\alpha_{i_1,\dots,i_{s-1}} \text{ if } i_{s-2} = k+1,\ i_{s-1} = k,\ i_s = k+1, \tag{2.18}$$

$$\alpha_{i_1,\dots,i_s} = (1+\eta_{k+1,k})\alpha_{i_1,\dots,i_{s-1}}, \quad \beta_{i_1,\dots,i_s}(\lambda, z_{i_{s-1}}) = (1+\eta_{k+1,k})\beta_{i_1,\dots,i_{s-1}}(\lambda, z_{i_{s-1}}),$$

$$\gamma_{i_1,\dots,i_s}(\lambda, z_{i_{s-1}}) = (1+\eta_{k+1,k})\gamma_{i_1,\dots,i_{s-1}}(\lambda, z_{i_{s-1}})$$
$$+\frac{\lambda^2}{4}\big(1-\eta_{k+1,k}^2\big)\big(c_{k+1}^2 - c_k^2\big)\alpha_{i_1,\dots,i_{s-1}} \text{ if } i_{s-2} = k+1,\ i_{s-1} = k,\ i_s = k-1. \tag{2.19}$$

The numbers $\eta_{k+1,k}$ in these formulas are defined by the equalities

$$\eta_{k+1,k} = \frac{d_{k+1,k}-1}{d_{k+1,k}+1} = -\eta_{k,k+1}.$$

Equalities (1.9) determine the conjugation conditions for w at $z = z_k$. They have the form

$$w|_{z=z_k+0} = w|_{z=z_k-0}, \quad w_z|_{z=z_k+0} = d_{k,k+1} w_z|_{z=z_k-0} + h_k(\lambda, t), \quad k = 1, 2, \dots. \tag{2.20}$$

Here $h_k(\lambda, t)$ is calculated by the formula

$$h_k(\lambda, t) = \frac{\lambda^2}{2}\sum_{s\ge 1}\big(c_k^2 - c_{k+1}^2\big)(1-\chi_i\eta_{k,k+1})(1-d_{k,k+1})\chi_i\beta_i(\lambda, z_k)\theta_2(t+\chi_i z_k - \tau_i) \tag{2.21}$$

in which $k = i_s$.

Thus, we have described all formulas necessary for calculation of the summands generating $v^*(\lambda, z, t)$. Summing over the multi-indices all traveling waves whose supports have nonempty intersection with $D(T) = \{(z,t) \mid z \ge 0,\ 0 \le t < T - z\}$, we construct the function v^* that accumulates all discontinuities of the solution to (1.7)–(1.10) lying in $D(T)$.

§ 3. Proof of Theorem 1.1 and an Algorithm for Computing the Medium Parameters

Consider the trace of $v^*(\lambda, z, t)$ for $z = 0$. Put $v^*(\lambda, 0, t) = F^*(\lambda, t)$. It follows from the construction of $v^*(\lambda, z, t)$ that $F^*(\lambda, t)$ is a piecewise quadratic function of t and is defined by the equality

$$F^*(\lambda, t) = \theta_0(t)\sum_{s\ge 1}[\alpha_i\theta_0(t-\tau_i) + \beta_i(\lambda, 0)\theta_1(t-\tau_i) + \gamma_i(\lambda, 0)\theta_2(t-\tau_i)], \tag{3.1}$$

in which summation is carried out over all multi-indices $i = (i_1, \dots, i_s)$ such that either $i_s = 0$ or $(i_{s-1}, i_s) = (0, 1)$. By Theorem 1.2, $F(\lambda, t) - F^*(\lambda, t) = v(\lambda, 0, t) - v^*(\lambda, 0, t) = w(\lambda, 0, t) \in \mathbf{C}^2(0, T)$. Therefore, $F(\lambda, t)$ and $F^*(\lambda, t)$ have discontinuities at the same points of $(0, T)$ and the values of these discontinuities as well as their first and second derivatives coincide. Since $F(\lambda, t)$ is known, the points at which $F^*(\lambda, t)$ and its first two derivatives are discontinuous as well as the values of these discontinuities. It follows from (3.1) that $F^*(\lambda, t) = 0$ for $t < z^*$. Therefore, if $F^*(\lambda, t) \not\equiv 0$, $t \in (0, T)$, then $z^* \in (0, T)$. Formulas (3.1), (2.5), (2.6), (2.10), and (2.11) lead to the equalities

$$\begin{gathered} F^*(\lambda, z^* + 0) = 2\alpha_0 = -b = F(\lambda, z^* + 0), \\ F_t^*(\lambda, z^* + 0) = 2\beta_0(\lambda, 0) = -g_0 - \lambda^2 c_1^2 \alpha_0 z^* = F_t(\lambda, z^* + 0), \qquad (3.2) \\ F_{tt}^*(\lambda, z^* + 0) = 2\gamma_0(\lambda, 0) = -g_1 + \frac{1}{2}\lambda^2 c_1^2 b - \lambda^2 c_1^2 \beta_0(z^*) z^* + \frac{1}{4}\lambda^4 c_1^4 \alpha_0 (z^*)^2 = F_{tt}(\lambda, z^* + 0). \end{gathered}$$

The first of these equalities means that $F(\lambda, t)$ has a nonzero jump at $t = z^*$ and therefore determines $z^* = \sup\{t^* \in (0, T) \mid F(\lambda, t) \equiv 0, t \in (0, t^*)\}$, and also the value b. The second equality considered for $\lambda = \lambda_1$ and $\lambda = \lambda_2$ determines $g_0 = g(+0)$ and c_1 (recall that $\lambda_1^2 \neq \lambda_2^2$), and the third equality enables us to calculate $g_1 = g'(+0)$.

Thus, the speed in the first layer becomes known, while the boundary $z = z_1$ remains unknown. Find a necessary and sufficient condition under which we can determine it. As we see from (3.1), the discontinuity of $F(\lambda, t)$ next to the point $t = z^*$ is possible only in result of the waves reflected once from the boundary $z = z_1$. These waves correspond to the multi-indices $i = (1, 0)$ and $i = (1, 0, 1)$. Show that either $F^*(\lambda, t)$ or the second derivative $F_{tt}^*(\lambda, t)$ of $F^*(\lambda, t)$ is discontinuous at $t = \tau_{1,0} = 2z_1 - z^*$. From (2.5), (2.6), (2.10), (2.11), and (2.14) we obtain the equalities

$$\begin{aligned} [F^*]_{t=\tau_{1,0}} &= 2\alpha_{1,0} = 2\eta_{1,2}\alpha_1 = -b\eta_{1,2} = [F]_{t=\tau_{1,0}}, \\ [F_t^*]_{t=\tau_{1,0}} &= 2\beta_{1,0}(\lambda, 0) = 2\eta_{1,2}\left(\beta_1(\lambda, z_1) - \frac{1}{2}\lambda^2 c_1^2 \alpha_1 z_1\right) \\ &= 2\eta_{1,2}\left(\beta_1(z^*) - \frac{1}{2}\lambda^2 c_1^2 \alpha_1 (2z_1 - z^*)\right) = [F_t]_{t=\tau_{1,0}}, \end{aligned}$$

$$\begin{aligned} [F_{tt}^*]_{t=\tau_{1,0}} &= 2\gamma_{1,0}(\lambda, 0) \\ &= 2\eta_{1,2}\left(\gamma_1(\lambda, z_1) - \frac{1}{2}\lambda^2 c_1^2 \beta_1(\lambda, z_1) z_1 + \frac{1}{8}\lambda^4 c_1^4 \alpha_1 z_1^2\right) + \frac{\lambda^2}{2}\left(1 - \eta_{1,2}^2\right)\left(c_1^2 - c_2^2\right)\alpha_1 \\ &= 2\eta_{1,2}\left(\gamma_1(\lambda, z^*) - \frac{1}{2}\lambda^2 c_1^2 \beta_1(z^*)(2z_1 - z^*) + \frac{1}{8}\lambda^4 c_1^4 \alpha_1 (2z_1 - z^*)^2\right) \\ &\quad + \frac{\lambda^2}{2}\left(1 - \eta_{1,2}^2\right)\left(c_1^2 - c_2^2\right)\alpha_1 = [F_{tt}]_{t=\tau_{1,0}}. \qquad (3.3) \end{aligned}$$

Here $[\cdot]_{t=t_0}$ denotes the value of the jump of the function in the brackets upon the passage through $t = t_0$. Formulas (3.3) demonstrate that we have the following alternative: either $\eta_{1,2} \neq 0$ and F^* is discontinuous at $t = \tau_{1,0}$ or $\eta_{1,2} = 0$, implying that F^* and F_t^* are continuous at $t = \tau_{1,0}$ and F_{tt}^* has finite discontinuity at this point. The latter follows from the fact that $\eta_{1,2} = 0$ only if $d_{1,2} = 1$; and in this case, as noticed above, $c_1 \neq c_2$. Thus, a necessary and sufficient condition for determining $t = \tau_{1,0}$ from the given function $F(\lambda, t)$ is that the function itself or its second derivative F_{tt} has discontinuity on (z^*, T). On the other hand, if the number of discontinuity points is greater than one then, obviously, the value $t = \tau_{1,0}$ corresponds to the nearest point of z^*, i.e., $\tau_{1,0} = \sup\{t^* \in (z^*, T) \mid F(\lambda, t) \in \mathbf{C}^2(z^*, t^*)\}$. That is the value $\tau_{1,0}$ which determines the boundary of the first layer $z_1 = (\tau_{1,0} + z^*)/2$. Moreover, the first equality in (3.3) determines $\eta_{1,2}$ and hence $d_{1,2}$. Then the first summand in the third equality having the factor $\eta_{1,2}$ becomes known and, since at least one of the values λ_1 and λ_2 is nonzero, the functions $F(\lambda_1, t)$

and $F(\lambda_2, t)$ uniquely determine the speed c_2 in the layer (z_1, z_2). The next question is about finding the next boundary $z = z_2$, the speed in the layer (z_2, z_3), and the value $d_{2,3}$, and so on.

Consider the general case: assume that the boundary $z = z_{n-1}$ and the numbers c_n and $d_{n-1,n}$, $n \geq 2$, are already available. We have to answer the following question: Can we find the next boundary $z = z_n$ from the data of the problem? If the answer to this question is positive then we need to give the formulas for computation of z_n and the parameters c_{n+1} and $d_{n,n+1}$. Introduce the function $F_n^*(\lambda, t)$ by the equality

$$F_n^*(\lambda, t) = \theta_0(t) \sum_{s \geq 1} [\alpha_i \theta_0(t - \tau_i) + \beta_i(\lambda, 0)\theta_1(t - \tau_i) + \gamma_i(\lambda, 0)\theta_2(t - \tau_i)], \tag{3.4}$$

where summation is carried out over all multi-indices $i = (i_1, \ldots, i_s)$ such that $i_k \leq n - 1$, $k = 1, \ldots, s$, and either $i_s = 0$ or $(i_{s-1}, i_s) = (0, 1)$ and $\tau_i < T$. The function $F_n^*(\lambda, t)$ contains the part of $F^*(\lambda, t)$ that corresponds to the waves multiply reflected and refracted inside the first $n - 1$ layers. Since all parameters of these layers are available by the above assumption, the summands of this function can be calculated by the formulas in § 2 and thus $F_n^*(\lambda, t)$ is known.

Introduce the difference $F(\lambda, t) - F_n^*(\lambda, t)$. In terms of this function, the above question is settled as follows: if the difference $F(\lambda, t) - F_n^*(\lambda, t)$ is a function of the class $\mathbf{C}^2(0, T)$ then the boundary $z = z_n$ cannot be found, since the data of the inverse problem contain no information about the waves reflected from this boundary. If this difference on $(0, T)$ contains discontinuities of either the function itself or its second derivative then the boundary z_n and the parameters c_{n+1} and $d_{n,n+1}$ are determined uniquely. Indeed, consider the wave corresponding to the multi-index $i = (1, 2, \ldots, n - 1, n, n + 1, \ldots, 1, 0)$. It is obvious that this is the earliest wave of those reflected from the boundary $z = z_n$ that attains the axes $z = 0$. Let τ_i be the corresponding traveling time of the signal from the source to the boundary $z = z_n$ and then from this boundary to the axis $z = 0$. Then the following hold:

$$[F^* - F_n^*]_{t=\tau_i} = 2\alpha_i = 2\kappa_n \alpha_1 = -b\kappa_n = [F - F_n^*]_{t=\tau_i},$$

$$[(F^* - F_n^*)_t]_{t=\tau_i} = 2\beta_i(\lambda, 0) = 2\kappa_n \left(\beta_1(z^*) - \frac{1}{2}\lambda^2 \nu_n \alpha_1 \right) = [(F - F_n^*)_t]_{t=\tau_i},$$

$$\begin{aligned} [(F^* - F_n^*)_{tt}]_{t=\tau_i} &= 2\gamma_{1,0}(\lambda, 0) \\ = 2\kappa_n \left(\gamma_1(\lambda, z^*) - \frac{1}{2}\lambda^2 \beta_1(z^*)\nu_n + \frac{1}{8}\lambda^4 \alpha_1 \nu_n^2 + \frac{1}{2}\lambda^2 \alpha_1 p_n \right) &+ \frac{1}{2}\lambda^2 \alpha_1 q_n = [(F - F_n^*)_{tt}]_{t=\tau_i}. \end{aligned} \tag{3.5}$$

Here

$$\kappa_n = \eta_{n,n+1} \prod_{k=1}^{n-1} (1 - \eta_{k,k+1}^2), \quad \nu_n = c_1^2(2z_1 - z^*) + 2 \sum_{k=2}^{n} c_k^2 (z_k - z_{k-1}),$$

$$p_n = \sum_{k=2}^{n-1} (c_k^2 - c_{k+1}^2)(1 - \eta_{k,k+1}), \quad q_n = \left(\sum_{k=2}^{n} (c_{k+1}^2 - c_k^2)(1 - \eta_{k,k+1}) \right) \prod_{k=1}^{n-1} (1 - \eta_{k,k+1}^2). \tag{3.6}$$

Formulas (3.5) demonstrate that we have the alternative: either $\kappa_n \neq 0$ and then $(F^* - F_n^*)$ is discontinuous at $t = \tau_i$ or $\kappa_n = 0$, implying that $(F^* - F_n^*)$ and $(F^* - F_n^*)_t^*$ are continuous at this point and $\eta_{n,n+1} = 0$ and $d_{n,n+1} = 1$. In the latter case $(F^* - F_n^*)_{tt}^*$ has finite discontinuity at $t = \tau_i$, $i = (1, 2, \ldots, n - 1, n, n + 1, \ldots, 1, 0)$. Thus, a necessary and sufficient condition for the point to be determined from $F - F_n^*$ is that either this function or its second derivative $(F - F_n^*)_{tt}$ on $(0, T)$ has discontinuity. If there is discontinuity then $\tau_i = \sup\{t^* \in (0, T) \mid F(\lambda, t) \in \mathbf{C}^2(0, t^*)\}$ and the boundary $z = z_n$ is found by the formula $z_n = (\tau_i + z^*)/2$. Otherwise the interval $(0, T)$ of observation of $F(\lambda, t)$ turns out to be insufficient for determination of this boundary. Once the boundary $z = z_n$ is found, the first equality in (3.5) determines the value κ_n and hence $\eta_{n,n+1}$ and $d_{n,n+1}$. Then the first summand of the third equality with κ_n becomes known. Therefore, the functions $F(\lambda_1, t) - F_n^*(\lambda_1, t)$ and $F(\lambda_2, t) - F_n^*(\lambda_2, t)$ uniquely determine q_n and consequently c_{n+1} in the layer (z_n, z_{n+1}).

Note that, in order to find c_{n+1}, it suffices to know $F(\lambda, t)$ only for one $\lambda \neq 0$. Recall that there is only a sole place in the proof of the theorem where we used the assumption that $F(\lambda, t)$ is known for two values λ; namely, at the initial step of finding the numbers g_0 and c_1. Therefore, if in the initial statement of the problem we assume that the speed c_1 in the first layer is known then all other parameters which determine the medium structure on $(0, (T + z^*)/2)$ are determined uniquely from the function $F(\lambda, t)$ for a single fixed $\lambda \neq 0$.

To complete the proof of Theorem 1.1, we are left with establishing that $\hat{g}(t)$ is found uniquely from the given information on $t \in [0, T - z^*)$. Fix $\lambda = \lambda_1$ (below we drop the index 1 for convenience) and consider $D'(T) = \{(z,t) \mid 0 < z < z^*, z^* - z < t \leq T - z\}$. In this domain v satisfies the differential equation (1.7), the Cauchy conditions at $z = 0$

$$v|_{z=0} = F(\lambda, t),\ t \in (0, T), \quad v_z|_{z=0} = 0, \tag{3.7}$$

and the following condition on the characteristic $t = z$:

$$v|_{t=z} = \alpha_0, \quad z \in (0, z^*). \tag{3.8}$$

These data uniquely determine $v(\lambda, z, t)$ in the whole domain $D'(T)$ and consequently the limit values of this function and its derivative with respect to z at $z = z^* - 0$, i.e., $v(\lambda, z^* - 0, t)$ and $v_z(\lambda, z^* - 0, t)$ for $t \in (0, T - z^*)$. Thereby the conjugation conditions (1.10) determine $v(\lambda, z^* + 0, t)$. Now, consider $D''(T) = \{(z,t) \mid z^* < z < (T + z^*)/2, z - z^* < t \leq T - z\}$. The function $v = v(\lambda, z^* + 0, t)$ is given on the boundary of this domain defined by the equality $z = z^*$. Moreover, v is given on the characteristic boundary $t = z - z^*$ of $D'(T)$ by the equalities

$$v|_{t=z-z^*+0} = \begin{cases} \alpha_1, & z \in (z^*, z_1), \\ \alpha_{1,2,\dots,k-1,k}, & z \in (z_{k-1}, z_k),\ k = 2, 3, \dots. \end{cases} \tag{3.9}$$

It is well known that $v(\lambda, z, t)$, a solution to (1.7), is determined uniquely in $D''(T)$ by these data and the conjugation conditions (1.9). Consequently, we can uniquely determine the limit values of its normal derivative for $z = z^*$, i.e., $v_z(\lambda, z^* + 0, t)$ for all $t \in (0, T - z^*)$. Consequently, the left and right limit values of the normal derivatives of v at $z = z^*$ become known and the sought function $\hat{g}(t)$ is found for all $t \in (0, T - z^*)$ from the second equality of (1.10).

§ 4. Proof of Theorem 1.2

Constructing $v^*(\lambda, z, t)$ in § 2 we also obtained a differential equation in the function $w(\lambda, z, t)$, the boundary and initial conditions, and the conjugation conditions at z^* and the discontinuities points $z = z_k$ of the medium parameters. They are determined by the equalities

$$w_{tt} - w_{zz} + \lambda^2 c_k^2 w = \varphi_k(\lambda, z, t), \quad (z,t) \in (z_{k-1}, z_k) \times \mathbb{R}, \quad k = 1, 2, \dots, \tag{4.1}$$

$$w|_{t<0} \equiv 0, \quad w_z|_{z=0} = 0, \tag{4.2}$$

$$w|_{z=z^*+0} = w|_{z=z^*-0}, \quad w_z|_{z=z^*+0} - w_z|_{z=z^*-0} = h(\lambda, t), \tag{4.3}$$

$$w|_{z=z_k+0} = w|_{z=z_k-0}, \quad w_z|_{z=z_k+0} = d_{k,k+1} w_z|_{z=z_k-0} + h_k(\lambda, t), \quad k = 1, 2, \dots, \tag{4.4}$$

in which $h(\lambda, t)$, $h_k(\lambda, t)$, and $\varphi_k(\lambda, z, t)$ are continuously differentiable functions of t. Moreover, $\varphi_k(\lambda, z, t)$ are piecewise polynomial in z.

The theory of (4.1)–(4.4) is well known. Consider a solution to this problem in $D(T) = \{(z,t) \mid z \geq 0,\ 0 \leq t < T - z\}$. The continuous conjugation conditions for the solution at the points z^* and z_k and the continuity properties of $h(\lambda, t)$, $h_k(\lambda, t)$, and $\varphi_k(\lambda, z, t)$ in t guarantee that the functions w and w_t belong to the class $\mathbf{C}(D(T))$. Since we can differentiate all relations of (4.1)–(4.4) with respect to the variable t and the functions $h_t(\lambda, t)$, $(h_k)_t(\lambda, t)$, and $(\varphi_k)_t(\lambda, z, t)$ remain continuous; hence, w_t also possesses a similar property, i.e., belongs to the class $\mathbf{C}(D(T))$ together with its derivative w_{tt}. Obviously, a solution to (4.1)–(4.4) satisfies the equality $w(\lambda, z, t) \equiv 0$ for $t \leq |z - z^*|$. Thereby Theorem 1.2 is established.

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