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ISOMORPHISM AND HAMILTON REPRESENTATION OF SOME NONHOLONOMIC SYSTEMS

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Abstract: We consider some questions connected with the Hamiltonian form of the two problems of nonholonomic mechanics: the Chaplygin ball problem and the Veselova problem. For these problems we find representations in the form of the generalized Chaplygin systems that can be integrated by the reducing multiplier method. We give a concrete algebraic form of the Poisson brackets which, together with an appropriate change of time, enable us to write down the equations of motion of the problems under study. Some generalization of these problems are considered and new ways of implementation of nonholonomic constraints are proposed. We list a series of nonholonomic systems possessing an invariant measure and sufficiently many first integrals for which the question about the Hamiltonian form remains open even after change of time. We prove a theorem on isomorphism of the dynamics of the Chaplygin ball and the motion of a body in a fluid in the Clebsch case.

Keywords: nonholonomic system, reducing multiplier, Hamiltonization, isomorphism

We start with considering some general results on the method for integration of nonholonomic systems which is named after S. A. Chaplygin [1] the *reducing multiplier method*. We modify this method in order to apply it to a broader class of the so-called generalized Chaplygin systems. In the next sections we apply these results and explicitly find the Poisson structure as well as isomorphisms with other integrable Hamiltonian systems.

1. Generalized Chaplygin Systems

Consider a mechanical system, with two degrees of freedom, whose equation of motion can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_1}\right) - \frac{\partial L}{\partial q_1} = \dot q_2 S, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_2}\right) - \frac{\partial L}{\partial q_2} = -\dot q_1 S, \qquad S = a_1(q)\dot q_1 + a_2(q)\dot q_2 + b(q), \tag{1}$$

where L is a function of the coordinates and velocities also called the *Lagrangian of the system.*

For a special choice of the function S we obtain the usual Chaplygin system (in this event, certainly $b(q) = 0$) [1]. As demonstrated by Chaplygin, the form (1) with $b(q) = 0$ is the form to which the so-called Chaplygin sledge equations can be reduced; these equations can be integrated by the reducing multiplier method to be presented below and solution of the Hamilton–Jacobi equation. As was demonstrated in [2], the Veselova problem is a Chaplygin system as well. Recall that the Veselova problem concerns a rigid body rotating around a fixed point subject to a nonintegrable constraint requiring that the projection of the angular velocity to a fixed axis is zero. As we will demonstrate later, the form (1) now with $b(q) \neq 0$ arises also in the problem on a dynamically asymmetric balanced Chaplygin rolling ball [3].

We will call (1) the *generalized Chaplygin system* (not to be confused with [4] proposing a somewhat different generalization of Chaplygin systems).

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As demonstrated [1] by Chaplygin, in the case $b(q) = 0$ the form of the equations is preserved under the changes of time

$$N(q)\,dt = d\tau,$$

if N is independent of the velocities. Let us show that this is also valid for (1).

Denoting differentiation with respect to the new time by $q_i' = \frac{dq_i}{d\tau}$, we find that

$$\dot{q}_i = Nq_i', \quad \frac{\partial L}{\partial \dot{q}_i} = \frac{1}{N}\frac{\partial \overline{L}}{\partial q_i'}, \quad \frac{\partial L}{\partial q_i} = \frac{\partial \overline{L}}{\partial q_i} - \frac{1}{N}\frac{\partial N}{\partial q_i}\sum_{k=1}^{2} q_k' \frac{\partial \overline{L}}{\partial q_k'},$$

where $\overline{L}(q, q') = L(q, Nq')$. Inserting this in (1), we obtain

$$\begin{gathered}\frac{d}{d\tau}\left(\frac{\partial \overline{L}}{\partial q_q'}\right) - \frac{\partial \overline{L}}{\partial q_1} = q_2'\overline{S}, \quad \frac{d}{d\tau}\left(\frac{\partial \overline{L}}{\partial q_2'}\right) - \frac{\partial \overline{L}}{\partial q_2} = -q_1'\overline{S},\\ \overline{S} = NS + \frac{1}{N}\left(\frac{\partial N}{\partial q_2}\frac{\partial \overline{L}}{\partial q_1'} - \frac{\partial N}{\partial q_1}\frac{\partial \overline{L}}{\partial q_2'}\right).\end{gathered} \tag{2}$$

For the usual Chaplygin system, it is well known [1] that if there is an invariant measure with density depending only on the coordinates then we can choose $N(q)$ so that $\overline{S} = 0$ and consequently write down the system with the new time τ in canonical Hamiltonian form. We expand this result to the case of generalized Chaplygin systems of the form (1) on assuming that the Lagrangian is a quadratic function of the velocities $\dot{q}_i$ (not necessarily homogeneous).

Theorem 1. *Suppose that*

$$\det\left\|\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right\| \neq 0$$

and (1) *admits an invariant measure with density depending only on the coordinates. Then there is a change of time* $N(q)\,dt = d\tau$ *such that*

(1) *the function* $\overline{S}$ *defined by* (2) *depends only on the coordinates*: $\overline{S} = \overline{S}(q)$,

(2) *the equations of motion in the new time are written in Hamiltonian form*

$$\frac{dq_i}{d\tau} = \{q_i, \overline{H}\}, \quad \frac{dp_i}{d\tau} = \{p_i, \overline{H}\},$$

where

$$p_i = \frac{\partial \overline{L}}{\partial q_i'}, \quad \overline{H} = \sum_{k=1}^{2} p_k q_k' - \overline{L}|_{q_i' \to p_i},$$

and the Poisson bracket is defined by the relations

$$\{q_i, p_j\} = \delta_{ij}, \quad \{p_1, p_2\} = \overline{S}(q), \quad \{q_1, q_2\} = 0. \tag{3}$$

Proof. Carry out the Legendre transform for the initial system (1):

$$P_i = \frac{\partial L}{\partial \dot{q}_i}, \quad H = \sum_i P_i \dot{q}_i - L|_{\dot{q}_i \to P_i};$$

moreover,

$$\begin{gathered}\dot{q}_i = \frac{\partial H}{\partial P_i}, \quad \dot{P}_1 = -\frac{\partial H}{\partial q_1} + \frac{\partial H}{\partial P_2}S, \quad \dot{P}_2 = -\frac{\partial H}{\partial q_2} - \frac{\partial H}{\partial P_1}S,\\ S = a_1(q)\dot{q}_1 + a_2(q)\dot{q}_2 + b(q) = A_1(q)P_1 + A_2(q)P_2 + B(q).\end{gathered} \tag{4}$$

The Liouville equation for the density of the invariant measure $\rho(q)\, dP_1 dP_2 dq_1 dq_2$ of (4) leads to the equality

$$\dot{q}_1 \left(\frac{1}{\rho} \frac{\partial \rho}{\partial q_1} - A_2(q) \right) + \dot{q}_2 \left(\frac{1}{\rho} \frac{\partial \rho}{\partial q_2} + A_1(q) \right) = 0,$$

and since ρ depends only on the coordinates, each of the expressions in parentheses must vanish:

$$\frac{1}{\rho} \frac{\partial \rho}{\partial q_1} - A_2(q) = 0, \quad \frac{1}{\rho} \frac{\partial \rho}{\partial q_2} + A_1(q) = 0.$$

Now, write (2) for $\overline{S}$ considering that $\frac{1}{N} \frac{\partial \overline{L}}{\partial q_i'} = P_i$:

$$\overline{S} = \left(N A_1(q) + \frac{\partial N}{\partial q_2} \right) P_1 + \left(N A_2(q) - \frac{\partial N}{\partial q_1} \right) P_2 + B(q).$$

Thus, if we choose $N(q) = \rho(q)$ then $\overline{S} = B(q)$, thereby the first assertion of the theorem is proven.

The second assertion is proven by straightforwardly checking the equations and the Jacobi identity. □

The Hamiltonian systems with (3) arise in description of generalized potential systems (for example, in description of motion of charged particles in a magnetic field) or systems with gyroscopic forces [5]. In this case the closed 2-form $\Omega = \overline{S}(q)\, dq_1 \wedge dq_2$ is called the 2-*form of gyroscopic forces*. Locally, it is representable as a total differential: $\Omega = d\omega$, $\omega = W_1(q)dq_1 + W_2(q)dq_2$; moreover, the equations of motion (2) can be written as the Lagrange–Euler equations

$$\frac{d}{d\tau} \left(\frac{\partial L_W}{\partial q_i'} \right) - \frac{\partial L_W}{\partial q_i} = 0,$$
$$L_W = \overline{L} + W(q, q'), \quad W(q, q') = W_1(q) q_1' + W_2(q) q_2',$$

and the Poisson bracket is reduced to canonical form by means of the new momenta $\tilde{p}_i = p_i + W_i(q)$.

If the manifold $\mathscr{M}$ on which the coordinates q_1 and q_2 are defined is compact then the criterion for exactness of Ω takes the form $\int_{\mathscr{M}} \Omega = 0$. Thus, if $\int \Omega \neq 0$, i.e., the 2-form is not exact then the generalized potential W and the corresponding Lagrangian and Hamiltonian have singularities (the so-called monopole) [6]. In this case sometimes one speaks of impossibility of global representation of the equations of motion in the (canonical) Hamiltonian form.

2. The Veselova System

The Veselova system describes the motion of a rigid body with a fixed point subject to a nonholonomic constraint of the form $(\boldsymbol{\omega}, \boldsymbol{\gamma}) = 0$, where $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ are the vector of the angular velocity of the body and the unit vector of a spatially fixed axis in the coordinate system rigidly connected with the body. Thus, in the case of the Veselova constraint the projection of the angular velocity to a spatially fixed axis is zero. This constraint is dual to the Suslov constraint [7] which requires that the projection be zero of the angular velocity to an axis fixed with respect to the body.

In the moving axes connected with the body, the equations of motion can be written as [8, 9]

$$\mathbf{I}\dot{\boldsymbol{\omega}} = \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \mu \boldsymbol{\gamma} + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \tag{5}$$

where μ is an undetermined multiplier, $\mathbf{I} = \operatorname{diag}(I_1, I_2, I_3)$ is the momentum tensor, and $U(\boldsymbol{\gamma})$ is the potential energy. The undetermined multiplier μ can be found by differentiation of the constraint:

$$\mu = -\frac{\left(\mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}}, \mathbf{I}^{-1}\boldsymbol{\gamma}\right)}{(\boldsymbol{\gamma}, \mathbf{I}^{-1}\boldsymbol{\gamma})}. \tag{6}$$

In the general case (5) admits the energy and geometric integral

$$H = \frac{1}{2}(\mathbf{I}\boldsymbol{\omega}, \boldsymbol{\omega}) + U(\boldsymbol{\gamma}), \quad \boldsymbol{\gamma}^2 = 1, \tag{7}$$

as well as the invariant measure $\rho_\omega \, d^3\boldsymbol{\omega} d^3\boldsymbol{\gamma}$ with the density

$$\rho_\omega = \sqrt{(\boldsymbol{\gamma}, \mathbf{I}^{-1}\boldsymbol{\gamma})}. \tag{8}$$

For $U = 0$ we have the additional integral

$$F = (\mathbf{I}\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega}) - (\mathbf{I}\boldsymbol{\omega}, \boldsymbol{\gamma})^2 = |\mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\gamma}|^2; \tag{9}$$

consequently, the system is integrable by the Euler–Jacobi theorem [8].

Remark 1. Integral (9) can be generalized if we add the Brun potential [8, 9] and some other potentials [2, 10]. Also, it can be generalized by addition of a gyrostat [8, 9].

Remark 2. System (5), (6) with the constraint $(\boldsymbol{\omega}, \boldsymbol{\gamma}) = 0$ was rediscovered in [11] almost ten years after the publication of [8, 9]. In [11] explicit integration was carried out by means of spheroconical coordinates.

Remark 3. A generalization of the Veselova constraint $(\boldsymbol{\omega}, \boldsymbol{\gamma}) = d \neq 0$ was considered in [12]. It was reduced to quadratures by the Chaplygin method for integration of the equations of motion of a dynamically asymmetric ball with a nonzero constant of areas [3].

As was demonstrated in [2], the Veselova system is the Chaplygin system (1) for $b(q) = 0$ and consequently it can be written in Hamiltonian form after the change of time $N\, dt = d\tau$ with the reducing multiplier $N = \rho_\omega^{-1}$. We will verify this explicitly taking the Euler angles θ, φ, and ψ as the local coordinates and then use the so-found canonical Poisson structure of the cotangent bundle of the sphere T^*S^2 for construction of the Poisson bracket with respect to the redundant variables $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$. Using this *algebraization* of the Poisson structure, we can establish most naturally an isomorphism with the Neumann system that describes the motion of a point over a sphere in a quadratic potential (it was established straightforwardly in [8, 9]). As we will see later, we can straightforwardly translate this analogy onto the Chaplygin ball and the general Clebsch system (whose particular case is the Neumann system).

The angular velocity $\boldsymbol{\omega}$ of the body and the unit vector $\boldsymbol{\gamma}$ are expressed in terms of the Euler angles by the formulas

$$\begin{gathered}\boldsymbol{\omega} = (\dot\psi \sin\theta \sin\varphi + \dot\theta \cos\varphi, \dot\psi \sin\theta \cos\varphi - \dot\theta \sin\varphi, \dot\psi \cos\theta + \dot\varphi),\\ \boldsymbol{\gamma} = (\sin\theta \sin\varphi, \sin\theta \cos\varphi, \cos\theta).\end{gathered} \tag{10}$$

The equation of constraint has the form

$$f = (\boldsymbol{\omega}, \boldsymbol{\gamma}) = \dot\psi + \cos\theta \dot\varphi = 0. \tag{11}$$

Eliminate the undetermined Lagrange multiplier from the equations of motion for the angles θ and φ and represent them as the Chaplygin system:

$$\begin{gathered}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = \dot\varphi S, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot\varphi}\right) - \frac{\partial T}{\partial \varphi} + \frac{\partial U}{\partial \varphi} = -\dot\theta S,\\ S = \frac{\partial T_0}{\partial \dot\psi}\bigg|_{\dot\psi = -\cos\theta\dot\varphi} = \sin^2\theta(\dot\theta(I_2 - I_1)\sin\varphi\cos\varphi - \dot\varphi(I_1\cos^2\varphi + I_2\sin^2\varphi + I_3)),\end{gathered} \tag{12}$$

where U is the potential energy of the body in the exterior field, $T_0 = \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega})$ is the kinetic energy without the constraint taken into account, and T is the kinetic energy after elimination of $\dot\psi$ by means of the constraint

$$\begin{aligned}T = T_0|_{\dot\psi = -\cos\theta\dot\varphi} = \frac{1}{2}I_1(\dot\theta\cos\varphi - \dot\varphi\sin\varphi\sin\theta\cos\theta)^2\\ + \frac{1}{2}I_2(\dot\theta\sin\varphi + \dot\varphi\cos\varphi\sin\theta\cos\theta)^2 + \frac{1}{2}I_3\dot\varphi^2\sin^4\theta.\end{aligned} \tag{13}$$

Remark 4. Obtaining (12) bases on simple application of the usual differentiation on considering (11):

$$\frac{\partial T}{\partial \dot\theta}=\frac{\partial T_0}{\partial \dot\theta},\quad \frac{\partial T}{\partial \dot\varphi}=\frac{\partial T_0}{\partial \dot\varphi}-\cos\theta\frac{\partial T_0}{\partial \dot\psi},\quad \frac{\partial T}{\partial \theta}=\frac{\partial T_0}{\partial \theta}+\dot\varphi\sin\theta\frac{\partial T_0}{\partial \dot\psi},\quad \frac{\partial T}{d\varphi}=\frac{\partial T_0}{\partial \varphi}.$$

Theorem 2 [2]. *After the change of time $N\,dt=d\tau$, $N=(\boldsymbol{\gamma},\mathbf{I}\boldsymbol{\gamma})^{-1/2}$, the equations of motion of the Veselova system are representable as the Euler–Lagrange equations*

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \theta'}\right)-\frac{\partial L}{\partial \theta}=0,\quad \frac{d}{d\tau}\left(\frac{\partial L}{\partial \varphi'}\right)-\frac{\partial L}{\partial \varphi}=0, \tag{14}$$

where $L=T-U|_{\dot\theta=N\theta',\,\dot\varphi=N\varphi'}$ is the Lagrangian representable as

$$L=\frac12\frac{(\boldsymbol{\gamma}'\times\boldsymbol{\gamma},\mathbf{I}(\boldsymbol{\gamma}'\times\boldsymbol{\gamma}))}{(\boldsymbol{\gamma},\mathbf{I}^{-1}\boldsymbol{\gamma})}-U(\boldsymbol{\gamma})$$

in view of the constraint and change of time.

The proof bases on simply checking that after the change of time the right-hand side $\overline{S}$ of (14), calculated by (2), vanishes. □

The canonical Hamiltonian form of the equations of motion (14) can be obtained by the Legendre transform

$$\begin{gathered} p_\theta=\frac{\partial L}{\partial \theta'},\quad p_\varphi=\frac{\partial L}{\partial \varphi'},\quad H=p_\theta\theta'+p_\varphi\varphi'-L,\\ \frac{d\theta}{d\tau}=\frac{\partial H}{\partial p_\theta},\quad \frac{d\varphi}{d\tau}=\frac{\partial H}{\partial p_\varphi},\quad \frac{dp_\theta}{d\tau}=-\frac{\partial H}{\partial \theta},\quad \frac{dp_\varphi}{d\tau}=-\frac{\partial H}{\partial \varphi}. \end{gathered} \tag{15}$$

Using the canonical variables (15), after the change of time $(\rho_\omega\sqrt{\det\mathbf{I}})^{-1}dt=d\tau$ we can represent the equations of motion of the Veselova problem in the Hamiltonian form on the (co)algebra $e(3)$ of the Poisson brackets:

$$\begin{gathered} \boldsymbol{M}=\rho_\omega\mathbf{I}^{1/2}\boldsymbol{\omega},\quad \boldsymbol{\Gamma}=\rho_\omega^{-1}\mathbf{I}^{-1/2}\boldsymbol{\gamma},\\ \frac{d\boldsymbol{M}}{d\tau}=\boldsymbol{M}\times\frac{\partial H}{\partial \boldsymbol{M}}+\boldsymbol{\Gamma}\times\frac{\partial H}{\partial \boldsymbol{\Gamma}},\quad \frac{d\boldsymbol{\Gamma}}{d\tau}=\boldsymbol{\Gamma}\times\frac{\partial H}{\partial \boldsymbol{M}}, \end{gathered} \tag{16}$$

$$H=\frac12(\boldsymbol{\Gamma},\mathbf{I}\boldsymbol{\Gamma})(\boldsymbol{M},\boldsymbol{M})+\widetilde{U}(\boldsymbol{\Gamma}), \tag{17}$$

where $\widetilde{U}(\boldsymbol{\Gamma})=U(\rho_\omega\mathbf{I}^{1/2}\boldsymbol{\Gamma})$ and respectively

$$\begin{gathered} \boldsymbol{\gamma}^2=\boldsymbol{\Gamma}^2=1,\quad (\boldsymbol{\omega},\boldsymbol{\gamma})=(\boldsymbol{M},\boldsymbol{\gamma})=0,\quad \rho_\omega=(\boldsymbol{\gamma},\mathbf{I}^{-1}\boldsymbol{\gamma})^{1/2}=(\boldsymbol{\Gamma},\mathbf{I}\boldsymbol{\Gamma})^{-1/2},\\ \{M_i,M_j\}=\varepsilon_{ijk}M_k,\quad \{M_i,\Gamma_j\}=\varepsilon_{ijk}\Gamma_k,\quad \{\Gamma_i,\Gamma_j\}=0. \end{gathered}$$

The Veselova system with a gyrostat. Attachment of a rotor with a constant angular velocity to the body changes the equations of motion (5) as follows:

$$\begin{gathered} \mathbf{I}\dot{\boldsymbol{\omega}}=(\mathbf{I}\boldsymbol{\omega}+\boldsymbol{k})\times\boldsymbol{\omega}+\mu\boldsymbol{\gamma}+\boldsymbol{\gamma}\times\frac{\partial U}{\partial \boldsymbol{\gamma}},\quad \dot{\boldsymbol{\gamma}}=\boldsymbol{\gamma}\times\boldsymbol{\omega},\\ \mu=-(\boldsymbol{\gamma},\mathbf{I}^{-1}\boldsymbol{\gamma})^{-1}\left((\mathbf{I}\boldsymbol{\omega}+\boldsymbol{k})\times\boldsymbol{\omega}+\boldsymbol{\gamma}\times\frac{\partial U}{\partial \boldsymbol{\gamma}},\,\mathbf{I}^{-1}\boldsymbol{\gamma}\right), \end{gathered}$$

where $\boldsymbol{k}$ is the constant vector of the gyrostatic momentum of the rotor. These equations admit an invariant measure (8) as well and Theorem 1 applies; moreover, the corresponding function $\overline{S}$ has the form

$$\overline{S}=\rho\omega^{-1}\sin\theta(k_1\sin\theta\sin\varphi+k_2\sin\theta\cos\varphi+k_3\cos\theta)=\sqrt{\frac{\gamma_1^2+\gamma_2^2}{(\boldsymbol{\gamma},\mathbf{I}^{-1}\boldsymbol{\gamma})}}(\boldsymbol{\gamma},\boldsymbol{k}).$$

For the variables (16) we obtain the respective Poisson structure of the form

$$\{M_i, M_j\} = \varepsilon_{ijk}\big(M_k - \rho_\omega^3 \det \mathbf{I}(\boldsymbol{k}, \mathbf{I}^{1/2}\boldsymbol{\Gamma})\Gamma_k\big), \quad \{M_i, \Gamma_j\} = \varepsilon_{ijk}\Gamma_k, \quad \{\Gamma_i, \Gamma_j\} = 0. \tag{18}$$

The result on existence of structure (18) is new.

Isomorphism with the Neumann and Braden systems. It is easy to see that for $\widetilde{U} = 0$ at the level $H = h$ the system is isomorphic to the Hamiltonian system on $e(3)$ with the Hamiltonian $\mathscr{H} = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{M}) - \frac{h}{(\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma})}$ provided that $\mathscr{H} = 0$. This system was indicated by Braden in [13]. Thus, the symplectic sheet of the structure $(\boldsymbol{M}, \boldsymbol{\gamma}) \in \mathbb{R}^6$ on which the real motion occurs is given by the conditions $\boldsymbol{\gamma}^2 = 1$ and $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$ which corresponds to the zero constant of areas in the classical Euler–Poisson equations [6]. Also, observe that the system of (16) and (17) determines some integrable potential system on the two-dimensional sphere and thereby it defines some geodesic flow.

The inverse change has the form

$$\boldsymbol{\omega} = (\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma})^{1/2}\mathbf{I}^{-1/2}\boldsymbol{M}, \quad \gamma = (\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma})^{-1/2}\mathbf{I}^{1/2}\boldsymbol{\Gamma}.$$

Thus, the search of integrable potentials in the Veselova problem reduces to the well-studied problem of finding integrable cases for the Hamiltonian system on $e(3)$ with Hamiltonian (17). For example, if $U = 0$ then the additional integral (9) takes the form

$$F = (\mathbf{I}\boldsymbol{M}, \boldsymbol{M})(\mathbf{I}\boldsymbol{\Gamma}, \boldsymbol{\Gamma}) - (\mathbf{I}\boldsymbol{M}, \boldsymbol{\Gamma})^2.$$

(Note that $\{H, F\} = 0$ only at the zero constant $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$.)

It was observed in [8, 9] that for $U = 0$ system (5) is equivalent to the Neumann problem. As we see, here this representation is not connected immediately with the natural reduction of the Veselova problem to Hamiltonian form (16), (17) on $e(3)$. It turns out that the isomorphism with the Neumann system is due to existence of a transformation not preserving the Poisson bracket but reducing the vector field to the required form on the level surface $H = \text{const}$.

Indeed, consider the Hamiltonian system on $e(3)$ under the condition $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$ defined by the Hamiltonian

$$H = \alpha\frac{1}{2}\boldsymbol{M}^2(\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma}) + \beta\frac{1}{2}((\boldsymbol{M}, \mathbf{I}\boldsymbol{M})(\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma}) - (\boldsymbol{M}, \mathbf{I}\boldsymbol{\Gamma})^2). \tag{19}$$

Clearly, both summands are the first integrals of the system. The following simple lemma is valid:

Lemma 1. *At a fixed level* $\frac{\boldsymbol{M}^2(\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma})}{\det \mathbf{I}} = c$ *the vector field generated by the Hamiltonian* (19) *is isomorphic to the vector field of the Kirchhoff equations in the Clebsch case at the zero constant of areas* $(\boldsymbol{L}, \boldsymbol{s}) = 0$ *which are representable as*

$$\dot{\boldsymbol{s}} = k(\alpha \boldsymbol{s} \times \boldsymbol{L} + \beta \boldsymbol{s} \times \mathbf{I}\boldsymbol{L}), \ \dot{\boldsymbol{L}} = k(\alpha c \boldsymbol{s} \times \mathbf{I}\boldsymbol{s} + \beta(\boldsymbol{L} \times \mathbf{I}\boldsymbol{L} - c(\det \mathbf{I})\boldsymbol{s} \times \mathbf{I}^{-1}\boldsymbol{s})), \ k = -\sqrt{\det \mathbf{I}}. \tag{20}$$

Proof. Carry out the change of variables

$$\boldsymbol{L} = \mathbf{I}^{-1/2}\boldsymbol{M}, \quad \boldsymbol{s} = (\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma})^{-1/2}\mathbf{I}^{1/2}\boldsymbol{\Gamma},$$

which preserves the relations $\boldsymbol{s}^2 = \boldsymbol{\Gamma}^2 = 1$, $(\boldsymbol{M}, \boldsymbol{\Gamma}) = (\boldsymbol{s}, \boldsymbol{L}) = 0$. By linearity, we can consider the cases $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$ separately. In the first case, writing down the equations of motion in the new variables, we find that

$$\dot{\boldsymbol{s}} = -\sqrt{\det \mathbf{I}}\left(\boldsymbol{s} \times \boldsymbol{L} + (\boldsymbol{s}, \boldsymbol{L})\frac{(\boldsymbol{s} \times \mathbf{I}^{-1}\boldsymbol{s})}{(\boldsymbol{s}, \mathbf{I}^{-1}\boldsymbol{s})}\right), \quad \dot{\boldsymbol{L}} = -\sqrt{\det \mathbf{I}}\frac{(\mathbf{I}\boldsymbol{L}, \boldsymbol{L})}{\det \mathbf{I}(\boldsymbol{s}, \mathbf{I}^{-1}\boldsymbol{s})}\boldsymbol{s} \times \mathbf{I}\boldsymbol{s}.$$

Hence, using the relations $(\boldsymbol{s}, \boldsymbol{L}) = 0$ and $\frac{(\mathbf{I}\boldsymbol{L}, \boldsymbol{L})}{(\boldsymbol{s}, \mathbf{I}^{-1}\boldsymbol{s})} = \boldsymbol{M}^2(\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma}) = c \det \mathbf{I}$, we arrive at the sought result.

The case $\alpha = 0$ and $\beta = 1$ is considered similarly. □

If we assume c to be a constant parameter then the vector field (20) is generated on $e(3)$ by the Hamiltonian of the form

$$H = k\alpha\left(\frac{1}{2}\boldsymbol{M}^2 + \frac{c}{2}(\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma})\right) + k\beta\left(\frac{1}{2}(\boldsymbol{M}, \mathbf{I}\boldsymbol{M}) - \frac{c}{2}\det\mathbf{I}(\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma})\right).$$

For $\alpha = 1$ and $\beta = 0$ this is the Hamiltonian of the Neumann case, while for $\alpha = 0$ and $\beta = 1$ this is the Hamiltonian of the Brun problem; for arbitrary α and β it corresponds to the general Clebsch case in the Kirchhoff equations [6, 14]. Using (16), (17), we easily obtain the following theorem for the Veselova system:

Theorem 3 [8, 9]. *The vector field of the Veselova problem (for $U = 0$) at a fixed level $H = h = \text{const}$ of the energy integral after the change of time is isomorphic to the vector field of the Neumann problem.*

REMARK 5. The multidimensional analog of the Veselova problem was considered in [2, 15]. In [15] an invariant measure and some first integrals were found. Integrability of the multidimensional analog was established in [2] by means of special invariant relations under some additional constraints on the momentum tensor. Moreover, isomorphism with the multidimensional Neumann problem is also used.

3. The Chaplygin Ball

Consider the problem on a balanced dynamically asymmetric ball rolling over a horizontal absolutely rough plane (i.e., the motion without sliding) in an axially symmetric potential force field. The equations of motion in the coordinate system connected with the principal axes of the ball are representable as [3]

$$\begin{gathered}\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\omega} + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \\ \boldsymbol{M} = \mathbf{I}\boldsymbol{\omega} + D\boldsymbol{\gamma} \times (\boldsymbol{\omega} \times \boldsymbol{\gamma}) = \mathbf{I}_Q\boldsymbol{\omega}, \quad D = mR^2,\end{gathered} \tag{21}$$

where $\boldsymbol{\omega}$ is the angular velocity of the ball, $\boldsymbol{\gamma}$ is the unit vertical vector in the moving axes, $\mathbf{I} = \operatorname{diag}(I_1, I_2, I_3)$ is momentum tensor of the ball with respect to its center, m and R are the mass and the radius of the ball, and $U = U(\boldsymbol{\gamma})$ is the potential of the exterior axially symmetric field. The vector $\boldsymbol{M}$ has the sense of the kinetic momentum of the ball with respect to the contact point and the tensor $\mathbf{I}_Q$ is representable as

$$\mathbf{I}_Q = \mathbf{J} - D\boldsymbol{\gamma} \otimes \boldsymbol{\gamma}, \quad \mathbf{J} = \mathbf{I} + D.$$

Equations (5) (for an arbitrary potential) possess the energy integral, the geometric integral, and the integral of areas:

$$H = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{\omega}) + U(\boldsymbol{\gamma}), \quad (\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 1, \quad (\boldsymbol{M}, \boldsymbol{\gamma}) = c = \text{const}; \tag{22}$$

they also admit the invariant measure $\rho_\mu\, d^3\boldsymbol{M} d^3\boldsymbol{\gamma}$ with the density

$$\rho_\mu = (\det \mathbf{I}_Q)^{-1/2} = [\det \mathbf{J}(1 - D(\boldsymbol{\gamma}, \mathbf{J}^{-1}\boldsymbol{\gamma}))]^{-1/2} \tag{23}$$

indicated by S. A. Chaplygin [3].

In the absence of the exterior field ($U = 0$) system (5) possesses the additional integral

$$F = (\boldsymbol{M}, \boldsymbol{M}) \tag{24}$$

and so it is integrable by the Euler–Jacobi theorem [3]. In [3] system (5) was integrated in terms of hyperelliptic functions.

REMARK 6. Integral (24) admits generalization to the case of the Brun field $U(\boldsymbol{\gamma}) = \frac{k}{2}(\boldsymbol{\gamma}, \mathbf{I}\boldsymbol{\gamma})$ [16] and to the case of the gyrostat [5]. The other integrable potentials (for the zero constant of areas $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$) can be found by means of the representation for the system on the algebra $e(3)$ indicated below.

Comments. Observe that, developing the general reducing multiplier method [1], Chaplygin himself did not use it for solving the rolling ball problem. The reader can find a discussion of possible obstacles for application of the reducing multiplier method in this case in [16]. On the other hand, already in [3] Chaplygin actually posed the question about the Hamiltonian form of the equations of a rolling ball. A more rigorous statement of the question about the Hamiltonian form (or Hamiltonization) of the Chaplygin ball was given by Kozlov [14] and Duistermaat [17, 18]. In [19] the authors demonstrated numerically that without change of time the equations of the Chaplygin rolling ball have no Hamiltonian form; the reason is the difference of the circulation periods on two-dimensional invariant resonance tori.

We supplemented this negative result of [19] by the "positive" result of [20] on the Hamilton property of the Chaplygin ball after the corresponding change of time and also presented the explicit form of the nonlinear degenerate Poisson structure and the Hamilton function. Unfortunately, the authors of the remarkable survey [18] could not verify explicitly our result, perhaps, in view of some publisher's misprints in [20]. Here we repeat again the result of [20] and also, using it, indicate an interesting *isomorphism of the Chaplygin ball with the Clebsch case* of the Kirchhoff equations. The existence of this isomorphism was conjectured in [9], but no explicit form was indicated in publications. This is closely connected with the above-indicated isomorphism between the Veselova and Neumann systems.

As demonstrated in [20], for an arbitrary potential, after the change of time and variables

$$\rho_\mu \, dt = d\tau, \quad \boldsymbol{L} = \rho_\mu \boldsymbol{M} \tag{25}$$

the equations of motion (5) take the Hamiltonian form

$$\frac{dM_k}{d\tau} = \{H, M_k\}, \quad \frac{d\gamma_k}{d\tau} = \{H, \gamma_k\}$$

with the nonlinear Poisson bracket

$$\{L_i, L_j\} = \varepsilon_{ijk}(L_k - (\boldsymbol{L}, \boldsymbol{\gamma}) D \rho_\mu^2 J_i J_j \gamma_k), \quad \{L_i, \gamma_j\} = \varepsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0, \tag{26}$$

where the Hamiltonian is the energy (6) which can be written as

$$H = \frac{\det \mathbf{J}}{2}((1 - D(\boldsymbol{\gamma}, \mathbf{J}^{-1}\boldsymbol{\gamma}))(\boldsymbol{L}, \mathbf{J}^{-1}\boldsymbol{L}) + D(\mathbf{J}^{-1}\boldsymbol{L}, \boldsymbol{\gamma})^2) + U(\boldsymbol{\gamma}). \tag{27}$$

It is easy to show [20] that for $(\boldsymbol{L}, \boldsymbol{\gamma}) = 0$ and $U(\boldsymbol{\gamma}) = 0$ the Hamiltonian (27) determines the phase flow conjugate to the flow of the integrable Braden problem [13]. This substantiates once again the conclusion of [12] that the Chaplygin and Veselova problems determine transversal flows on the same invariant pairs.

We show now that the problem on the Chaplygin rolling ball is the generalized Chaplygin system (1) and bracket (26) can be obtained by the reducing multiplier method (see Theorem 1).

As in the Veselova problem, we use the local coordinates: the Euler angles θ, φ, and ψ and the Cartesian coordinates x and y of the center of the cylinder. In the moving coordinate system connected with the principal axes of the ball, the angular velocity vector and the normal to the plane have the form (10).

The equations of constraints, expressing the condition of absence of sliding at the contact point, are representable as

$$f_x = \dot{x} - R\dot{\theta}\sin\psi + R\dot{\varphi}\sin\theta\cos\psi = 0, \quad f_y = \dot{y} + R\dot{\theta}\cos\psi + R\dot{\varphi}\sin\theta\sin\psi = 0. \tag{28}$$

The equations of motion with undetermined Lagrange multipliers have the form

$$\begin{gathered} \frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{x}}\right) = \lambda_x, \quad \frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{y}}\right) = \lambda_y, \quad \frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{\psi}}\right) = 0, \\ \frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{\theta}}\right) - \frac{\partial T_0}{\partial \theta} = \lambda_x \frac{\partial f_x}{\partial \dot{\theta}} + \lambda_y \frac{\partial f_y}{\partial \dot{\theta}}, \quad \frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{\varphi}}\right) - \frac{\partial T_0}{\partial \varphi} = \lambda_x \frac{\partial f_x}{\partial \dot{\varphi}} + \lambda_y \frac{\partial f_y}{\partial \dot{\varphi}}, \end{gathered} \tag{29}$$

where T_0 is the kinetic energy of the ball without constraints (28) (which is obviously independent of x, y, and ψ):

$$T_0 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega}).$$

Eliminating the undetermined multipliers λ_x and λ_y by means of the first two equations in (29) and constraints (28), we find that

$$\begin{gathered}\lambda_x \frac{\partial f_x}{\partial \dot{\theta}} + \lambda_y \frac{\partial f_y}{\partial \dot{\theta}} = -mR^2(\ddot{\theta} + \dot{\psi}\dot{\varphi}\sin\theta),\\ \lambda_x \frac{\partial f_x}{\partial \dot{\varphi}} + \lambda_y \frac{\partial f_y}{\partial \dot{\varphi}} = -mR^2(\ddot{\varphi}\sin\theta + \dot{\theta}\dot{\varphi}\cos\theta - \dot{\theta}\dot{\psi})\sin\theta.\end{gathered}$$

Consequently, the equations of motion for the angles θ and φ are independent of ψ and depend only on $\dot{\psi}$. Thus, we see that ψ is a cyclic variable and can be eliminated by the Routh reduction; afterwards we can rewrite the equations of motion for θ and φ as

$$\frac{d}{dt}\left(\frac{\partial \mathscr{R}}{\partial \dot{\theta}}\right) - \frac{\partial \mathscr{R}}{\partial \theta} = -\dot{\varphi}S, \quad \frac{d}{dt}\left(\frac{\partial \mathscr{R}}{\partial \dot{\varphi}}\right) - \frac{\partial \mathscr{R}}{\partial \varphi} = \dot{\theta}S, \quad S = mR^2\sin\theta(\dot{\varphi}\cos\theta + \dot{\psi}). \tag{30}$$

Here $\mathscr{R}$ is the Routh function:

$$\mathscr{R}(\theta, \varphi, \dot{\theta}, \dot{\varphi}) = T_0 - \dot{\psi}\frac{\partial T_0}{\partial \dot{\psi}}$$

where we should insert $\dot{x}$ and $\dot{y}$ from the equations of constraints and eliminate $\dot{\psi}$ by means of the equation for the cyclic integral

$$\begin{gathered}\frac{\partial T_0}{\partial \dot{\psi}} = (I_1 - I_2)\dot{\theta}\sin\theta\sin\varphi\cos\varphi + I_3\dot{\varphi}\cos\theta\\ +((I_1\sin^2\varphi + I_2\cos^2\varphi)\sin^2\theta + I_3\cos^2\theta)\dot{\psi} = c = \mathrm{const}\,.\end{gathered} \tag{31}$$

We have thus represented the equations of motion in the form of a generalized Chaplygin system (1) and, since the system possesses a measure, we can represent equations (30) in Hamiltonian form with bracket (3).

Carry out the change of time of the form

$$N(\theta, \varphi)dt = d\tau, \tag{32}$$

where $N = \rho_\mu$ is the density of the invariant measure (23) in the case under consideration.

By (2), the equations of motion with the new time take the form

$$\begin{gathered}\frac{d}{d\tau}\left(\frac{\partial \overline{\mathscr{R}}}{\partial \theta'}\right) - \frac{\partial \overline{\mathscr{R}}}{\partial \theta} = -\varphi'\overline{S}, \quad \frac{d}{d\tau}\left(\frac{\partial \overline{\mathscr{R}}}{\partial \varphi'}\right) - \frac{\partial \overline{\mathscr{R}}}{\partial \varphi} = \theta'\overline{S},\\ \overline{S} = cDJ_3N^3\sin\theta(J_1\cos^2\varphi + J_2\sin^2\varphi),\end{gathered} \tag{33}$$

where

$$\theta' = \frac{d\theta}{d\tau} = N^{-1}\dot{\theta}, \quad \varphi' = \frac{d\varphi}{d\tau} = N^{-1}\dot{\varphi}, \quad \text{and} \quad \overline{\mathscr{R}} = \mathscr{R}(\theta, \dot{\theta}, \varphi, \dot{\varphi})|_{\dot{\theta}=N\theta',\ \dot{\varphi}=N\varphi'}.$$

Carrying out the Legendre transform for the system with the new time τ, we obtain the following

Theorem 4 (on the Hamilton property of the Chaplygin ball). *After the change of time $N dt = d\tau$ the equations of motion* (30) *for the Chaplygin ball are representable in Hamiltonian form:*

$$\frac{d\theta}{d\tau} = \frac{\partial H}{\partial p_\theta}, \quad \frac{dp_\theta}{d\tau} = -\frac{\partial H}{\partial \theta} - \overline{S}\frac{\partial H}{\partial p_\varphi}, \quad \frac{d\varphi}{d\tau} = \frac{\partial H}{\partial p_\varphi}, \quad \frac{dp_\varphi}{d\tau} = -\frac{\partial H}{\partial \varphi} + \overline{S}\frac{\partial H}{\partial p_\theta}$$

with the Poisson bracket of the form

$$\{\theta, p_\theta\} = \{\varphi, p_\varphi\} = 1, \quad \{p_\varphi, p_\theta\} = \overline{S}(\theta, \varphi), \quad \{\theta, \varphi\} = 0, \tag{34}$$

where

$$p_\theta = \frac{\partial \overline{\mathscr{R}}}{\partial \theta'}, \quad p_\varphi = \frac{\partial \overline{\mathscr{R}}}{\partial \varphi'},$$

$$H = \theta' p_\theta + \varphi' p_\varphi - \overline{\mathscr{R}} = \frac{1}{2}p_\theta^2(I_3\widetilde{I}_{12} - D(\boldsymbol{\gamma}, \mathbf{I}\boldsymbol{\gamma})) + \frac{1}{2}\frac{p_\varphi^2}{\sin^2\theta}(I_1 I_2 \sin^2\theta + I_3\widetilde{I}_{12}\cos^2\theta - D(\boldsymbol{\gamma}, \mathbf{I}\boldsymbol{\gamma}))$$
$$+ \frac{p_\theta p_\varphi}{\sin\theta} I_3(I_1 - I_2)\cos\theta\sin\theta\sin\varphi\cos\varphi - \frac{Ncp_\theta}{\sin\theta}(I_1 - I_2)(I_3 + D\sin^2\theta)\sin\varphi\cos\varphi$$
$$- \frac{Ncp_\varphi}{\sin^2\theta} I_3(\widetilde{I}_{21} + D) + \frac{N^2c^2}{\sin^2\theta}(I_3 + D\sin^2\theta)(\widetilde{I}_{21} + D),$$
$$\widetilde{I}_{12} = I_1\sin^2\varphi + I_2\cos^2\varphi, \quad \widetilde{I}_{21} = I_1\cos^2\varphi + I_2\sin^2\varphi.$$

Expressing now the variables $\boldsymbol{L} = \rho_\mu \boldsymbol{M}$ of (25) in terms of the local variables θ, φ, p_θ, and p_φ, we find that

$$L_1 = p_\theta\cos\varphi - p_\varphi\frac{\cos\theta\sin\varphi}{\sin\theta} + cN\frac{\sin\varphi}{\sin\theta},$$
$$L_2 = -p_\theta\sin\varphi - p_\varphi\frac{\cos\theta\cos\varphi}{\sin\theta} + cN\frac{\cos\varphi}{\sin\theta}, \quad L_3 = p_\varphi.$$

We check immediately that this mapping takes the Poisson bracket (34) into the above-indicated structure (26).

The Chaplygin ball with a gyrostat (see [5]). If we add a rotor having a constant angular velocity to a ball rolling over the plane then the equations of motion (21) take the form

$$\dot{\boldsymbol{M}} = (\boldsymbol{M} + \boldsymbol{k}) \times \boldsymbol{\omega} + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \quad \boldsymbol{M} = \mathbf{I}_Q\boldsymbol{\omega},$$

where $\boldsymbol{k}$ is the constant gyrostatic momentum vector. Similarly, for (30) we have to choose

$$T_0 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega}) + (\boldsymbol{\omega}, \boldsymbol{k}).$$

This system possesses the same invariant measure (23) and, obviously, we can also apply the reducing multiplier method to the system (Theorem 1); moreover, for the corresponding function $\overline{S}$ we have

$$\overline{S} = N^3 D\sin\theta(cJ_3(J_1\cos^2\varphi + J_2\sin^2\varphi) + \det\mathbf{J}(\boldsymbol{\gamma}, \mathbf{J}^{-1}\boldsymbol{k}).$$

Using (34) with the new function $\overline{S}$, we find the following commutation relations (similar to (26)) for the components of the vectors $\boldsymbol{L} = \rho_\mu(\boldsymbol{M} + \boldsymbol{k})$ and $\boldsymbol{\gamma}$:

$$\{L_i, L_j\} = \varepsilon_{ijk}\big(L_k - D\det\mathbf{J}\big(\rho_\mu^2(\boldsymbol{L}, \boldsymbol{\gamma}) - (\boldsymbol{\gamma}, \mathbf{J}^{-1}\boldsymbol{k})\big)\gamma_k\big),$$
$$\{L_i, \gamma_j\} = \varepsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0.$$

Isomorphism with the Clebsch system. Consider the integrable case $U = 0$ at the zero constant of areas $(\boldsymbol{M}, \boldsymbol{\gamma}) = (\boldsymbol{L}, \boldsymbol{\gamma}) = 0$ in more detail, since in this case bracket (26) corresponds to the algebra $e(3)$. Write the Hamiltonian (22) and the additional integral (24) in the variables $\boldsymbol{L}$ and $\boldsymbol{\gamma}$ in the form (dropping down some inessential factors)

$$H = \frac{1}{2}\boldsymbol{L}^2(\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma}) - \frac{1}{2}[(\boldsymbol{L}, \mathbf{B}\boldsymbol{L})(\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma}) - (\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{L})^2],$$
$$F = \boldsymbol{L}^2(\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma}), \quad \mathbf{B} = 1 - D\mathbf{J}^{-1} = \mathbf{I}\mathbf{J}^{-1}.$$

Using Lemma 1, we obtain

Theorem 5. *After the changes of time* (25) *and variables* $\boldsymbol{s} = (\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma})^{-1/2}\mathbf{B}^{-1/2}\boldsymbol{\gamma}$ *and* $\widetilde{\boldsymbol{L}} = \mathbf{B}^{-1/2}\boldsymbol{L}$ *the vector field* (21) *for* $U = 0$ *at a fixed level* $(\boldsymbol{M}, \boldsymbol{M}) = \mathrm{const}$ *and* $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$ *reduces to the vector field of the Clebsch case in the Kirchhoff equations for the zero constant of areas.*

Remark 7. Perhaps the indicated isomorphism can be used for proving integrability of the n-dimensional Chaplygin ball at least on special orbits similar to the corresponding result on the Veselova problem in [2].

Remark 8. In [15, 21] the equations were considered of the n-dimensional Chaplygin ball rolling over the $(n-1)$-dimensional plane. Although the question of integrability of this problem is still open, in [15, 21] an invariant measure was found together with a series of the first integrals which generalize the energy integrals, the integral of areas, and the geometric integral.

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