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THE ISOMETRY GROUPS OF RIEMANNIAN ORBIFOLDS

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Abstract: We prove that the isometry group $\mathfrak{I}(\mathscr{N})$ of an arbitrary Riemannian orbifold $\mathscr{N}$, endowed with the compact-open topology, is a Lie group acting smoothly and properly on $\mathscr{N}$. Moreover, $\mathfrak{I}(\mathscr{N})$ admits a unique smooth structure that makes it into a Lie group. We show in particular that the isometry group of each compact Riemannian orbifold with a negative definite Ricci tensor is finite, thus generalizing the well-known Bochner's theorem for Riemannian manifolds.

Keywords: orbifold, isometry group, Lie group of transformations, Ricci tensor

Introduction

Satake introduced orbifolds in [1] as a generalization of the concept of a manifold. They arise naturally in the various areas of mathematics and theoretical physics [2]. The orbifolds appear in foliation theory as the spaces of leaves of foliations locally stable in the sense of Reeb [3]. Orbifolds are used in string theory [4] as the spaces of string propagation. In [5] a theory is developed of deformation quantization on the symplectic orbispaces which include symplectic orbifolds.

In the first article [6] on the Riemannian geometry of orbifolds, Satake proved the Gauss–Bonnet theorem for orbifolds. The famous results of Thurston on the classification of 3-dimensional manifolds rest on the classification of 2-dimensional compact Riemannian orbifolds of constant curvature [7]. The structure of Riemannian orbifolds with Ricci curvature bounded below was studied by Borzellino [8, 9], also by Borzellino and Zhu [10]. In this article we prove the following theorems:

Theorem 1. *The isometry group $\mathfrak{I}(\mathscr{N})$ of an n-dimensional Riemannian orbifold $\mathscr{N}$ endowed with the compact-open topology is a Lie group of dimension at most $n(n+1)/2$. The action of $\mathfrak{I}(\mathscr{N})$ on $\mathscr{N}$ is smooth and proper, and the equality $\dim\mathfrak{I}(\mathscr{N}) = n(n+1)/2$ holds if and only if $\mathscr{N}$ is isometric to one of the following n-dimensional Riemannian manifolds of constant curvature:* (a) *the Euclidean space $\mathbb{E}^n$;* (b) *the sphere S^n;* (c) *the projective space $\mathbb{RP}^n$;* (d) *the simply-connected hyperbolic space $\mathbb{H}^n$.*

Theorem 2. *The isometry group of a Riemannian orbifold admits a unique smooth structure that makes it into a Lie group.*

Theorem 3. *If $\mathscr{N}$ is a compact Riemannian orbifold with nonpositive definite Ricci tensor and the Ricci tensor is negative definite at some point of $\mathscr{N}$ then the isometry group of $\mathscr{N}$ is finite.*

In the case that $\mathscr{N}$ is a manifold, Theorem 1 covers the classical theorem of Myers and Steenrod [11]. Theorem 3 generalizes the well-known theorem of Bochner [12] for Riemannian manifolds. Theorem 2 implies that the topology on the isometry Lie group $\mathfrak{I}(\mathscr{N})$ of a Riemannian orbifold $\mathscr{N}$, which we introduced in [13], coincides with the compact-open topology.

We indicate the specific character of the isometry groups of good Riemannian orbifolds (Section 6). We illustrate the content of the article with examples.

1. The Category of Orbifolds

Throughout this article we understand by smoothness the smoothness of class C^∞. Given some smooth map of manifolds $f : M_1 \to M_2$, denote by f_* and f^* the differential and codifferential of f.

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Recall the definition of a smooth orbifold [6, 14]. Let $\mathscr{N}$ be a connected Hausdorff topological space with a countable base, let U be an open subset of $\mathscr{N}$, and let n be a fixed natural number. A *chart* on $\mathscr{N}$ is a triple (Ω, Γ, p) consisting of a connected open subset Ω of the n-dimensional arithmetic space $\mathbb{R}^n$, a finite group Γ of diffeomorphisms of Ω, and the composition $p : \Omega \to U \subset \mathscr{N}$ of the quotient map $r : \Omega \to \Omega/\Gamma$ with a homeomorphism $q : \Omega/\Gamma \to U$ of the quotient space Ω/Γ onto U. The subset U is called a *coordinate neighborhood* of (Ω, Γ, p). Note that, unlike Satake [6], we do not require the dimension of the fixed-point set $\operatorname{Fix}\Gamma$ of Γ to be smaller than $n - 1$.

Let U and U' be coordinate neighborhoods of charts (Ω, Γ, p) and (Ω', Γ', p'), with $U \subset U'$. An embedding $\phi : \Omega \to \Omega'$ such that $p' \circ \phi = p$ is called an *injection of the chart* (Ω, Γ, p) *into the chart* (Ω', Γ', p') *corresponding to the inclusion* $U \subset U'$. It is known [15] that each injection ϕ induces a (unique) monomorphism of groups $\psi : \Gamma \to \Gamma'$ for which $\phi \circ \gamma = \psi(\gamma) \circ \phi$ for all $\gamma \in \Gamma$, and if ϕ is a diffeomorphism then ψ is an isomorphism between the groups Γ and Γ'.

Two charts $(\Omega_1, \Gamma_1, p_1)$ and $(\Omega_2, \Gamma_2, p_2)$ with coordinate neighborhoods U_1 and U_2 are called *compatible* if in the case $U_1 \cap U_2 \neq \varnothing$ for each point $x \in U_1 \cap U_2$ there exist: (a) a chart (Ω, Γ, p) with coordinate neighborhood U such that $x \in U \subset U_1 \cap U_2$; (b) injections of charts $\phi_1 : \Omega \to \Omega_1$ and $\phi_2 : \Omega \to \Omega_2$ corresponding to inclusions $U \subset U_1$ and $U \subset U_2$.

A set $\mathscr{A} = \{(\Omega_i, \Gamma_i, p_i) \mid i \in J\}$ of charts is called an *atlas* if the family $\{U_i := p_i(\Omega_i) \mid i \in J\}$ is an open covering of $\mathscr{N}$ and each pair of charts in $\mathscr{A}$ is compatible. An atlas $\mathscr{A}$ is called *maximal* if $\mathscr{A}$ coincides with every atlas that includes it. A maximal atlas is called the *structure of a smooth n-dimensional orbifold* on $\mathscr{N}$. A pair $(\mathscr{N}, \mathscr{A})$, where $\mathscr{A}$ is a maximal atlas on $\mathscr{N}$, is called a *smooth n-dimensional orbifold*. Note that each atlas is included in a unique maximal atlas, and thus defines the structure of a smooth orbifold.

Henceforth we assume all orbifolds $\mathscr{N}$ smooth and denote by $\mathscr{A} = \{(\Omega_i, \Gamma_i, p_i) \mid i \in J\}$ the maximal atlas of $\mathscr{N}$. The injection ϕ_{ij} of a chart $(\Omega_i, \Gamma_i, p_i)$ into a chart $(\Omega_j, \Gamma_j, p_j)$ corresponding to the inclusion of coordinate neighborhoods $U_i \subset U_j$ is called an *injection of charts* and is denoted by $\phi_{ij} : \Omega_i \to \Omega_j$, $i, j \in J$.

For charts (Ω, Γ, p) and (Ω', Γ', p') in $\mathscr{A}$ with coordinate neighborhoods containing $x \in \mathscr{N}$ the isotropy subgroups Γ_y and Γ'_z of $y \in p^{-1}(x)$ and $z \in p'^{-1}(x)$ are respectively isomorphic. Therefore, to each point x of $\mathscr{N}$ there corresponds a unique (up to a group isomorphism) abstract group Γ_y called the *orbifold group* of x. A point x is called *regular* if its orbifold group is trivial. *Singular* we call a point that is not regular. It is known, see [16] for instance, that the set Δ_n of all regular points of an n-dimensional orbifold $\mathscr{N}$ with the induced topology is a connected open n-dimensional manifold dense in $\mathscr{N}$. For each point $x \in \mathscr{N}$ there exists [6] a chart $(\Omega, \Gamma, p) \in \mathscr{A}$ such that Ω is an n-dimensional arithmetic space $\mathbb{R}^n$, $p(0) = x$ with $0 = (0, \dots, 0) \in \mathbb{R}^n$, and Γ is a finite group of orthogonal transformations of $\mathbb{R}^n$. Such a chart $(\mathbb{R}^n, \Gamma, p)$ is called a *linearized chart* at x.

A continuous map $f : \mathscr{N} \to \mathscr{N}'$ of an orbifold $(\mathscr{N}, \mathscr{A})$ into an orbifold $(\mathscr{N}', \mathscr{A}')$ is called [15] *smooth* if for each point $x \in \mathscr{N}$ there exist: (a) a chart $(\Omega, \Gamma, p) \in \mathscr{A}$ with coordinate neighborhood $U \ni x$; (b) a chart $(\Omega', \Gamma', p') \in \mathscr{A}'$ with coordinate neighborhood U' such that $f(U) \subset U'$; (c) a smooth map $\tilde{f} : \Omega \to \Omega'$ of Ω into Ω' such that $p' \circ \tilde{f} = f|_U \circ p$. In this case the smooth map $\tilde{f}$ is called a *local lift* of f.

The *category of orbifolds* is the category whose morphisms are given by the smooth maps of orbifolds and the composition of morphisms is the composition of smooth maps. We denote this category by $\mathfrak{Orb}$. The category of smooth manifolds with smooth maps of manifolds as morphisms is a full subcategory of $\mathfrak{Orb}$.

Call some action $\Phi : G \times \mathscr{N} \to \mathscr{N}$ of some Lie group G on some orbifold $\mathscr{N}$ *smooth* if Φ is a smooth map of the product orbifold $G \times \mathscr{N}$ into $\mathscr{N}$. An orbifold $(\mathscr{N}, \mathscr{A})$ is called [6, 14] *oriented* if for all $i \in J$ the manifolds Ω_i are oriented so that each transformation $\gamma \in \Gamma_i$ as well as each injection $\phi_{ij} : \Omega_i \to \Omega_j$, $i, j \in J$, preserves orientation.

The following example shows that, in contrast to manifolds and 2-dimensional orbifolds, for $n \geq 3$ the underlying topological spaces of n-dimensional orbifolds need not in general be locally Euclidean.

EXAMPLE 1. Define the action of the generator $f : \mathbb{R}^n \to \mathbb{R}^n$ of the group $\Gamma = \langle f \mid f^2 \rangle \cong \mathbb{Z}_2$ by the equality $f(x) := -x\ \forall x \in \mathbb{R}^n$, where $n \geq 3$. The quotient space $\mathscr{N} := \mathbb{R}^n/\Gamma$ is a smooth n-dimensional orbifold with the unique singular point $a := p(0)$, where 0 is the origin, and $p : \mathbb{R}^n \to \mathbb{R}^n/\Gamma$ is the quotient map. Note that the topological space $\mathscr{N}$ is contractible. Check that the point a has no neighborhood homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+$, including a proof for $\mathbb{R}^n$ and omitting the analogous proof for $\mathbb{R}^n_+$. Suppose to the contrary that there exist some neighborhood U of a and a homeomorphism $\chi : \mathbb{R}^n \to U$ onto U. Without loss of generality, we may assume that $\chi(0) = a$. Then the restriction $\chi|_{\mathbb{R}^n\backslash\{0\}} : \mathbb{R}^n\backslash\{0\} \to U\backslash\{a\}$ is also a homeomorphism; consequently, the topological spaces $U\backslash\{a\}$ and $\mathbb{R}^n\backslash\{0\}$ have isomorphic fundamental groups. Since $U\backslash\{a\}$ is homeomorphic to the direct product of the line $\mathbb{R}^1$ and the $(n-1)$-dimensional projective space $\mathbb{RP}^{n-1}$, it follows that $\pi_1(U\backslash\{a\}) \cong \mathbb{Z}_2$, while $\pi_1(\mathbb{R}^n\backslash\{0\}) \cong 0$ for $n \geq 3$. This contradiction shows that the underlying topological space of $\mathscr{N}$ is not locally Euclidean.

2. Fiber Bundles over Orbifolds

Recall that an *antihomomorphism* of some group Γ into some group G is a map $b : \Gamma \to G$ such that $b(\gamma_1\gamma_2) = b(\gamma_2)b(\gamma_1)$ for all $\gamma_1, \gamma_2 \in \Gamma$. If b is also injective then b is called an *antimonomorphism*.

Let F be a smooth manifold and let H be a Lie group. Following [14], we say that a *fiber bundle with standard fiber F and structure group H* is defined over some orbifold $(\mathscr{N}, \mathscr{A})$ if

(1) for each chart $(\Omega_i, \Gamma_i, p_i) \in \mathscr{A}$ there are given:

(a) a fiber bundle P_i with projection $\pi_i : P_i \to \Omega_i$, standard fiber F, and structure group H;

(b) an antimonomorphism $b_i : \Gamma_i \to \operatorname{Aut} P_i$ of Γ_i into the automorphism group $\operatorname{Aut} P_i$ of the fiber bundle such that $\gamma^{-1} \circ \pi_i = \pi_i \circ b_i(\gamma)\ \forall \gamma \in \Gamma_i$;

(2) for each injection of charts $\phi_{ij} : \Omega_i \to \Omega_j$, $i, j \in J$, an isomorphism $\bar{\phi}_{ij} : P_j|_{\phi_{ij}(\Omega_i)} \to P_i$ of fiber bundles is defined, where $P_j|_{\phi_{ij}(\Omega_i)}$ is the restriction of P_j to $\phi_{ij}(\Omega_i)$), satisfying the following conditions:

(a) $b_i(\gamma) \circ \bar{\phi}_{ij} = \bar{\phi}_{ij} \circ b_j(\psi_{ij}(\gamma))$ for all $\gamma \in \Gamma_i$, where $\psi_{ij} : \Gamma_i \to \Gamma_j$ is a monomorphism of groups induced by ϕ_{ij};

(b) if $U_i \subset U_j \subset U_k$ with the corresponding injections of charts ϕ_{ij} and ϕ_{jk} then $\overline{\phi_{jk} \circ \phi_{ij}} = \bar{\phi}_{ij} \circ \bar{\phi}_{jk}$.

Denote by $\xi = \{P_i, b_i, \bar{\phi}_{ij}\}_{i,j \in J}$ the fiber bundle over $\mathscr{N}$ described above.

A fiber bundle over an orbifold can be defined starting from an arbitrary atlas; see [6]. For each orbifold $\mathscr{N}$ there exists some atlas $\mathscr{B} = \{(\Omega_\beta, \Gamma_\beta, p_\beta) \mid \beta \in B\}$ with contractible coordinate neighborhoods of all charts. For such an atlas the fiber bundles P_β are trivial; i.e, $P_\beta = \Omega_\beta \times F$ and $\pi_\beta : P_\beta \to \Omega_\beta$ is the canonical projection onto the first factor. If for each $i \in J$ the fiber bundle P_i is a principal H-bundle then call $\xi = \{P_i, b_i, \bar{\phi}_{ij}\}_{i,j \in J}$ a *principal bundle over $\mathscr{N}$ with structure group H*.

Let $\xi = \{P_i, b_i, \bar{\phi}_{ij}\}_{i,j \in J}$ be a fiber bundle with standard fiber F and structure group H over some orbifold $\mathscr{N}$. For each chart $(\Omega_i, \Gamma_i, p_i) \in \mathscr{A}$ the antimonomorphism b_i determines the smooth left action $\Phi_i : \Gamma_i \times P_i \to P_i : (\gamma, z) \mapsto b_i(\gamma^{-1})(z)$ of Γ_i on P_i. Since Γ_i is a finite group, the quotient space $\overline{P}_i := P_i/\Gamma_i$ is a smooth orbifold of dimension $\dim \mathscr{N} + \dim F$, and the equality $\bar{\pi}_i \circ \bar{p}_i = p_i \circ \pi_i$ holds, where $\bar{p}_i : P_i \to P_i/\Gamma_i$ is the quotient map, and $\bar{\pi}_i : \overline{P}_i \to U_i$ takes the orbit of $z \in P_i$ into $p_i(\pi_i(z)) \in U_i = p_i(\Omega_i)$. Denote by $\overline{P}$ the disjoint union $\bigsqcup_{i \in J} \overline{P}_i$. Define on $\overline{P}$ the equivalence relation ρ: Say that two points $\bar{z}_i \in \overline{P}_i$ and $\bar{z}_j \in \overline{P}_j$ are *ρ-equivalent* if: (a) $\bar{\pi}_i(\bar{z}_i) = \bar{\pi}_j(\bar{z}_j) = x \in U_i \cap U_j$; (b) there exist two points $z_i \in (\bar{p}_i)^{-1}(\bar{z}_i)$ and $z_j \in (\bar{p}_j)^{-1}(\bar{z}_j)$ and a chart $(\Omega_k, \Gamma_k, p_k) \in \mathscr{A}$ with coordinate neighborhood U_k such that $x \in U_k \subset U_i \cap U_j$ and $z_j = (\bar{\phi}_{kj})^{-1} \circ \bar{\phi}_{ki}(z_i)$. In [16] we showed that the relation ρ is indeed an equivalence, the quotient space $\mathscr{P} = \overline{P}/\rho$ is naturally equipped with the structure of a smooth orbifold, and the projections $\pi_i : P_i \to \Omega_i$ define a smooth map $\pi : \mathscr{P} \to \mathscr{N}$ of orbifolds.

Therefore, given a fiber bundle with standard fiber F and structure group H over some orbifold $\mathscr{N}$, a smooth orbifold $\mathscr{P}$ of dimension $\dim \mathscr{N} + \dim F$ and a smooth map of orbifolds $\pi : \mathscr{P} \to \mathscr{N}$ are naturally defined. The orbifold $\mathscr{P}$ is called the *total space*; and the map $\pi : \mathscr{P} \to \mathscr{N}$, the *projection*.

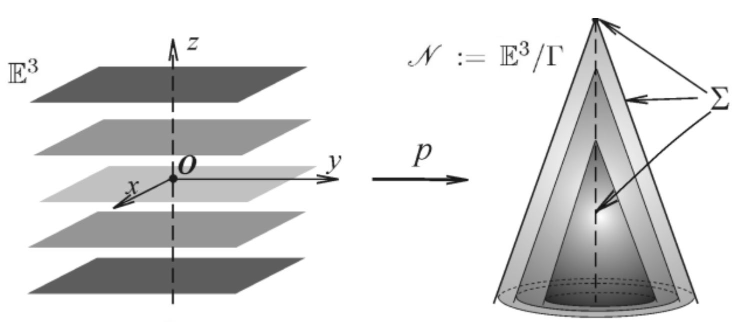
Suppose that $\xi = \{P_i, b_i, \bar{\phi}_{ij}\}_{i,j \in J}$ is a principal bundle over $\mathscr{N}$ with structure group H. Show that a smooth right action of the Lie group H is defined on the total space $\mathscr{P}$. For each $i \in J$ the smooth right action $\Upsilon_i : P_i \times H \to P_i$ with $(z, h) \mapsto z \cdot h$, where $z \in P_i$ and $h \in H$, of H is defined on the total space P_i of the principal H-bundle $\pi_i : P_i \to \Omega_i$. Since $b_i(\gamma)$ for $\gamma \in \Gamma_i$ is an automorphism of the principal bundle P_i, it follows that $b_i(\gamma)(z \cdot h) = (b_i(\gamma)(z)) \cdot h$; consequently, the map $\overline{\Upsilon}_i : \overline{P}_i \times H \to \overline{P}_i : (\bar{z}, h) \mapsto \bar{p}_i(z \cdot h)$, where $\bar{z} \in \overline{P}_i$, $z \in \bar{p}_i^{-1}(\bar{z})$, and $h \in H$, defines a smooth right action of H on $\overline{P}_i = P_i/\Gamma_i$. Denote by $q : \overline{P} \to \overline{P}/\rho = \mathscr{P}$ the natural projection. The composition $q_i := q \circ j : \overline{P}_i \to \mathscr{P}$ of the inclusion $j : \overline{P}_i \hookrightarrow \overline{P}$ with q is a homeomorphism onto the image. Take $z' \in \mathscr{P}$, $x = \pi(z')$ and some chart $(\Omega_i, \Gamma_i, p_i) \in \mathscr{A}$ with coordinate neighborhood $U_i \ni x$. The formula $\Upsilon(z', h) := q_i \circ \bar{p}_i(z \cdot h)$, where $z \in (q_i \circ \bar{p}_i)^{-1}(z')$ and $h \in H$, defines a smooth right action $\Upsilon : \mathscr{P} \times H \to \mathscr{P}$ of H on $\mathscr{P}$. The orbit space $\mathscr{P}/H$ of the action Υ is the orbifold $\mathscr{N}$. The following diagram is commutative:

$$\begin{array}{ccccc}
P_i \times H & \xrightarrow{(\bar{p}_i, \mathrm{id}_H)} & \overline{P}_i \times H & \xrightarrow{(q_i, \mathrm{id}_H)} & \mathscr{P} \times H \\
\downarrow \Upsilon_i & & \downarrow \overline{\Upsilon}_i & & \downarrow \Upsilon \\
P_i & \xrightarrow{\bar{p}_i} & \overline{P}_i & \xrightarrow{q_i} & \mathscr{P} \\
\downarrow \pi_i & & \downarrow \bar{\pi}_i & & \downarrow \pi \\
\Omega_i & \xrightarrow{p_i} & U_i & \hookrightarrow & \mathscr{N}.
\end{array}$$

According to [14], a *smooth section* of a fiber bundle $\xi = \{P_i, b_i, \bar{\phi}_{ij}\}_{i,j \in J}$ with standard fiber F and structure group H over some orbifold $(\mathscr{N}, \mathscr{A})$ is defined as a family $\{s_i\}_{i \in J}$ of smooth sections $s_i : \Omega_i \to P_i$ of the fiber bundles P_i if the following are satisfied: (a) $b_i(\gamma) \circ s_i \circ \gamma = s_i$ for all $\gamma \in \Gamma_i$, $i \in J$; (b) $\bar{\phi}_{ij} \circ s_j \circ \phi_{ij} = s_i$ for each injection of charts $\phi_{ij} : \Omega_i \to \Omega_j$, $i, j \in J$. Note that a family $\{s_i\}_{i \in J}$ determines a smooth map $s : \mathscr{N} \to \mathscr{P}$ of orbifolds satisfying the equality $\pi \circ s = \mathrm{id}_{\mathscr{N}}$.

Let $(\mathscr{N}, \mathscr{A})$ be an n-dimensional orbifold. Denote by $\pi_i : T\Omega_i \to \Omega_i$ the tangent bundle of Ω_i. Given $\gamma \in \Gamma_i$, define a map $b_i(\gamma) : T\Omega_i \to T\Omega_i$ by the equality $b_i(\gamma)(X_x) := (\gamma^{-1})_{*x}(X_x)$, where $X_x \in T_x\Omega_i$ is a tangent vector at some point $x \in \Omega_i$. For each injection of charts $\phi_{ij} : \Omega_i \to \Omega_j$, $i, j \in J$, define a map $\bar{\phi}_{ij} : T\Omega_j|_{\phi_{ij}(\Omega_i)} \to T\Omega_i$ by the formula $\bar{\phi}_{ij}(X_{\phi_{ij}(x)}) := (\phi_{ij})^{-1}_{*x}(X_{\phi_{ij}(x)})$ for $X_{\phi_{ij}(x)} \in T_{\phi_{ij}(x)}\Omega_j$ and $x \in \Omega_i$. Therefore, we have defined the fiber bundle with standard fiber a vector space isomorphic to $\mathbb{R}^n$ and structure group $G = GL(n, \mathbb{R})$, which is called the *tangent bundle to the orbifold* $\mathscr{N}$. The total space $T\mathscr{N}$ of this bundle is a smooth $2n$-dimensional orbifold.

Similarly, the cotangent bundle and the tensor bundle of type (p, q) over an orbifold are defined in [6, 14]. A smooth section of the tensor bundle of type (p, q) is called a *tensor field of type* (p, q) on the orbifold. In particular, a *smooth vector field* on an orbifold $(\mathscr{N}, \mathscr{A})$ is a smooth section of the tangent bundle of $\mathscr{N}$; i.e., a family $\{X_i\}_{i \in J}$ of Γ_i-invariant vector fields X_i on Ω_i such that for each injection of charts $\phi_{ij} : \Omega_i \to \Omega_j$, $i, j \in J$, the equality $(\phi_{ij})_*(X_i) = X_j$ holds.



$\Gamma = \langle \alpha, \beta \mid \alpha^2, \beta^m \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_m$ Σ is the set of singular points of $\mathscr{N}$

Fig. 1. A 3-Dimensional Noncompact Orbifold.

We say that a symmetric bilinear form $t = \{t_i\}_{i \in J}$ on an orbifold $\mathscr{N}$ is *negative* (or *nonpositive*) *definite at* $x \in \mathscr{N}$ if there is some chart $(\Omega_i, \Gamma_i, p_i) \in \mathscr{A}$ with coordinate neighborhood $U_i \ni x$ such that the form t_i is negative (nonpositive) definite at $x' \in p_i^{-1}(x)$. Conditions (a) and (b) in the definition of a section imply that this definition is independent of the choice of a chart $(\Omega_i, \Gamma_i, p_i)$, the coordinate neighborhood $U_i \ni x$, and a point $x' \in p_i^{-1}(x)$. We say also that a symmetric bilinear form t is *negative* (*nonpositive*) *definite on* $\mathscr{N}$ if t possesses this property at each $x \in \mathscr{N}$. Similarly we define the vanishing of an arbitrary tensor at a point and on the orbifold.

EXAMPLE 2. Consider the action on the Euclidean space $\mathbb{E}^3$ of the finite group Γ generated by two isometries α and β, where α is the reflection in the plane Oxy and β is the rotation about the axis Oz through $2\pi/m$ for $m \in \mathbb{N}$ and $m \geq 2$. Then $\Gamma = \langle \alpha, \beta \mid \alpha^2, \beta^m \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_m$. The orbit space $\mathscr{N} := \mathbb{E}^3/\Gamma$ is a 3-dimensional orbifold (see Fig. 1). Denote by p the quotient map $\mathbb{E}^3 \to \mathbb{E}^3/\Gamma$.

In order to construct the tangent bundle of $\mathscr{N}$, use the atlas consisting of the single chart (Ω, Γ, p), where $\Omega = \mathbb{E}^3$ and both the group Γ and the map p are defined above.

Consider the tangent bundle $\bar{\pi} : P = T\mathbb{E}^3 \to \mathbb{E}^3$ of $\mathbb{E}^3$. The contractibility of $\mathbb{E}^3$ implies that $T\mathbb{E}^3 = \mathbb{E}^3 \times \mathbb{E}^3$ and $\bar{\pi}$ is identified with the projection $T\mathbb{E}^3 = \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{E}^3$ onto the first factor.

Define an antimonomorphism $b : \Gamma \to \mathrm{Aut}(T\mathbb{E}^3)$ by the formula $b(\gamma) := (\gamma^{-1}, (\gamma^{-1})_*)$, where $(\gamma^{-1}, (\gamma^{-1})_*)(x, v) := (\gamma^{-1}(x), (\gamma^{-1})_*(v))$ for all $\gamma \in \Gamma$, $x \in \mathbb{E}^3$, and $v \in T_x\mathbb{E}^3 = \mathbb{E}^3$ and $(\gamma^{-1})_*$ is the differential of the diffeomorphism γ^{-1} at x. The equality $\gamma^{-1} \circ \bar{\pi} = \bar{\pi} \circ b(\gamma)$ holds for all $\gamma \in \Gamma$. The map

$$\Phi : \Gamma \times \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{E}^3 \times \mathbb{E}^3 : (\gamma, (x, v)) \mapsto b(\gamma^{-1})(x, v) = (\gamma(x), \gamma_*(v))$$

determines a smooth left action of Γ on $T\mathbb{E}^3$. The orbit space

$$T\mathscr{N} := T\mathbb{E}^3/\Gamma = \{\Gamma \cdot (x, v) \mid (x, v) \in T\mathbb{E}^3 = \mathbb{E}^3 \times \mathbb{E}^3\}$$

of this action is the total space of the tangent bundle of $\mathscr{N}$; and so it is a smooth 6-dimensional orbifold, and $\pi : T\mathscr{N} \to \mathscr{N} : \Gamma \cdot (x, v) \mapsto p(x)$ is the projection of the tangent bundle onto $\mathscr{N}$.

The fiber $\pi^{-1}(y)$ of the tangent bundle over some point $y \in \mathscr{N}$ is equal to

$$\pi^{-1}(y) = \{\Gamma \cdot (x, v) \mid p(x) = y,\ v \in T_x\mathbb{E}^3\};$$

consequently, it is homeomorphic to the 3-dimensional orbifold $\mathbb{E}^3/\Gamma_x$, where Γ_x is the stationary subgroup of Γ at $x \in p^{-1}(y)$. This implies that for each singular point $y \in \mathscr{N}$ the fiber $\pi^{-1}(y)$ is not a vector space, and for each regular point $z \in \mathscr{N}$ the group Γ_x, with $x \in p^{-1}(z)$, is trivial, and so the fiber $\pi^{-1}(z) = \mathbb{E}^3$ is naturally equipped with the structure of the 3-dimensional vector space.

Take some smooth Γ-invariant vector field X on $\mathbb{E}^3$; i.e., $\gamma_*(X_x) = X_{\gamma(x)}$ for each $x \in \mathbb{E}^3$ and $\gamma \in \Gamma$. One example of a nonzero Γ-invariant vector field is $Y : \mathbb{E}^3 \to T\mathbb{E}^3 = \mathbb{E}^3 \times \mathbb{E}^3 : x \mapsto Y_x = (x, x)$. The map $s = s(X) : \mathscr{N} \to T\mathscr{N} = T\mathbb{E}^3/\Gamma : z \mapsto \Gamma \cdot (X_x)$, where $x \in p^{-1}(z)$, is well-defined and it is a smooth section of the tangent bundle $\pi : T\mathscr{N} \to \mathscr{N}$; i.e., it is a smooth vector field on $\mathscr{N}$. All smooth vector fields on $\mathscr{N}$ arise in this way.

3. Riemannian Orbifolds

We assume all Riemannian metrics positive definite.

In accord with [6, 14], we call a *Riemannian metric* g on some orbifold $(\mathscr{N}, \mathscr{A})$ a family $\{g_i\}_{i \in J}$ of Γ_i-invariant Riemannian metrics g_i on the manifolds Ω_i such that each injection of charts $\phi_{ij} : \Omega_i \to \Omega_j$, $i, j \in J$, is an isometry of the Riemannian manifolds (Ω_i, g_i) and (Ω_j, g_j). It is known that each smooth orbifold admits a Riemannian metric.

Let $(\mathscr{N}, g)$ be an n-dimensional Riemannian orbifold, and let $O(n, \mathbb{R})$ be the group of orthogonal matrices. Denote by $\pi_i : \mathscr{R}_i \to \Omega_i$ the bundle of orthonormal frames over the Riemannian manifold (Ω_i, g_i), which is a principal $O(n, \mathbb{R})$-bundle, regarding an orthonormal frame $z \in \mathscr{R}_i$ at $x \in \Omega_i$ as a linear isomorphism $z : \mathbb{R}^n \to T_x\Omega_i$ of vector spaces $\mathbb{R}^n$ and $T_x\Omega_i$. Define an antimonomorphism b_i of the group Γ_i into the automorphism group of the fiber bundle $\mathscr{R}_i$ by the equality $b_i(\gamma)(z) := (\gamma^{-1})_{*x} \circ z$, where $z \in \mathscr{R}_i$ is an orthonormal frame at $x \in \Omega_i$. Given an injection $\phi_{ij} : \Omega_i \to \Omega_j$, $i, j \in J$, define $\bar{\phi}_{ij}$ by the formula $\bar{\phi}_{ij}(z) := (\phi_{ij}^{-1})_{*\phi_{ij}(x)} \circ z$, where z is some orthonormal frame at $\phi_{ij}(x) \in \phi_{ij}(\Omega_i) \subset \Omega_j$.

Since the Riemannian metrics in the family $g = \{g_i\}_{i\in J}$ are compatible, it follows that $(\phi_{ij})^* g_j = g_i$, and hence $\bar{\phi}_{ij}$ is an isomorphism of the fiber bundles $\mathscr{R}_i$ and $\mathscr{R}_j|_{\phi_{ij}(\Omega_i)}$. The so-defined family $\xi = \{\mathscr{R}_i, b_i, \bar{\phi}_{ij}\}_{i,j\in J}$ determines a principal bundle with structure group $O(n,\mathbb{R})$ which is called in [6, 14] the *bundle of orthonormal frames* over the Riemannian orbifold $(\mathscr{N}, g)$.

For each chart $(\Omega_i, \Gamma_i, p_i)$ the smooth left action $\Phi_i : \Gamma_i \times \mathscr{R}_i \to \mathscr{R}_i$ with $(\gamma, z) \mapsto b_i(\gamma^{-1})(z)$ of the group Γ_i on the manifold $\mathscr{R}_i$ is defined. If the automorphism $b_i(\gamma)$, $\gamma \in \Gamma_i$, fixes some point $z \in \mathscr{R}_i$ then $(\gamma)_{*x}$ is an identity map of the tangent space $T_x\Omega_i$ at the point $x = \pi_i(z)$. Hence, as the topological space Ω_i is connected and the $O(n,\mathbb{R})$-structure of $\mathscr{R}_i$ over the manifold Ω_i is a G-structure of first order, the isometry γ is equal to the identity map id_{Ω_i}. Thus, the group Γ_i acts freely on $\mathscr{R}_i$. Consequently, the quotient space $\mathscr{R}_i/\Gamma_i$ is a smooth manifold, and the quotient map $\bar{p}_i : \mathscr{R}_i \to \mathscr{R}_i/\Gamma_i$ is a regular covering with the deck transformation group isomorphic to Γ_i. Thus, the total space $\mathscr{R}$ of the bundle of orthonormal frames over $(\mathscr{N}, g)$ is a smooth manifold of dimension $n(n+1)/2$.

In accordance with what we recalled in Section 2, there is a smooth right action of the Lie group $O(n,\mathbb{R})$ on $\mathscr{R}$ whose orbit space is the orbifold $\mathscr{N}$. The connected components of the orbits of this action determine a smooth foliation $\mathscr{F}$ of $\mathscr{R}$ of codimension n. If the orbifold $\mathscr{N}$ is orientable then the manifold $\mathscr{R}$ has two diffeomorphic connected components $\mathscr{R}^+$ and $\mathscr{R}^-$. In this case we denote by $\mathscr{R}$ one of those components. In the case of a nonorientable orbifold $\mathscr{N}$ the total space of the bundle $\mathscr{R}$ is connected.

4. A Lie Group Structure on the Isometry Group of a Riemannian Orbifold

Let $(\mathscr{N}, g)$ be a Riemannian orbifold. An automorphism $f : \mathscr{N} \to \mathscr{N}'$ of the orbifold $(\mathscr{N}, \mathscr{A})$ is an *isometry* if for each point $x \in \mathscr{N}$ and each pair $(\Omega_i, \Gamma_i, p_i)$ and $(\Omega_j, \Gamma_j, p_j)$ of charts of the maximal atlas $\mathscr{A}$ with coordinate neighborhoods U_i and U_j such that $x \in U_i$ and $f(U_i) = U_j$ there exists a local lift $f_{ij} : \Omega_i \to \Omega_j$ that is an isometry of the Riemannian manifolds (Ω_i, g_i) and (Ω_j, g_j). The definition of the Riemannian metric g on $\mathscr{N}$ implies that this definition in correct; i.e., it is independent of the choice of charts at x and $f(x)$ and of the choice of a local lift.

Throughout this article we denote by $\mathfrak{I}(\mathscr{N})$ the isometry group of the Riemannian orbifold $(\mathscr{N}, g)$. Recall that the *compact-open topology* on some group H of homeomorphisms of some topological space X is the topology with subbasis composed of the sets of the form $W(V, V') := \{f \in H \mid f(V) \subset V'\}$, where V is compact and V' is an open subset of X.

An *absolute parallelism* of an n-dimensional manifold $\mathscr{M}$ is a tuple of n smooth vector fields on $\mathscr{M}$ that are linearly independent at each point of $\mathscr{M}$.

Let G be a group of automorphisms of an orbifold $\mathscr{N}$ which admits the structure of a Lie group. The group G is called a *Lie group of transformations* of $\mathscr{N}$ if the map $\Phi : G \times \mathscr{N} \to \mathscr{N}$ with $(f, x) \mapsto f(x)$ is a smooth map of the product orbifold $G \times \mathscr{N}$ into $\mathscr{N}$. If the map $\Pi = (\Phi, \mathrm{id}_{\mathscr{N}}) : G \times \mathscr{N} \to \mathscr{N} \times \mathscr{N}$ with $(f, x) \mapsto (f(x), x)$ is proper, which means that the preimage $\Pi^{-1}(K)$ of each compact subset K in $\mathscr{N} \times \mathscr{N}$ is compact; then we will say that the action Φ of G on $\mathscr{N}$ is *proper.*

Proof of Theorem 1. Let $\pi : \mathscr{R} \to \mathscr{N}$ be the bundle of orthonormal frames over an n-dimensional orbifold $(\mathscr{N}, g)$, and let $o(n,\mathbb{R})$ be the Lie algebra of the Lie group $O(n,\mathbb{R})$. Define the two 1-forms θ and ω on the manifold $\mathscr{R}$ with values in $\mathbb{R}^n$ and $o(n,\mathbb{R})$ respectively as follows: Take $z' \in \mathscr{R}$, $x := \pi(z')$, $X_{z'} \in T_{z'}\mathscr{R}$, and some chart $(\Omega_i, \Gamma_i, p_i) \in \mathscr{A}$ with coordinate neighborhood $U_i \ni x$. Denote by ω_i the form of the Riemannian connection and by θ_i, the canonical form [17] on the bundle of orthonormal frames $\pi_i : \mathscr{R}_i \to \Omega_i$. Take the maps $\bar{p}_i : \mathscr{R}_i \to \mathscr{R}_i/\Gamma_i = \overline{\mathscr{R}}_i$ and $q_i : \overline{\mathscr{R}}_i \to \mathscr{R}$ defined in Section 2, which are respectively a regular covering and a diffeomorphism onto the image. Put $\theta_{z'}(X_{z'}) := (\theta_i)_z(\overline{X}_z)$ and $\omega_{z'}(X_{z'}) := (\omega_i)_z(\overline{X}_z)$, where $z \in (q_i \circ \bar{p}_i)^{-1}(z')$ and $\overline{X}_z \in T_z\mathscr{R}_i$ is some tangent vector satisfying the equality $(q_i \circ \bar{p}_i)_{*z}(\overline{X}_z) = X_{z'}$. Since for all $\gamma \in \Gamma_i$ the automorphism $b_i(\gamma)$ of the fiber bundle $\mathscr{R}_i$ preserves ω_i and θ_i, and for each injection of charts $\phi_{ij} : \Omega_i \to \Omega_j$, $i, j \in J$, we have the equalities $(\bar{\phi}_{ij})^*\omega_i = \omega_j$ and $(\bar{\phi}_{ij})^*\theta_i = \theta_j$, it follows that the forms θ and ω are well-defined. Fix some Euclidean scalar products

d_0 and d_1 on the vector spaces $\mathbb{R}^n$ and $o(n,\mathbb{R})$ respectively; then, the formula

$$d(X,Y) := d_0(\theta(X),\theta(Y)) + d_1(\omega(X),\omega(Y)),$$

where X and Y are smooth vector fields on the manifold $\mathscr{R}$, defines a Riemannian metric on $\mathscr{R}$.

Each isometry $f \in \mathfrak{I}(\mathscr{N})$ induces an isometry $\hat{f}$ of the Riemannian manifold $(\mathscr{R}, d)$ as follows. Take $z' \in \mathscr{R}$ and $x := \pi(z')$. Since f is an automorphism of $\mathscr{N}$, it follows that for $x \in \mathscr{N}$ there exist charts $(\Omega_i, \Gamma_i, p_i)$ and $(\Omega_j, \Gamma_j, p_j)$ in $\mathscr{A}$ with such coordinate neighborhoods U_i and U_j that $x \in U_i$ and $f(U_i) = U_j$. Since according to the definition of f its local lift $f_{ij} : \Omega_i \to \Omega_j$ is an isometry of the Riemannian manifolds (Ω_i, g_i) and (Ω_j, g_j), the diffeomorphism $\hat{f}_{ij} : \mathscr{R}_i \to \mathscr{R}_j : z \mapsto (f_{ij})_* \circ z$, $z \in \mathscr{R}_i$, is an isomorphism of fiber bundles satisfying the equalities $(\hat{f}_{ij})^*\omega_j = \omega_i$ and $(\hat{f}_{ij})^*\theta_j = \theta_i$. It is easy to verify that the formula $\hat{f}(z') := q_j \circ \bar{p}_j \circ \hat{f}_{ij}(z)$, where $z \in (q_i \circ \bar{p}_i)^{-1}(z')$, determines a diffeomorphism $\hat{f} : \mathscr{R} \to \mathscr{R}$ of the manifold $\mathscr{R}$ onto itself possessing the properties

$$\hat{f}^*\theta = \theta, \quad \hat{f}^*\omega = \omega, \quad \pi \circ \hat{f} = f \circ \pi. \tag{1}$$

The first two equalities in (1) imply that $\hat{f}^*d = d$; i.e., $\hat{f}$ is an isometry of the Riemannian manifold $(\mathscr{R}, d)$. By the Myers–Steenrod theorem [11, 17] the isometry group $\mathfrak{I}(\mathscr{R})$ of the Riemannian manifold $(\mathscr{R}, d)$ endowed with the compact-open topology is a Lie group of transformations. Note that the isometry $\hat{f} \in \mathfrak{I}(\mathscr{R})$ is induced by the isometry $f \in \mathfrak{I}(\mathscr{N})$ if and only if $\hat{f}$ satisfies the equalities in (1). Owing to that, the set $\mathfrak{K}$ of all such isometries $\hat{f}$ is a closed subset of $\mathfrak{I}(\mathscr{R})$; consequently, $\mathfrak{K}$ admits the structure of a Lie group that makes it into a closed Lie subgroup of the Lie group $\mathfrak{I}(\mathscr{R})$. Since $\pi \circ \hat{f} = f \circ \pi$, the equality $\hat{f} = \mathrm{id}_{\mathscr{R}}$ yields $f = \mathrm{id}_{\mathscr{N}}$. This defines the group isomorphism $\sigma : \mathfrak{I}(\mathscr{N}) \to \mathfrak{K} : f \mapsto \hat{f}$. The bijection σ induces on $\mathfrak{I}(\mathscr{N})$ the structure of a smooth manifold. Since σ is a group isomorphism, with respect to the induced smooth structure $\mathfrak{I}(\mathscr{N})$ is a Lie group.

The action $\widehat{\Psi} : \mathfrak{K} \times \mathscr{R} \to \mathscr{R} : (h, z) \mapsto h(z)$ of the Lie group $\mathfrak{K}$ on $\mathscr{R}$ is smooth because it is a restriction of a smooth action of the Lie group $\mathfrak{I}(\mathscr{R})$ on $\mathscr{R}$. Define a map $\Psi : \mathfrak{I}(\mathscr{N}) \times \mathscr{N} \to \mathscr{N}$ by the rule $\Psi(f, x) := f(x)$ for all $f \in \mathfrak{I}(\mathscr{N})$ and $x \in \mathscr{N}$. Then the smoothness of π, σ, $\widehat{\Psi}$ and the equality $\pi \circ \widehat{\Psi} = \Psi \circ (\sigma \times \pi)$ imply the smoothness of Ψ.

Every closed subgroup of the isometry Lie group of a Riemannian manifold acts properly [18]; thus, the action $\widehat{\Psi}$ of $\mathfrak{K}$ on $\mathscr{R}$ is proper. Since $\pi : \mathscr{R} \to \mathscr{N} = \mathscr{R}/O(n,\mathbb{R})$ is the quotient map onto the orbit space of the compact group $O(n,\mathbb{R})$, it is proper by Theorem 3.1 of Chapter I in [19]. Since π and $\widehat{\Psi}$ are proper continuous maps, the equality $\pi \circ \widehat{\Psi} = \Psi \circ (\sigma \times \pi)$ implies that Ψ is also proper; i.e., the action Ψ of the group $\mathfrak{I}(\mathscr{N})$ on $\mathscr{N}$ is proper.

Take orthonormal bases $\{e_m \mid m = 1, \dots, n\}$ and $\{E_{kl} \mid 1 \le k \le l \le n\}$ in the Euclidean spaces $(\mathbb{R}^n, d_0)$ and $(o(n,\mathbb{R}), d_1)$ respectively. Since ω is a smooth form on the manifold $\mathscr{R}$, the correspondence $\mathfrak{M} : z \mapsto \mathfrak{M}_z := \ker \omega_z$, $z \in \mathscr{R}$, determines a smooth distribution $\mathfrak{M}$ on $\mathscr{R}$. It is transversal to the smooth distribution $\mathfrak{V}$ tangent to the orbits of $O(n,\mathbb{R})$, and $\mathfrak{M}_z \oplus \mathfrak{V}_z = T_z\mathscr{R}$ for $z \in \mathscr{R}$. Note that $\mathfrak{V}_z = \ker \theta_z$, and the restrictions $\theta|_{\mathfrak{M}_z} : \mathfrak{M}_z \to \mathbb{R}^n$ and $\omega|_{\mathfrak{V}_z} : \mathfrak{V}_z \to o(n,\mathbb{R})$ are vector space isomorphisms. The smooth vector fields $\{Z_{(m)}, Z_{(kl)} \mid m = 1, \dots, n, 1 \le k \le l \le n\}$, where $(Z_{(m)})_z := (\theta_z|_{\mathfrak{M}_z})^{-1}(e_m)$ and $(Z_{(kl)})_z := (\omega_z|_{\mathfrak{V}_z})^{-1}(E_{kl})$ for $z \in \mathscr{R}$, determine a basis of the tangent vector space $T_z\mathscr{R}$ at each point z of $\mathscr{R}$. This means that the family $\{Z_{(m)}, Z_{(kl)}\}$ defines an absolute parallelism on $\mathscr{R}$. Since each isometry h in $\mathfrak{K}$ preserves the forms θ and ω, it follows that h also preserves the absolute parallelism $\{Z_{(m)}, Z_{(kl)}\}$. Note that the automorphism group $\mathfrak{A}(\mathscr{R})$ of the absolute parallelism of $\mathscr{R}$ is a closed Lie subgroup of the isometry Lie group $\mathfrak{I}(\mathscr{R})$ of the Riemannian manifold $(\mathscr{R}, d)$. It is known that the group $\mathfrak{A}(\mathscr{R})$ acts on $\mathscr{R}$ freely and that $\dim \mathfrak{A}(\mathscr{R}) \le \dim \mathscr{R} = n(n+1)/2$. Thus, the dimension of the closed Lie subgroup $\mathfrak{K}$ of the Lie group $\mathfrak{A}(\mathscr{R})$ satisfies the inequality $\dim \mathfrak{K} \le n(n+1)/2$, which implies that $\dim \mathfrak{I}(\mathscr{N}) = \dim \mathfrak{K} \le n(n+1)/2$. Therefore, the action of the Lie group $\mathfrak{K}$ on $\mathscr{R}$ is proper and free; consequently, each orbit $\widehat{\Psi}(\mathfrak{K}, z)$, $z \in \mathscr{R}$, is a closed embedded submanifold of $\mathscr{R}$ diffeomorphic to $\mathfrak{K}$, and the orbit $\hat{\Psi}(\mathscr{R}, z)$ is not connected in general. Hence, in the case where $\dim \mathfrak{I}(\mathscr{N}) = \dim \mathfrak{K} = n(n+1)/2 =$

$\dim \mathscr{R}$, each orbit $\widehat{\Psi}(\mathfrak{K}, z)$, $z \in \mathscr{R}$, of $\mathfrak{K}$ is open in $\mathscr{R}$. Because $\mathscr{R}$ is connected, the orbit $\widehat{\Psi}(\mathfrak{K}, z)$ coincides with $\mathscr{R}$; i.e., the group $\mathfrak{K}$ acts on $\mathscr{R}$ transitively. Consequently, the group $\mathfrak{I}(\mathscr{N})$ acts transitively on $\mathscr{N}$, which is only possible in the case that $\mathscr{N}$ is a manifold. It is known [17, Remark 10, Theorem 1] that for an n-dimensional Riemannian manifold $(\mathscr{N}, g)$ the equality $\dim \mathfrak{I}(\mathscr{N}) = n(n+1)/2$ is attained if and only if $(\mathscr{N}, g)$ is isometric to one of the following n-dimensional Riemannian manifolds of constant curvature: (a) $\mathbb{E}^n$; (b) S^n; (c) $\mathbb{R}\mathrm{P}^n$; (d) $\mathbb{H}^n$, where S^n and $\mathbb{R}\mathrm{P}^n$ are considered with some positive constant curvature.

In order to verify that the topology τ of the Lie group $\mathfrak{I}(\mathscr{N})$ coincides with the compact-open topology τ^{co}, we will use the following assertion:

Lemma 1. *Let $\mathscr{N}$ be a Riemannian orbifold with metric g, and let Δ_n be the manifold of its regular points with the induced metric $g|_{\Delta_n}$. Let $\mathfrak{I}(\Delta_n)$ be the isometry Lie group of the Riemannian manifold $(\Delta_n, g|_{\Delta_n})$ with the compact-open topology and let $\mathfrak{I}(\mathscr{N})$ be the isometry group of $(\mathscr{N}, g)$. Then there is a map $\nu : \mathfrak{I}(\mathscr{N}) \to \mathfrak{I}(\Delta_n) : f \mapsto f|_{\Delta_n}$ inducing an isomorphism of the algebraic group $\mathfrak{I}(\mathscr{N})$ onto the closed Lie subgroup $\operatorname{im} \nu$ of $\mathfrak{I}(\Delta_n)$.*

Proof. Since each isometry $f \in \mathfrak{I}(\mathscr{N})$ is an automorphism of $\mathscr{N}$; therefore, f necessarily takes each regular point into a regular point: $f(\Delta_n) = \Delta_n$. Consequently, there is a map $\nu : \mathfrak{I}(\mathscr{N}) \to \mathfrak{I}(\Delta_n) : f \mapsto f|_{\Delta_n}$. The Riemannian metric g naturally induces the Riemannian metric $g|_{\Delta_n}$ on the manifold Δ_n of regular points. By the Myers–Steenrod theorem the isometry group $\mathfrak{I}(\Delta_n)$ of the Riemannian manifold $(\Delta_n, g|_{\Delta_n})$ endowed with the compact-open topology is a Lie group. Show that the image $\operatorname{im} \nu$ is a closed subgroup of $\mathfrak{I}(\Delta_n)$.

Take a sequence $\{h_n\}$ of isometries in $\operatorname{im} \nu$ converging to $h \in \mathfrak{I}(\Delta_n)$. Then there exists some sequence $\{f_n\} \subset \mathfrak{I}(\mathscr{N})$ such that $f_n|_{\Delta_n} = h_n$. Pick some point $x \in \Delta_n$; then $y := h(x) \in \Delta_n$. There exists some coordinate neighborhood V of y in the manifold Δ_n whose closure $\overline{V}$ is compact in $\mathscr{N}$. Without loss of generality we may assume that $f_n(x) = h_n(x) \in \overline{V}$ for all $n \in \mathbb{N}$. It was shown above that the action $\Psi : \mathfrak{I}(\mathscr{N}) \times \mathscr{N} \to \mathscr{N}$ is proper; thus, the sequence $\{f_n\}$ belongs to the compact subset $\mathrm{pr}_1 \circ \Psi^{-1}(\overline{V})$ of $\mathfrak{I}(\mathscr{N})$, where $\mathrm{pr}_1 : \mathfrak{I}(\mathscr{N}) \times \mathscr{N} \to \mathfrak{I}(\mathscr{N})$ is the projection to the first factor. Consequently, the sequence $\{f_n\}$ includes a converging subsequence $\{f_{n_k}\}$. Suppose that $f_{n_k} \to f$ as $k \to \infty$. On the other hand, $f_{n_k}|_{\Delta_n} \to h$. Because Δ_n is Hausdorff, the limit of the sequence $\{f_{n_k}(y)\}$ is unique for all $y \in \Delta_n$. Hence, $f(y) = h(y)$; i.e., $f|_{\Delta_n} = h \in \operatorname{im} \nu$. Therefore, $\operatorname{im} \nu$ is a closed Lie subgroup in the isometry Lie group $\mathfrak{I}(\Delta_n)$, and so Lemma 1 is proved.

The group isomorphism $\nu^{-1}|_{\operatorname{im} \nu} : \operatorname{im} \nu \to \mathfrak{I}(\mathscr{N})$ defines on $\mathfrak{I}(\mathscr{N})$ some topology τ^0 and the structure of a smooth manifold, with respect to which $\mathfrak{I}(\mathscr{N})$ is a Lie group. Since the topology on the Lie group $\mathfrak{I}(\Delta_n)$ is compact-open, we have the subbasis of τ^0 consisting of the sets of the form

$$W(V, V') := \{f \in \mathfrak{I}(\mathscr{N}) \mid f(V) \subset V'\},$$

where V is compact and V' is an open subset of Δ_n. Since Δ_n is an open subset of $\mathscr{N}$, we obtain the inclusion $\tau^0 \subset \tau^{co}$, where τ^{co} is the compact-open topology on $\mathfrak{I}(\mathscr{N})$. The Lie group $(\mathfrak{I}(\mathscr{N}), \tau)$ acts on $\mathscr{N}$ smoothly and, hence, continuously. By Theorem 2 in [20] the topology of each group of homeomorphisms of a Hausdorff locally compact topological space X acting continuously on X contains all subsets open in the compact-open topology. This implies that $\tau^{co} \subset \tau$. Therefore, $\tau^0 \subset \tau$. Thus, the identity map $\mathrm{id}_{\mathfrak{I}(\mathscr{N})} : (\mathfrak{I}(\mathscr{N}), \tau) \to (\mathfrak{I}(\mathscr{N}), \tau^0)$ is a continuous isomorphism of Lie groups. It is known that each continuous homomorphism of Lie groups is an analytic map. Consequently, $\mathrm{id}_{\mathfrak{I}(\mathscr{N})}$ is an analytic map. According to [21, p. 21] each bijective (smooth) homomorphism of Lie groups is an isomorphism of Lie groups; thus, $\mathrm{id}_{\mathfrak{I}(\mathscr{N})}$ is an isomorphism of Lie groups and $\tau^0 = \tau$. Hence, the inclusions $\tau^0 \subset \tau^{co} \subset \tau$ yield $\tau^0 = \tau^{co} = \tau$. □

Corollary 1. *If a Riemannian orbifold is compact then so is its isometry group.*

Proof. As mentioned in the proof of Theorem 1, the map $\pi : \mathscr{R} \to \mathscr{N}$ is proper. As the preimage of the compact topological space $\mathscr{N}$ under the proper map π, the manifold $\mathscr{R}$ is compact. Since the isometry group $\mathfrak{I}(\mathscr{R})$ of the compact Riemannian manifold $(\mathscr{R}, d)$ is compact [17, Chapter VI, Theorem 3.4], its closed subgroup $\mathfrak{K}$, as well as the group $\mathfrak{I}(\mathscr{N})$ isomorphic to the latter, is compact too. □

5. Uniqueness of the Lie Group Structure

Proposition 1. *Let G be an algebraic group, let $\mathscr{M}$ be a Riemannian manifold, and let $\mathfrak{I}(\mathscr{M})$ be the isometry group of $\mathscr{M}$ endowed with the compact-open topology. If there exists a group isomorphism of G onto a closed subgroup of $\mathfrak{I}(\mathscr{M})$ then G admits a unique smooth structure that makes it into a Lie group.*

Proof. Consider the compact-open topology τ^{co} on the isometry group $\mathfrak{I}(\mathscr{M})$ of a Riemannian manifold $\mathscr{M}$. By the Myers–Steenrod theorem the topological space $(\mathfrak{I}(\mathscr{M}), \tau^{co})$ possesses the structure of a smooth manifold with respect to which $\mathfrak{I}(\mathscr{M})$ is a Lie group. It is known that each closed subgroup G of $\mathfrak{I}(\mathscr{M})$ with the induced topology is a Lie group. Suppose that there exist another topology τ on G and the structure of a smooth manifold on the topological space (G, τ) with respect to which G is a Lie group. In order to distinguish this Lie group from the previous, denote it by G_1. Since the topological space (G, τ) of the manifold G_1 is locally compact and G acts on $\mathscr{M}$ effectively by the Montgomery–Zippin theorem [22, pp. 208, 212], also see [17, Chapter I, Theorem 4.6], on (G, τ) there is the structure of a Lie group G_2 of transformations. Since at most one structure of a Lie group may exist on a second countable locally Euclidean topological group, $G_1 = G_2$. Therefore, G_1 acts smoothly, thus continuously, on $\mathscr{M}$. By Theorem 2 in [20] the topology τ of the Lie group G_1 contains all subsets that are open in the compact-open topology. Hence, $\tau^{co} \subset \tau$, and so the identity map $\mathrm{id}_G : (G, \tau) \to (G, \tau^{co})$ is continuous. Since id_G is a continuous Lie group isomorphism, as in the proof of Theorem 1 we deduce that id_G is a Lie group isomorphism, and $G = G_1$. □

Corollary 2. *The isometry group $\mathfrak{I}(\mathscr{M})$ of a Riemannian manifold $\mathscr{M}$ admits a unique Lie group structure, and the topology on the Lie group $\mathfrak{I}(\mathscr{M})$ coincides with the compact-open topology.*

Corollary 3. *If a Lie group G acts effectively, smoothly, and properly on some manifold $\mathscr{M}$ then the algebraic group G admits no other Lie group structure.*

Proof. It is known that for each smooth proper action $\Phi : G \times \mathscr{M} \to \mathscr{M}$ of a Lie group G on some manifold $\mathscr{M}$ there exists a Riemannian metric on $\mathscr{M}$ with respect to which each transformation $\Phi_h := \Phi(h, \cdot)$, $h \in G$, is an isometry. Since Φ is an effective action, the isometry group $\widetilde{G} := \{\Phi_h \mid h \in G\}$ is isomorphic to G. Since Φ is a proper action, it follows that the subgroup $\widetilde{G}$ is closed in the isometry Lie group $\mathfrak{I}(\mathscr{M})$ of $\mathscr{M}$ endowed with the compact-open topology.

Therefore, G is isomorphic to a closed subgroup $\widetilde{G}$ of the isometry Lie group $\mathfrak{I}(\mathscr{M})$ of $\mathscr{M}$; thus, Proposition 1 implies the required claim. □

Corollary 4. *If G is a compact Lie group then the algebraic group G admits no other Lie group structure.*

Proof. The left action $\Phi : G \times G \to G : (h_1, h_2) \mapsto h_1 h_2$ of the Lie group G on G is smooth and effective. Because G is compact, the action Φ is proper; thus, the claim follows from Corollary 3. □

Proof of Theorem 2. Take some Riemannian orbifold $\mathscr{N}$ with metric g. By Lemma 1 there exists an isomorphism of the isometry group $\mathfrak{I}(\mathscr{N})$ of the Riemannian orbifold $\mathscr{N}$ onto a closed subgroup of the isometry Lie group $\mathfrak{I}(\Delta_n)$ of the Riemannian manifold $(\Delta_n, g|_{\Delta_n})$ endowed with the compact-open topology. Thus, the claim of Theorem 2 follows from Proposition 1. □

6. Coverings of Orbifolds

A smooth map of orbifolds $\kappa : \mathscr{N}' \to \mathscr{N}$ is called a *covering* [7] if for each point $x \in \mathscr{N}$ there is a chart (Ω, Γ, p) with coordinate neighborhood $U \ni x$ such that for each connected component U' of $\kappa^{-1}(U)$ there exists a homeomorphism $q' : \Omega/\Gamma' \to U'$ such that $\kappa|_{U'} \circ p' = p$, where Γ' is some subgroup of the group Γ, and $p' : \Omega \to U'$ is the composition of q' with the quotient map $\Omega \to \Omega/\Gamma'$. The chart (Ω, Γ, p) in this definition is said to be *regularly covered*.

Recall that a *deck transformation* of a covering $\kappa : \mathscr{N}' \to \mathscr{N}$ is an automorphism $f : \mathscr{N}' \to \mathscr{N}'$ of the covering orbifold $\mathscr{N}'$ such that $\kappa \circ f = \kappa$. The set $G(\kappa)$ of all deck transformations of the covering $\kappa : \mathscr{N}' \to \mathscr{N}$ forms a group. The covering $\kappa : \mathscr{N}' \to \mathscr{N}$ is called *regular* if $\mathscr{N} = \mathscr{N}'/G(\kappa)$. In the case that $G(\kappa)$ is isomorphic to $\mathbb{Z}_2$ we call the regular covering $\kappa : \mathscr{N}' \to \mathscr{N}$ *two-sheeted*. An orbifold $\mathscr{N}$ is *good* if there exists some regular covering $\kappa : \mathscr{N}' \to \mathscr{N}$, where $\mathscr{N}'$ is a smooth manifold.

Take some covering $\kappa : \mathscr{N}' \to \mathscr{N}$ of an orbifold $(\mathscr{N}, \mathscr{A})$ by an orbifold $(\mathscr{N}', \mathscr{A}')$ and some tensor t of type (p, q) on $\mathscr{N}$. Recall that if $\mathscr{A} = \{(\Omega_i, \Gamma_i, p_i) \mid i \in J\}$ is the maximal atlas of $\mathscr{N}$ then the definition of t implies that for each $i \in J$ a tensor t_i of type (p, q) is defined on Ω_i. The tensor t induces some tensor t' of type (p, q) on $\mathscr{N}'$ as follows: Pick a point x' of $\mathscr{N}'$. Then for the point $x := \kappa(x')$ there is some regularly covered chart $(\Omega_i, \Gamma_i, p_i) \in \mathscr{A}$ with coordinate neighborhood $U_i \ni x$. Denote by U' the connected component of $\kappa^{-1}(U_i)$ containing x'. By the definition of a covering there exist a subgroup Γ' of Γ_i and a homeomorphism $q' : \Omega_i/\Gamma' \to U'$ such that $\kappa|_{U'} \circ p' = p$, where $p' : \Omega_i \to U'$ is the composition of q' with the quotient map $\Omega_i \to \Omega_i/\Gamma'$.

Note that $(\Omega_i, \Gamma', p') \in \mathscr{A}'$ is a chart with coordinate neighborhood U'. Since $\Gamma' \subset \Gamma_i$ it follows that t_i is a Γ'-invariant tensor of type (p, q) of the manifold Ω_i, which we denote by $t_{x'}$. Consequently, for each point $x' \in \mathscr{N}'$ there are a chart $(\Omega_i, \Gamma', p') \in \mathscr{A}'$ with coordinate neighborhood $U' \ni x'$ and a tensor $t_{x'}$ of type (p, q) on Ω_i. Because the tensors in $t = \{t_i\}_{i \in J}$ are compatible, the family of tensors $\{t_{x'}\}_{x' \in \mathscr{N}'}$ correctly defines a tensor t' of type (p, q) on the orbifold $\mathscr{N}'$.

Therefore, we have proved the following statement:

Lemma 2. *Given some covering $\kappa : \mathscr{N}' \to \mathscr{N}$ of an orbifold $\mathscr{N}$, each tensor t of type (p, q) on $\mathscr{N}$ naturally induces a tensor t' of type (p, q) on $\mathscr{N}'$.*

Proposition 2. *Let $\kappa : \mathscr{N}' \to \mathscr{N}$ be a regular covering of a Riemannian orbifold $\mathscr{N}$ by an orbifold $\mathscr{N}'$ with the deck transformation group Γ. Then:* (a) *the orbifold $\mathscr{N}'$ is equipped with an induced Riemannian metric with respect to which Γ is a subgroup of the isometry group $\mathfrak{I}(\mathscr{N}')$;* (b) *the group $\mathfrak{I}(\mathscr{N})$ is isomorphic to the quotient group $\mathbf{N}(\Gamma)/\Gamma$ of the normalizer $\mathbf{N}(\Gamma)$ of Γ in the isometry group $\mathfrak{I}(\mathscr{N}')$.*

Proof. The first claim follows from Lemma 2. Define a group homomorphism $\chi : \mathbf{N}(\Gamma) \to \mathfrak{I}(\mathscr{N}) : f' \mapsto f$ by the equality $f(x) := \kappa \circ f'(y)$ for $y \in \kappa^{-1}(x)$ and $x \in \mathscr{N}$. The definitions of the deck transformation group Γ and the homomorphism χ imply that $\ker \chi$ coincides with Γ. Note that for each isometry $f \in \mathfrak{I}(\mathscr{N})$ there exists some isometry $f' \in \mathfrak{I}(\mathscr{N}')$ covering f; i.e., $\kappa \circ f' = f \circ \kappa$. The covering isometry f' takes each orbit of the action of Γ into another orbit, $f'(\Gamma(x)) = \Gamma(f'(x))$, $x \in \mathscr{N}'$. This implies that $f'\Gamma f'^{-1} = \Gamma$; i.e., $f' \in \mathbf{N}(\Gamma)$. Hence, χ is surjective. Since Γ is a closed discrete subgroup of $\mathfrak{I}(\mathscr{N}')$, the normalizer $\mathbf{N}(\Gamma)$ is a closed subgroup of $\mathfrak{I}(\mathscr{N}')$. Consequently, $\mathbf{N}(\Gamma)$ is a closed Lie subgroup of the Lie group $\mathfrak{I}(\mathscr{N}')$. Therefore, $\mathfrak{I}(\mathscr{N})$ is isomorphic to the quotient Lie group $\mathbf{N}(\Gamma)/\Gamma$. □

Proposition 3. *Each nonorientable orbifold $\mathscr{N}$ possesses some regular two-sheeted covering $\kappa : \mathscr{N}' \to \mathscr{N}$ by an orientable orbifold $\mathscr{N}'$. In addition, if $\mathscr{N}$ is a Riemannian orbifold then a Riemannian metric is induced on $\mathscr{N}'$ with respect to which the deck transformation group $\Gamma \cong \mathbb{Z}_2$ is a group of isometries, and the isometry group $\mathfrak{I}(\mathscr{N})$ of the Riemannian orbifold $\mathscr{N}$ is isomorphic to the quotient group $\mathbf{N}(\Gamma)/\Gamma$ of the normalizer $\mathbf{N}(\Gamma)$ of Γ in the isometry group $\mathfrak{I}(\mathscr{N}')$ of the orientable Riemannian orbifold $\mathscr{N}'$.*

Proof. Let $\mathscr{N}$ be a nonorientable orbifold, let g be a Riemannian metric on $\mathscr{N}$, and let $\pi : \mathscr{R} \to \mathscr{N}$ be the bundle of orthonormal frames over the Riemannian orbifold $(\mathscr{N}, g)$. As mentioned above, the manifold $\mathscr{R}$ carries a smooth right action $\Upsilon : \mathscr{R} \times O(n, \mathbb{R}) \to \mathscr{R}$ of the orthogonal group $O(n, \mathbb{R})$ whose orbit space $\mathscr{R}/O(n, \mathbb{R})$ coincides with $\mathscr{N}$. The restriction $\widetilde{\Upsilon} := \Upsilon|_{\mathscr{R} \times SO(n,\mathbb{R})} : \mathscr{R} \times SO(n, \mathbb{R}) \to \mathscr{R}$ is a smooth action of the special orthogonal group $SO(n, \mathbb{R})$ on $\mathscr{R}$. All stationary subgroups of this action are finite. Its orbit space $\mathscr{N}' := \mathscr{R}/SO(n, \mathbb{R})$ naturally possesses the structure of a smooth n-dimensional orbifold with respect to which the map $\kappa : \mathscr{N}' \to \mathscr{N}$, taking each orbit $\widetilde{\Upsilon}(z, SO(n, \mathbb{R}))$, $z \in \mathscr{R}$, into the orbit $\Upsilon(z, O(n, \mathbb{R}))$, is a regular covering with the deck transformation group $\Gamma \cong \mathbb{Z}_2$. Since $\mathscr{R}$ admits an absolute parallelism, $\mathscr{R}$ is orientable. Since each transformation $\widetilde{\Upsilon}(\,\cdot\,, h)$, $h \in SO(n, \mathbb{R})$, preserves the orientation on $\mathscr{R}$, the orbifold $\mathscr{N}' := \mathscr{R}/SO(n, \mathbb{R})$ is orientable.

The second claim follows from Proposition 2. □

7. An Analog of Bochner's Theorem

Integration on orbifolds. Let $\alpha = \{\alpha_i\}_{i \in J}$ be an exterior form on an orbifold $(\mathscr{N}, \mathscr{A})$, where $\mathscr{A} = \{(\Omega_i, \Gamma_i, p_i) \mid i \in J\}$ is the maximal atlas. The closure of the set of those points of $\mathscr{N}$ where α does not vanish is called the *support* of α and is denoted by $\operatorname{supp}\alpha$.

Let $\mathscr{N}$ be an oriented n-dimensional orbifold, and let α be an exterior n-form with compact support. If $\operatorname{supp}\alpha$ lies inside the coordinate neighborhood U_i of some chart $(\Omega_i, \Gamma_i, p_i) \in \mathscr{A}$ then by definition $\int_{U_i} \alpha := \frac{1}{|\Gamma_i|} \int_{\Omega_i} \alpha_i$, where $|\Gamma_i|$ is the order of Γ_i. In general the compactness of $\operatorname{supp}\alpha$ yields the existence of a finite covering $\mathscr{W} = \{W_k\}_{k=1,\dots,m}$ of the support α by charts in the atlas $\mathscr{A}$ with the coordinate neighborhoods W_k and a finite partition of unity subordinate to $\mathscr{W}$; i.e., a family $\{f_k\}_{k=1,\dots,m}$ of smooth functions on $\mathscr{N}$ such that: (a) $0 \le f_k(x) \le 1$ for all $x \in \mathscr{N}$ and $k \in \{1, \dots, m\}$; (b) $\operatorname{supp} f_k \subset W_k$ for all $k \in \{1, \dots, m\}$; (c) $\sum_{k=1}^m f_k(x) = 1$ for all $x \in \operatorname{supp}\alpha$. The integral of an exterior n-form with compact support α over the orbifold $\mathscr{N}$ is defined by the equality

$$\int_{\mathscr{N}} \alpha := \sum_{k=1}^{m} \int_{W_k} f_k\, \alpha. \tag{2}$$

The number $\int_{\mathscr{N}} \alpha$ determined by (2) is independent of the choice of a covering $\mathscr{W}$ and the subordinate partition of unity. Thus, the integral is well-defined. If $\mathscr{N}$ is compact, the support of each form α is compact. Consequently, the integral over an oriented n-dimensional compact orbifold $\mathscr{N}$ is defined for each exterior n-form α.

Let $\alpha = \{\alpha_i\}_{i \in J}$ be the volume form of an oriented n-dimensional compact Riemannian orbifold $(\mathscr{N}, g)$ determined by the metric tensor g and let $X = \{X_i\}_{i \in J}$ be a smooth vector field on $\mathscr{N}$. Then the family $\operatorname{div} X = \{\operatorname{div} X_i\}_{i \in J}$, where $\operatorname{div} X_i$ is the divergence of a vector field X_i on the Riemannian manifold (Ω_i, g_i), is a smooth function on $\mathscr{N}$. The Stokes's Formula [6, 14] for the orbifold $\mathscr{N}$ implies the equality

$$\int_{\mathscr{N}} (\operatorname{div} X)\alpha = 0. \tag{3}$$

Recall [17] that a smooth vector field Y on some n-dimensional Riemannian manifold $(\mathscr{M}, g)$ is called a *Killing field* (or an *infinitesimal isometry*) if the Lie derivative $L_Y g$ of the metric tensor g with respect to Y is identically zero. The *Ricci tensor* S on $(\mathscr{M}, g)$ is the tensor field of type $(0, 2)$ defined by the equality

$$S_x(X, Y) := \sum_{l=1}^{n} R_x(V_l, Y, V_l, X)$$

for all $X, Y \in T_x\mathscr{M}$ and all $x \in \mathscr{M}$, where R_x is the Riemann curvature tensor, and $\{V_l\}_{l=1,\dots,n}$ is some orthonormal frame at x.

A vector field $X = \{X_i\}_{i \in J}$ on some Riemannian orbifold $(\mathscr{N}, g)$ is called a *Killing field* if X_i is a Killing field on the Riemannian manifold (Ω_i, g_i) for each $i \in J$. Take the Ricci tensor S_i on the Riemannian manifold (Ω_i, g_i). The definition of the Riemannian metric $g = \{g_i\}_{i \in J}$ implies that the family $S = \{S_i\}_{i \in J}$ of tensors is a tensor of type $(0, 2)$ on the orbifold $\mathscr{N}$. The tensor S is called the *Ricci tensor* of $(\mathscr{N}, g)$.

Proof of Theorem 3. Suppose that the orbifold $\mathscr{N}$ is oriented. Take some Killing field $X = \{X_i\}_{i \in J}$ on the compact Riemannian orbifold $(\mathscr{N}, g)$. Then the equality

$$\operatorname{div}(A_{X_i} X_i) = -S_i(X_i, X_i) - \operatorname{trace}(A_{X_i} A_{X_i}),$$

where A_{X_i} is the Kobayashi operator, $A_{X_i} A_{X_i}$ is the composition of A_{X_i} with A_{X_i}, and S_i is the Ricci tensor of the Riemannian manifold (Ω_i, g_i) [17, Chapter VI, Proposition 5.1], yields

$$\int_{\mathscr{N}} \operatorname{div}(A_X X)\, \alpha = -\int_{\mathscr{N}} (S(X, X) + \operatorname{trace}(A_X A_X))\alpha. \tag{4}$$

Since the operator A_{X_i} is skew-symmetric on the Riemannian manifold (Ω_i, g_i) [17, Chapter VI, Proposition 3.2], it follows that $\operatorname{trace}(A_{X_i}A_{X_i}) \leq 0$. By the hypothesis of Theorem 3 the Ricci tensor S is nonpositive definite, and so $S_i(X_i, X_i) \leq 0$. Consequently, the function $S(X, X) + \operatorname{trace}(A_X A_X) = \{S_i(X_i, X_i) + \operatorname{trace}(A_{X_i}A_{X_i})\}_{i\in J}$ is nonpositive on $\mathscr{N}$. By (3), the left-hand side of (4) is identically zero. Hence, the inequalities $S_i(X_i, X_i) \leq 0$ and $\operatorname{trace}(A_{X_i}A_{X_i}) \leq 0$ yield $S_i(X_i, X_i) = 0$ and $\operatorname{trace}(A_{X_i}A_{X_i}) = 0$ for all $i \in J$.

Denote by ∇ the Levi-Civita connection on the Riemannian manifold $(\Delta_n, g|_{\Delta_n})$. Note that the restriction $\widetilde{X} := X|_{\Delta_n}$ of the vector field X is a smooth vector field on the manifold Δ_n. Since the torsion tensor T of the connection ∇ is equal to zero, the derivatives of arbitrary vector fields Y and Z on Δ_n satisfy [17, Chapter VI, Proposition 2.5] the equality $A_Y Z = -\nabla_Z Y$. The equality $\operatorname{trace}(A_{X_i}A_{X_i}) = 0$ implies that $A_{X_i} = 0$, and consequently, $A_{\widetilde{X}} = 0$. Therefore, $\nabla\widetilde{X} = 0$; i.e., the vector field $\widetilde{X}$ is absolutely parallel on Δ_n.

By the hypothesis of Theorem 3 there is a point $x \in \mathscr{N}$ at which the Ricci tensor S is negative definite. Hence, by the smoothness of S there is some neighborhood U of x in $\mathscr{N}$ on which S is negative definite. Since Δ_n is dense in $\mathscr{N}$, there is some point $y \in U \cap \Delta_n$. The vanishing of $S(X, X) = \{S_i(X_i, X_i)\}_{i\in J}$ at $y \in \Delta_n$ and the negative definiteness of S at y imply that the vector field $\widetilde{X}$ vanishes at y. Since X is a parallel Killing vector field on Δ_n, by the connectedness of Δ_n the vanishing of $\widetilde{X}$ at y implies the vanishing of $\widetilde{X}$ on Δ_n. Hence, for each chart $(\Omega_i, \Gamma_i, p_i) \in \mathscr{A}$ and its coordinate neighborhood U_i, because the vector field X_i on the manifold Ω_i is continuous, the vanishing of X_i on the dense subset $p_i^{-1}(U_i \cap \Delta_n)$ of Ω_i implies the vanishing of X_i everywhere on Ω_i. Consequently, every Killing vector field on an arbitrary oriented Riemannian orbifold $(\mathscr{N}, g)$ is identically zero.

Since all Killing vector fields on $(\mathscr{N}, g)$ are identically zero, the Lie algebra of the Lie group $\mathfrak{I}(\mathscr{N})$ is equal to zero. Thus, the group $\mathfrak{I}(\mathscr{N})$ is at most countable. By Proposition 1 the group $\mathfrak{I}(\mathscr{N})$ is compact, and so it must be finite.

Suppose now that the Riemannian orbifold $\mathscr{N}$ is nonorientable. By Proposition 3 there exists some two-sheeted covering $\kappa : \mathscr{N}' \to \mathscr{N}$ of the orbifold $\mathscr{N}$ by an orientable orbifold $\mathscr{N}'$. Assume that $\mathscr{N}'$ is oriented. Since $\mathscr{N}$ is compact; therefore, its two-sheeted covering orbifold $\mathscr{N}'$ is compact too. According to the proof of Lemma 2 the Riemannian metric is induced on $\mathscr{N}'$ so that the Ricci tensor S' is nonpositive definite on $\mathscr{N}'$, and at $x' \in \kappa^{-1}(x)$ the Ricci tensor S' is negative definite. We proved above that the isometry group $\mathfrak{I}(\mathscr{N}')$ of the oriented compact Riemannian orbifold $\mathscr{N}'$ is finite. By Proposition 3 the group $\mathfrak{I}(\mathscr{N})$ is isomorphic to the quotient group $\mathbf{N}(\Gamma)/\Gamma$, where $\mathbf{N}(\Gamma) \subset \mathfrak{I}(\mathscr{N}')$. Thus, the group $\mathfrak{I}(\mathscr{N})$ is also finite. $\square$

Remark 1. Since the Ricci tensor of a one-dimensional Riemannian manifold is identically zero, the Riemannian orbifolds satisfying Theorem 3 have dimension $n \geq 2$.

Corollary 5. *The isometry group of every compact Riemannian orbifold with a negative definite Ricci tensor is finite.*

Remark 2. In the case that the Riemannian orbifold is a Riemannian manifold the claim of Corollary 3 coincides with Bochner's theorem [12]; also see [17, Chapter VI, Corollary 5.4].

A Riemannian orbifold $(\mathscr{N}, g)$ is called [7] *hyperbolic* if for each $i \in J$ the Riemannian manifold (Ω_i, g_i) has constant negative curvature $\mathbf{k}$.

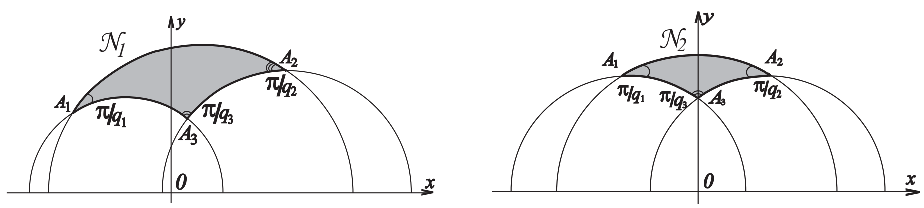
It is known that for each n-dimensional Riemannian manifold of constant curvature $\mathbf{k}$ we have the formula $S_{ab} = \mathbf{k}(n-1)g_{ab}$, $a, b = 1, \dots, n$, where g_{ab} and S_{ab} are the components of the metric tensor g and the Ricci tensor S in a local coordinate system. This equality implies that for $\mathbf{k} = \text{const} < 0$ the Ricci tensor is negative definite. Thus, if $\mathscr{N}$ is some hyperbolic orbifold, the Ricci tensor $S = \{S_i\}_{i\in J}$ of the Riemannian orbifold $\mathscr{N}$ is negative definite. Therefore, Corollary 5 implies the following assertion:

Corollary 6. *The isometry group of every compact hyperbolic orbifold is finite.*

8. Examples

Example 3. Let $\mathscr{N} = \mathbb{E}^3/\Gamma$ be the 3-dimensional orbifold of Example 2. Since Γ is a group of isometries of the Euclidean space $\mathbb{E}^3$, it follows that $\mathscr{N}$ is a flat Riemannian orbifold. By Proposition 2 the isometry group $\mathfrak{I}(\mathscr{N})$ is isomorphic to the quotient group $\mathbf{N}(\Gamma)/\Gamma$. The normalizer $\mathbf{N}(\Gamma) := \{f \in \mathfrak{I}(\mathbb{E}^3) \mid f \circ \Gamma = \Gamma \circ f\}$ consists of the isometries of $\mathbb{E}^3$ of the form $A = \begin{pmatrix} A' & 0 \\ 0 & a_{33} \end{pmatrix}$, where $A' \in O(2,\mathbb{R})$ and $a_{33} \in \{-1, 1\}$. Since $\mathfrak{I}(\mathscr{N}) \cong O(2,\mathbb{R})/G$, where $G \cong \mathbb{Z}_m$ is the subgroup of $O(2,\mathbb{R})$ generated by the rotation of the plane Oxy through angle $2\pi/m$, it follows that the Lie groups $\mathfrak{I}(\mathscr{N})$ and $O(2,\mathbb{R})$ are isomorphic.

Example 4. Consider the hyperbolic plane $\mathbb{H}^2$ realized as the upper half-plane with the coordinates $\{x, y\}$ and the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$, where $y > 0$. Choose positive integers q_i for $i = 1, 2, 3$ so that $\sum_{i=1}^{3} \frac{1}{q_i} < 1$. It is known [23] that for such q_i there exists a geodesic triangle T in $\mathbb{H}^2$ with angles π/q_i. The group Γ of the isometries of $\mathbb{H}^2$ generated by the reflections in the geodesics bounding T is a discrete group of isometries, and T is its fundamental domain. The quotient space $\mathscr{N}(q_i) := \mathbb{H}^2/\Gamma$ is a 2-dimensional compact hyperbolic orbifold which can be identified with T. The interior points of T are the regular points of $\mathscr{N}$, and the points on the boundary ∂T are singular. The orbifold group of the vertex $A_i \in T$ is isomorphic to the dihedral group of order q_i, and the orbifold groups of the points on the edges (excluding the vertices) are isomorphic to the reflection group $\mathbb{Z}_2$.



(a) (b)

Fig. 2. Compact Hyperbolic Orbifolds.

Case 1. Consider $\mathscr{N}_1 := \mathscr{N}(q_1, q_2, q_3)$, where q_i are pairwise distinct, as in Fig. 2(a).

Case 2. Consider $\mathscr{N}_2 := \mathscr{N}(q_1, q_2, q_3)$, where $q_1 = q_2 \neq q_3$, as in Fig. 2(b).

Since each isometry preserves angles between vectors, in case 1 the isometry group $\mathfrak{I}(\mathscr{N}_1)$ of the hyperbolic orbifold $\mathscr{N}_1$ is trivial, and in case 2 the unique nontrivial isometry of $\mathscr{N}_2$ is the restriction to $\mathscr{N}_2$ of the global isometry $f : \mathbb{H}^2 \to \mathbb{H}^2 : (x, y) \mapsto (-x, y)$ for all $(x, y) \in \mathbb{H}^2$. Therefore, the group $\mathfrak{I}(\mathscr{N}_2)$ is isomorphic to $\mathbb{Z}_2$, which agrees with Corollary 6.

The next example shows that the isometry group of a flat compact Riemannian orbifold can be finite.

Example 5. Take $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and the group Γ generated by the isometries $\gamma_1, \gamma_2, \gamma_3$ of the flat torus $T^3 = S^1 \times S^1 \times S^1 \subset \mathbb{C}^3$ defined by the equalities $\gamma_1(z_1, z_2, z_3) := (z_1, \bar{z}_2, -\bar{z}_3)$, $\gamma_2(z_1, z_2, z_3) := (-\bar{z}_1, z_2, \bar{z}_3)$, $\gamma_3(z_1, z_2, z_3) := (\bar{z}_1, -\bar{z}_2, z_3)$, $z_i \in S^1$, $i = 1, 2, 3$, where $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. Clearly, $\gamma_m^2 = \mathrm{id}_{T^3}$, $m = 1, 2, 3$, and $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The isometry γ_m leaves invariant the set $Q_{(m)}$ consisting of four circles; in particular, $Q_{(1)} := \{S^1 \times \{\pm 1\} \times \{\pm 1\}\}$. The quotient map $\pi : T^3 \to T^3/\Gamma$ takes the circles of the family $Q_{(m)}$ into one circle. The quotient space $\mathscr{N} := T^3/\Gamma$ is a 3-dimensional flat compact orbifold homeomorphic to the 3-dimensional sphere S^3, and the set Σ of singular points is the disjoint union of three linked circles known as the "Borromean rings" [7].

By Proposition 2 the isometry group $\mathfrak{I}(\mathscr{N})$ is isomorphic to the quotient group $\mathbf{N}(\Gamma)/\Gamma$ of the normalizer $\mathbf{N}(\Gamma)$ of Γ in the isometry group $\mathfrak{I}(T^3)$ of the torus T^3. It is not difficult to check that $\mathbf{N}(\Gamma)$ is a finite subgroup of $\mathfrak{I}(T^3)$. Thus, the isometry group $\mathfrak{I}(\mathscr{N}) \cong \mathbf{N}(\Gamma)/\Gamma$ is finite.

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