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ON Σ-SUBSETS OF NATURALS OVER ABELIAN GROUPS

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Abstract: We obtain conditions for the Σ-definability of a subset of the set of naturals in the hereditarily finite admissible set over a model and for the computability of a family of such subsets. We prove that: for each e-ideal I there exists a torsion-free abelian group A such that the family of e-degrees of Σ-subsets of ω in $\mathbb{HF}(A)$ coincides with I; there exists a completely reducible torsion-free abelian group in the hereditarily finite admissible set over which there exists no universal Σ-function; for each principal e-ideal I there exists a periodic abelian group A such that the family of e-degrees of Σ-subsets of ω in $\mathbb{HF}(A)$ coincides with I.

Keywords: admissible set, e-reducibility, computability, Σ-definability, abelian group

Problems of Σ-definability of subsets of the set of finite ordinals in admissible sets were addressed in articles [1–5]. Connections between T-reducibility and Σ-definability were studied in [2, 3, 5], and relations between e-reducibility and the family of Σ-subsets of ω in admissible sets, in [1, 4]. There are examples in [1] of the models in whose hereditarily finite extensions the family of Σ-definable subsets of ω coincides with $I^* = \{S \subseteq \omega \mid d_e(S) \in I\}$, where I is an arbitrary e-ideal. The present article is inspired by [1].

The necessary background on admissible sets can be found in [6, 7]. The fundamentals of the classical computability theory and group theory can be obtained from [8] and [9] respectively. In this article we consider hereditarily finite admissible sets over models of finite signatures.

Our notation is standard. Denote by W_n the nth computably enumerable set; and by D_n, the nth finite set: $D_n = \{a_1, \dots, a_k\}$ for $n = \sum_{i=1}^{k} 2^{a_i}$. Enumeration reducibility, or e-reducibility for brevity, is defined by

$$A \leq_e B \Leftrightarrow \exists n \forall t \ (t \in A \Leftrightarrow \exists m \ (\langle t, m\rangle \in W_n \& D_m \subseteq B)).$$

Define the enumeration operators Φ_n by

$$\Phi_n(S) = \{x \mid \exists m(\langle x, m\rangle \in W_n \& D_m \subseteq S)\}.$$

This gives another definition of e-reducibility:

$$A \leq_e B \Leftrightarrow \exists n \ (\Phi_n(B) = A).$$

In this case we say that W_n *determines* Φ_n. Call a sequence $\{\Theta_n\}_{n\in\omega}$ of enumeration operators *computable* whenever there exists a computable sequence $\{A_n\}_{n\in\omega}$ of computably enumerable sets that determine Θ_n.

An arbitrary nonempty family I of e-degrees of sets of naturals is called an *e-ideal* whenever the following are fulfilled:

1) $a \leq_e b$ and $b \in I \Rightarrow a \in I$;

2) $a, b \in I \Rightarrow a \sqcup b \in I$.

Denote by $(M^n)_{\neq}$ the set of all n-tuples of pairwise distinct elements of M; i.e., $(M^n)_{\neq} = \{\bar{a} \in M^n \mid a_i \neq a_j \text{ for } i < j\}$.

We will assume that if $\mathfrak{M}_0$ and $\mathfrak{M}_1$ are models of distinct signatures then $\mathfrak{M}_0$ cannot be embedded into $\mathfrak{M}_1$.

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§ 1. The Condition of Σ-Definability of Subsets of Naturals

Take a model $\mathfrak{M}$ of finite signature σ, some subset $M_0 \subseteq M$, and assume that the following are fulfilled:

1. For each $\bar{a} = \langle a_0, \dots, a_{n-1}\rangle \in \left(M_0^{<\omega}\right)_{\neq}$ the Σ-subset $S_{\bar{a}} \subseteq \omega$ is defined in $\mathbb{HF}(\mathfrak{M})$. If $n = 0$ then $S_{\bar{a}} \rightleftharpoons S_{\varnothing}$.

2. For each $n \in \omega$ a computable class $K_n = \{\langle \mathfrak{M}_r^n, \bar{b}\rangle \mid r \in \omega\}$ of constructive models is defined, with $\bar{b} = \langle b_0, \dots, b_{n-1}\rangle$, $\bar{b} \in \left(M_r^n\right)_{\neq}$, and for each $r \in \omega$ a finite set $S_r^n \subseteq \omega$ is defined effectively so that $\langle \mathfrak{M}_r^n, \bar{b}\rangle$ can be isomorphically embedded into $\langle \mathfrak{M}, \bar{a}\rangle$ if and only if $S_r^n \subseteq S_{\bar{a}}$, $\bar{a} \in \left(M_0^n\right)_{\neq}$.

3. For each finitely generated submodel $\langle \mathfrak{M}', \bar{a}\rangle \subseteq \langle \mathfrak{M}, \bar{a}\rangle$, $\bar{a} \in \left(M_0^n\right)_{\neq}$, there exists a number r such that $S_r^n \subseteq S_{\bar{a}}$ and $\langle \mathfrak{M}', \bar{a}\rangle$ can be isomorphically embedded into $\langle \mathfrak{M}_r^n, \bar{b}\rangle$.

We then have

Proposition 1. *Take a Σ-formula $\varphi(x, y_0, \dots, y_{n-1})$ of signature $\sigma' = \{\sigma, \in, \varnothing\}$ without parameters and a tuple $\bar{a} = \langle a_0, \dots, a_{n-1}\rangle \in \left(M_0^n\right)_{\neq}$. If $A \subseteq \omega$ can be defined by the formula $\varphi(x, \bar{a})$ in $\mathbb{HF}(\mathfrak{M})$ then $A \leq_e S_{\bar{a}}$; conversely, if $A \leq_e S_{\bar{a}}$ then A is Σ-definable in $\mathbb{HF}(\mathfrak{M})$.*

Proof. The second part of the proposition is already proved in [1], and we will demonstrate the first part. Take $A = \varphi^{\mathbb{HF}(\mathfrak{M})}[x, \bar{a}] \subseteq \omega$ and

$$W_s = \{\langle m, r\rangle \mid \mathbb{HF}(\mathfrak{M}_r^n, \bar{b}) \models \varphi(m, \bar{b})\}. \tag{1}$$

Condition 2 yields that W_s is computably enumerable.

To verify the equality

$$A = \left\{m \mid \exists r \left(\langle m, r\rangle \in W_s \;\&\; S_r^n \subseteq S_{\bar{a}}\right)\right\}, \tag{2}$$

denote its right-hand side by B. Take $m \in A$. Then

$$\mathbb{HF}(\mathfrak{M}) \models \varphi(m, \bar{a}). \tag{3}$$

There exists a finitely generated submodel $\langle \mathfrak{M}', \bar{a}\rangle \subseteq \langle \mathfrak{M}, \bar{a}\rangle$ such that

$$\mathbb{HF}(\mathfrak{M}', \bar{a}) \models \varphi(m, \bar{a}). \tag{4}$$

By condition 3 there exists a number r such that $S_r^n \subseteq S_{\bar{a}}$ and $\langle \mathfrak{M}', \bar{a}\rangle$ can be isomorphically embedded into $\langle \mathfrak{M}_r^n, \bar{b}\rangle$. Together with (4) this yields

$$\mathbb{HF}(\mathfrak{M}_r^n, \bar{b}) \models \varphi(m, \bar{b}). \tag{5}$$

Consequently, $\langle m, r\rangle \in W_s$ & $S_n^r \subseteq S_{\bar{a}}$; i.e., $m \in B$.

Take now $m \in B$. Then (1) implies (5). By condition 2 the model $\langle \mathfrak{M}_r^n, \bar{b}\rangle$ can be isomorphically embedded into $\langle \mathfrak{M}, \bar{a}\rangle$. Thus, we have (3), and so $m \in A$, which proves (2).

Take a computable function f such that $S_r^n = D_{f(r)}$ and put

$$W_s' = \{\langle m, f(r)\rangle \mid \langle m, r\rangle \in W_s\}. \tag{6}$$

It is clear that W_s' is computably enumerable. From (2) and (6)

$$A = \{m \mid \exists t \; (\langle m, t\rangle \in W_s' \;\&\; D_t \subseteq S_{\bar{a}})\};$$

i.e., $A \leqslant_e S_{\bar{a}}$. □

Remark 1. Suppose that for $\bar{a} \in \left(M_0^n\right)_{\neq}$ the set $S_{\bar{a}}$ can be defined by some Σ-formula with parameters $\bar{a}$. Then $A \subseteq \omega$ can be Σ-defined by some formula $\varphi(x, \bar{a})$ if and only if $A \leq_e S_{\bar{a}}$.

Introduce the following condition:

2′. For each $n \in \omega$ a computable class $K_n = \{\langle \mathfrak{M}_r^n, \bar{b}\rangle \mid r \in \omega\}$ of constructive models is defined uniformly in n, with $\bar{b} = \langle b_0, \dots, b_{n-1}\rangle$, $\bar{b} \in \left(M_r^n\right)_{\neq}$, and for all n and r a finite set $S_r^n \subseteq \omega$ is defined effectively and uniformly in n and r so that $\langle \mathfrak{M}_r^n, \bar{b}\rangle$ can be isomorphically embedded into $\langle \mathfrak{M}, \bar{a}\rangle$ if and only if $S_r^n \subseteq S_{\bar{a}}$, $\bar{a} \in \left(M_0^n\right)_{\neq}$.

Remark 2. If conditions 1, 2′, and 3 hold for a model $\mathfrak{M}$ and a set M_0 then in Proposition 1 the required enumeration operator is determined from φ effectively.

Suppose that for $\mathfrak{M}$ and M_0 the following condition is fulfilled in addition to conditions 1–3.

4. For each $x \in M$ there exist $\bar{a} \in \left(M_0^{<\omega}\right)_{\neq}$ and a parameter-free Σ-formula $\varphi(x, \bar{y})$ such that x is determined by the formula $\varphi(x, \bar{a})$ in $\mathbb{HF}(\mathfrak{M})$. In this case, say that $\mathfrak{M}$ is *$S\Sigma$-generated by M_0*. If $\mathfrak{M}$ is $S\Sigma$-generated by its inverse M then we say that $\mathfrak{M}$ is *$S\Sigma$-generated*.

Proposition 1 implies

Corollary 1. *Suppose that $\mathfrak{M}$ is $S\Sigma$-generated by M_0. A set $A \subseteq \omega$ is Σ-definable in $\mathbb{HF}(\mathfrak{M})$ if and only if there exists $\bar{a} \in \left(M_0^{<\omega}\right)_{\neq}$ such that $A \leq_e S_{\bar{a}}$.*

Proof. Take $A = \varphi^{\mathbb{HF}(\mathfrak{M})}[x, \bar{b}]$, where $\bar{b} = \langle b_0, \dots, b_{m-1}\rangle \in (M^n)_{\neq}$. By condition 4 there exists $\bar{a} \in \left(M_0^{<\omega}\right)_{\neq}$ such that for each $i < m$ there exists a formula $\varphi_i(y, \bar{a})$ of signature $\sigma' \cup \bar{a}$, $\sigma' = \sigma \cup \{U, \in, \varnothing\}$, that defines b_i in $\mathbb{HF}(\mathfrak{M})$. Consider the formula

$$\psi(x, \bar{a}) = \exists y_0 \dots \exists y_{m-1}\Big(\varphi(x, \bar{y}) \wedge \bigwedge_{i<m} \varphi_i(y_i, \bar{a})\Big).$$

It is easy to check that

$$\varphi^{\mathbb{HF}(\mathfrak{M})}[x, \bar{b}] = \psi^{\mathbb{HF}(\mathfrak{M})}[x, \bar{a}].$$

Together with Proposition 1 this yields the corollary. □

Lemma 1. *Every model $\mathfrak{M}$ of a finite purely predicate signature σ is $S\Sigma$-generated.*

Proof. Check conditions 1–4. Pick $n \in \omega$. Denote by K_n the class of all finite models of signature $\sigma \cup \bar{b}$, with $\bar{b} = \langle b_0, \dots, b_{n-1}\rangle$, whose inverses are the initial segments of the ordered set $\langle \{b_i \mid i \in \omega\}, <\rangle$, where $b_i < b_j$ for $i < j$. Suppose that Γ_n is an effective enumeration of this class, $\mathfrak{M}_r^n = \Gamma_n(r)$, and $S_r^n = \{r\}$. It is easy to check that K_n is a sequence of constructive models computable uniformly in n.

Given $\bar{a} \in (M^{<\omega})_{\neq}$, $a = \langle a_0, \dots, a_{n-1}\rangle$, denote by $S_{\bar{a}}$ the set of numbers r such that $\langle \mathfrak{M}_r^n, \bar{b}\rangle$ can be isomorphically embedded into $\langle \mathfrak{M}, \bar{a}\rangle$, and show that $S_{\bar{a}} \subseteq \omega$ is Σ-definable in $\mathbb{HF}(\mathfrak{M})$. Pick $r \in \omega$. Suppose that $M_r^n = \{b_0, \dots, b_{m-1}\}$. Denote by $\Phi'_r(b_0, \dots, b_{n-1}, \dots, b_{m-1})$ the open diagram of $\langle \mathfrak{M}_r^n, \bar{b}\rangle$. Let F be the set of all formulas of signature σ, and let $\gamma : \omega \to F$ be an effective enumeration of it. Put

$$\gamma(r') = \Phi_{r'}(\bar{x}) = \exists y_n \dots \exists y_{m-1} \Phi'_r(x_0, \dots, x_{n-1}, y_n, \dots, y_{m-1}).$$

The sequence $\langle \gamma^{-1} F_0^n \mid n \in \omega\rangle$ of sets, where $F_0^n = \{\Phi_{r'}(\bar{x}) \mid r \in \omega\}$, is computable. The function $f(n, r) = r'$ is computable, thus, Σ-definable in $\mathbb{HF}(\mathfrak{M})$. There exists a Σ-function $h : M^{<\omega} \to \omega$ such that $h(\bar{a}) = n$, where $\bar{a} = \langle a_0, \dots, a_{n-1}\rangle$. Thus, $S_{\bar{a}} = \{r \mid \mathbb{HF}(\mathfrak{M}) \models Tr_{\mathfrak{M}}(f(h(\bar{a}), r), \bar{a})\}$, where

$$Tr_{\mathfrak{M}}(m, \bar{a}) = \{\langle m, \bar{a}\rangle \mid \ m \text{ is the number of the } \exists\text{-formula } \Phi_m(\bar{x}) \text{ of signature } \sigma,$$
$$\bar{a} \in M^{<\omega}, \text{ and } \mathbb{HF}(\mathfrak{M}) \models \Phi_m(\bar{a})\},$$

is a Σ-set in $\mathbb{HF}(\mathfrak{M})$.

By the definition of S_r^n this implies that conditions 1 and 2 hold for $\mathfrak{M}$ and M. Let us check condition 3. Given a finite submodel $\langle \mathfrak{M}', \bar{a}\rangle \subseteq \langle \mathfrak{M}, \bar{a}\rangle$, there exists $r \in \omega$ such that $\langle \mathfrak{M}_r^n, \bar{b}\rangle \simeq \langle \mathfrak{M}', \bar{a}\rangle$. Hence, $S_r^n = \{r\} \subseteq S_{\bar{a}}$; i.e., condition 3 holds for $\mathfrak{M}$ and M. The verification of condition 4 is obvious. Consequently, $\mathfrak{M}$ is $S\Sigma$-generated. □

Remark 3. The proof of Lemma 1 also implies that

1) condition 2′ holds for $\mathfrak{M}$ and its inverse M;

2) $S_{\bar{a}}$ can be Σ-defined by a formula with parameters $\bar{a}$.

Hence, Lemma 1 and Remarks 1 and 2 imply

Corollary 2. *Take a model $\mathfrak{M}$ of a purely predicate signature σ. A set $A \subseteq \omega$ can be defined by some Σ-formula $\varphi(x, \bar{a})$ in $\mathbb{HF}(\mathfrak{M})$ if and only if $A \leq_e S_{\bar{a}}$. Moreover, the enumeration operator can be found effectively from φ.*

Given a model $\mathfrak{M}$, denote by $I_e(\mathfrak{M})$ the ideal of e-degrees of Σ-subsets of naturals in $\mathbb{HF}(\mathfrak{M})$.

Corollary 3 [1]. *Given a model $\mathfrak{M}$ of a finite purely predicate signature σ, the ideal $I_e(\mathfrak{M})$ is generated by the e-degrees $d_e(Th_\exists(\mathfrak{M}, \bar{a}))$, where $\bar{a} \in M^{<\omega}$.*

Indeed, $\mathfrak{M}$ is $S\Sigma$-generated by Lemma 1. Suppose that $A \subseteq \omega$ is Σ-defined by $\varphi(x, \bar{a})$. Proposition 1 gives $A \leq_e S_{\bar{a}}$. The proof of Lemma 1 implies that $S_{\bar{a}} \equiv_e Th_\exists(\mathfrak{M}, \bar{a})$. □

It was proved in [9] that every abelian p-group and every Ershov algebra is locally constructivizable. Thus, Corollary 3 implies

Corollary 4. *If $\mathfrak{M}$ is an abelian p-group or a Ershov algebra then each Σ-subset of the set of naturals in $\mathbb{HF}(\mathfrak{M})$ is computably enumerable.*

Given a nonempty family U of nonempty sets of naturals and a sequence $\Lambda = \langle \alpha_S \mid S \in U \rangle$ of infinite cardinals, a model $\mathfrak{M}'_{\langle U, \Lambda \rangle}$ is constructed in [1]. In fact, the construction applies also in the case that U contains the empty set. Given a nonempty family U of sets of naturals and a sequence Λ, we have

Lemma 2. *If U contains all finite sets then the model $\mathfrak{M} \rightleftharpoons \mathfrak{M}'_{\langle U, \Lambda \rangle}$ is $S\Sigma$-generated by the set $M_0 = \{\langle S, \gamma \rangle \mid S \in U, \gamma < \alpha_S\}$.*

Proof. We have to check conditions 1–4 for $S\Sigma$-generation of $\mathfrak{M}$ by M_0.

1. Take $\bar{a} = \langle a_0, \dots, a_{n-1} \rangle \in \left(M_0^n\right)_{\neq}$, $a_i = \langle S_i, \alpha_i \rangle$, and put $S_{\bar{a}} = S_0 \oplus \dots \oplus S_{n-1}$. It is easy to check that $S_{\bar{a}}$ is a Σ-subset in $\mathbb{HF}(\mathfrak{M})$.

2. Given some number n, fix an effective enumeration $\gamma^n : \omega \to A$, where A is the set of all m-tuples of pairwise distinct pairs of numbers for $m \geq n$, $m \in \omega$. Let $\gamma^n r = \langle \langle e_j, p'_j \rangle \mid j < m \rangle$. Put $U^n_{r'} = \{D_{e_0}, \dots, D_{e_{m-1}}\}$. Let $U^n_r = \{D_{r_0}, \dots, D_{r_{t-1}}\}$, where $D_{r_j} \neq D_{r_{j'}}$, $j < j' < t$, $p_{r_j} = \max\{p'_k \mid D_{r_j} = D_{e_k}, k < m\} + 1$, $\Lambda^n_r = \{p_{r_0}, \dots, p_{r_{t-1}}\}$, $\mathfrak{M}^n_r = \mathfrak{M}'_{\langle U^n_r, \Lambda^n_r \rangle}$, $b_i = \langle D_{e_i}, p'_i \rangle$. Put $S^n_r = D_{e_0} \oplus \dots \oplus D_{e_{n-1}}$.

It is easy to check that $\langle \mathfrak{M}^n_r, \bar{b} \rangle$ can be isomorphically embedded into $\langle \mathfrak{M}, \bar{a} \rangle$ if and only if $S^n_r \subseteq S_{\bar{a}}$.

3. Take some finitely generated submodel

$$\langle \mathfrak{M}', \bar{a} \rangle \subseteq \langle \mathfrak{M}, \bar{a} \rangle, \quad \bar{a} = \langle a_0, \dots, a_{n-1} \rangle \in \left(M_0^n\right)_{\neq}.$$

Without loss of generality we may assume that if $\langle S, \gamma, n \rangle \in M'$ then $\langle S, \gamma \rangle \in M'$. Suppose that $\{a_j \mid j < m\}$, with $m \geq n$, is the set of all elements in M' belonging to M_0, $a_j = \langle S_j, \alpha_j \rangle$, and $D_{e'_j} = \{n \mid \langle S_j, \alpha_j, n \rangle \in M'\}$.

It is easy to check that there exist numbers e_j, p'_j, such that for each $j < m$

1) $D_{e'_j} \subseteq D_{e_j} \subseteq S_j$;

2) $\langle D_{e_j}, p'_j \rangle \neq \langle D_{e_{j'}}, p'_{j'} \rangle$, $j < j' < m$.

Put $\gamma r = \langle \langle e_j, p'_j \rangle \mid j < m \rangle$.

It is easy to check that $\langle \mathfrak{M}', \bar{a} \rangle$ can be isomorphically embedded into $\langle \mathfrak{M}^n_r, \bar{b} \rangle$, where $b_i = \langle D_{e_i}, p'_i \rangle$ for $i < n$, and $S^n_r \subseteq S_{\bar{a}}$.

4. Take $x \in M$. If $x \in \omega$ then it is obvious that x is Σ-definable with respect to signature $\langle 0, s \rangle$. Suppose that $x = \langle S, \gamma, n \rangle$. Then

$$x = \langle S, \gamma, n \rangle \Leftrightarrow Q(x, \langle S, \gamma \rangle, n).$$

Thus, each x is Σ-definable in the model $\mathfrak{M}$ with constants in M_0. Therefore, all conditions are fulfilled, and so the model $\mathfrak{M}'_{\langle U, \Lambda \rangle}$ is $S\Sigma$-generated by M_0. □

Remark 4. The second part of the proof of Lemma 2 implies that $\mathfrak{M}$ satisfies condition $2'$.

Suppose that a model $\mathfrak{M}$ of signature σ and a set $M_0 \subseteq M$ satisfy conditions 1, 2′, and 3, and the following condition is fulfilled.

5. There exists a computable sequence $\varphi_r^e(x, x_0, \ldots, x_{r-1})$, with $r, e \in \omega$, of parameter-free Σ-formulas of signature σ' such that the following conditions are fulfilled:

(a) for all numbers r, e and each tuple $\bar{a} \in \left(M_0^r\right)_{\neq}$ the set $\{x \in M \mid \mathbb{HF}(\mathfrak{M}) \models \varphi_r^e(x, \bar{a})\}$ contains at most one element;

(b) for each $x \in M$ there exist numbers r_x, e_x and a tuple $\bar{a}_x \in \left(M_0^{r_x}\right)_{\neq}$ such that x is defined in $\mathbb{HF}(\mathfrak{M})$ by the formula $\varphi_{r_x}^{e_x}(x, \bar{a}_x)$.

Say then that $\mathfrak{M}$ is *computably $S\Sigma$-generated by M_0*.

It is easy to observe the validity of

Lemma 3. *Given a model $\mathfrak{M}$ and a formula $\psi(x, y_0, \ldots, y_{n-1}, \bar{b})$, $\bar{b} \in (M^t)_{\neq}$, we can define effectively by ψ the set of formulas*

$$\{\psi_i(x, y_0, \ldots, y_{m_i-1}, \bar{b}) \mid i < s,\ m_i < n\}$$

so that

$$\{\psi^{\mathfrak{M}}[x, \bar{a}, \bar{b}] \mid \bar{a} \in M^n\} = \{\psi_i^{\mathfrak{M}}[x, \bar{a}, \bar{b}] \mid i < s,\ \bar{a} \in (M^{m_i})_{\neq},\ a_j \neq b_k\}. \tag{7}$$

Theorem 1. *Suppose that $\mathfrak{M}$ is computably $S\Sigma$-generated by M_0. If a family S of subsets of the set of naturals is computable in $\mathbb{HF}(\mathfrak{M})$ then the family $S \cup \{\varnothing\}$ can be represented in the form*

$$\left\{\Theta_i(S_{\langle \bar{a}, \bar{b}\rangle}) \mid i \in \omega,\ \bar{a} \in \left(M_0^{m_i}\right)_{\neq},\ a_j \neq b_k\right\}$$

for some computable sequence of enumeration operators Θ_i, some $\bar{b} \in \left(M_0^t\right)_{\neq}$, and the function $m(i) = m_i$ is computable.

Proof. Take some computable family S of subsets of the set of naturals in $\mathbb{HF}(\mathfrak{M})$. By Proposition 4.4 in [1] and condition 5 there exists a Σ-formula $\Psi(x_0, x_1, \bar{b})$ such that

$$S \cup \{\varnothing\} = \{\Psi^{\mathbb{HF}(\mathfrak{M})}[x_0, c, \bar{b}] \mid c \in \mathbb{HF}(\mathfrak{M})\} \tag{8}$$

for some fixed $\bar{b} \in \left(M_0^t\right)_{\neq}$.

Following Ershov [7], introduce the sets $\mathfrak{H}_n = \mathbb{HF}(n = \{i \mid i < n\})$, $\mathfrak{H}_0 = \mathbb{HF}(\varnothing)$. Then $\mathbb{HF}(\omega) = \bigcup_n \mathfrak{H}_n$. There exists a Σ-enumeration $\alpha : \omega \to HF(\omega)$ of $HF(\omega)$ in $\mathbb{HF}(\mathfrak{M})$. Denote by t_m the element of $HF(\omega)$ with index m. Suppose that $t_m \in \mathfrak{H}_{k_m}$, where $k_m = \min\{k \mid t_m \in \mathfrak{H}_k\}$, and

$$\Psi^{(m)}(x_0, z_0, \ldots, z_{k_m-1}, \bar{b}) = \Psi(x, t_m(z_0, \ldots, z_{k_m-1}), \bar{b}). \tag{9}$$

The function $k(m) = k_m$ is computable. Denote by R the set of all finite tuples of naturals of the form $\alpha = \langle m, r_0, e_0, \ldots, r_{k_m-1}, e_{k_m-1}\rangle$. It is obvious that R is computable; let ν be a computable function that enumerates R.

For each $i \in \omega$, define a formula Ψ_i. If $\nu(i) = \alpha$ and $k_m \rightleftharpoons q$ then put

$$\begin{aligned}&\Psi_i\big(x, y_0^0, \ldots, y_{r_0-1}^0, y_0^{q-1}, \ldots, y_{r_{q-1}-1}^{q-1}, \bar{b}\big)\\ &= \exists z_0 \ldots \exists z_{q-1}\Big(\bigwedge_{j<q} U(z_j) \wedge \Psi^{(m)}(x_0, z_0, \ldots, z_{q-1}, \bar{b}) \wedge \bigwedge_{j<q} \varphi_{r_j}^{e_j}\big(z_j, y_0^j, \ldots, y_{r_j-1}^j\big)\Big),\end{aligned}$$

where $\varphi_{r_j}^{e_j}$ are the same formulas as in condition 5.

Condition 5 implies that for each tuple $\bar{c} = \langle c_0, \ldots, c_{q-1}\rangle \in M^{<\omega}$ there exist tuples $\alpha \in R$ and $\bar{a} = \langle \bar{a}_{c_0}, \ldots, \bar{a}_{c_{q-1}}\rangle \in M_0^{<\omega}$ such that $\nu(i) = \alpha$ and

$$\Psi^{\mathbb{HF}(\mathfrak{M})}[x_0, t_m(\bar{c}), \bar{b}] = \Psi_i^{\mathbb{HF}(\mathfrak{M})}[x_0, \bar{a}, \bar{b}].$$

Together with (8) and (9) this yields

$$S \cup \{\varnothing\} = \big\{\Psi_i^{\mathbb{HF}(\mathfrak{M})}[x_0, \bar{a}, \bar{b}] \mid i \in \omega, \bar{a} \in M_0^{m_i}\big\}. \tag{10}$$

By Lemma 3 we may assume that in (10) we have $\langle \bar{a}, \bar{b}\rangle \in \big(M_0^{m_i+t}\big)_{\neq}$. Then Remark 2 shows that

$$S \cup \{\varnothing\} = \big\{\Theta_i(S_{\langle \bar{a}, \bar{b}\rangle}) \mid i \in \omega,\ \bar{a} \in \big(M_0^{m_i}\big)_{\neq},\ a_j \neq b_k\big\}$$

for some computable sequence of enumeration operators Θ_i, $i \in \omega$, some $\bar{b} \in \big(M_0^t\big)_{\neq}$, and the function $m(i) = m_i$ is computable. □

Introduce the following conditions for a model $\mathfrak{M}$ and a set $M_0 \subseteq M$.

6. The set M_0 is a Σ-subset in $\mathbb{HF}(\mathfrak{M})$.

7. There exists a Σ-formula $\Phi_1(x, \bar{y})$, possibly with parameters, such that $\Phi_1^{\mathbb{HF}(\mathfrak{M})}[x, \bar{a}] = S_{\bar{a}}$ for each $\bar{a} \in \big(M_0^{<\omega}\big)_{\neq}$.

Remark 5. The proof of Lemma 1 implies that for a model $\mathfrak{M}$ of a purely predicate signature and its inverse M conditions 6 and 7 are fulfilled.

Corollary 5. *If $\mathfrak{M}$ is computably $S\Sigma$-generated by M_0 and conditions 6, 7 are fulfilled then a family S of subsets of the set of naturals is computable in $\mathbb{HF}(\mathfrak{M})$ if and only if $S \cup \{\varnothing\}$ can be represented in the form*

$$\big\{\Theta_i(S_{\langle \bar{a}, \bar{b}\rangle}) \mid i \in \omega,\ \bar{a} \in \big(M_0^{m_i}\big)_{\neq},\ a_j \neq b_k\big\} \tag{11}$$

for some computable sequence of enumeration operators Θ_i and some $\bar{b} \in \big(M_0^t\big)_{\neq}$, and the function $m(i) = m_i$ is computable.

Proof. By Theorem 1 we have to prove the *sufficiency*. Suppose that $S \cup \{\varnothing\}$ can be represented in the form (11) and a computable sequence $\{A_i\}$ of computably enumerable sets defines the operators $\{\Theta_i\}$. Then there exists a Σ-formula Φ_2 such that $\Phi_2^{\mathbb{HF}(\mathfrak{M})}[x, i] = A_i$.

This, together with (11), implies by condition 7 that

$$S \cup \{\varnothing\} = \big\{\lambda x.\exists t\ \Phi_2(\langle x, t\rangle, i)\ \&\ \forall y \in t\ (y \in D_t \to \Phi_1(y, \langle \bar{a}, \bar{b}\rangle)) \mid i \in \omega,\ \bar{a} \in \big(M_0^{m_i}\big)_{\neq},\ a_j \neq b_k\big\}.$$

Since the set $\big\{\bar{a} \in \big(M_0^{m_i}\big)_{\neq} \mid i \in \omega,\ a_j \neq b_k\big\}$ is Σ-definable in $\mathbb{HF}(\mathfrak{M})$, the family $S \cup \{\varnothing\}$ is computable. Thus, S is computable. □

Consider the model $\mathfrak{M}'_{\langle U, \Lambda\rangle}$ constructed in [1].

Corollary 6. *Given some e-ideal I and $\mathfrak{M} = \mathfrak{M}'_{\langle I^*, \Lambda\rangle}$, a family S of subsets of the set of naturals is computable in $\mathbb{HF}(\mathfrak{M})$ if and only if $S \cup \{\varnothing\}$ can be represented in the form*

$$\{\Theta_i(R, A) \mid i \in \omega,\ R \in I^*\}$$

for some computable sequence of enumeration operators Θ_i and some $A \in I^*$.

Proof. The *sufficiency* is easy from the existence of a universal Σ-predicate for admissible sets of finite signature. To prove the *necessity*, suppose that a family S is computable. By Theorem 1

$$S \cup \{\varnothing\} = \big\{\Theta_i^1(S_{\langle \bar{a}, \bar{b}\rangle}) \mid i \in \omega,\ \bar{a} \in \big(M_0^{m_i}\big)_{\neq},\ a_j \neq b_k\big\}$$

for some computable sequence of enumeration operators Θ_i^1 and some $\bar{b} \in \big(M_0^t\big)_{\neq}$, and the function $m(i) = m_i$ is computable. The proof of Lemma 2 implies the equality

$$S_{\langle \bar{a}, \bar{b}\rangle} = S_0 \oplus \dots \oplus S_{m_i - 1} \oplus A_0 \oplus \dots \oplus A_{t-1},$$

where $a_j = \langle S_j, \gamma_j\rangle$ for $j < m_i$, and $b_k = \langle A_k, \beta_k\rangle$ for $k < t$. Hence, there exists a computable sequence of enumeration operators Θ_i such that

$$S \cup \{\varnothing\} = \{\Theta_i(S_0 \oplus \dots \oplus S_{m_i-1}, A_0 \oplus \dots \oplus A_{t-1}) \mid S_j \in I^*, j < m_i,\ i \in \omega\}.$$

It remains to notice that $\{S_0 \oplus \dots \oplus S_{m_i-1} \mid S_j \in I^*,\ j < m_i\} = I^*$. □

§ 2. Σ-Subsets of Naturals over Abelian Groups

Let P be the set of all primes, and take some nonempty family $S = \{S_\alpha \mid \alpha < \beta\}$ of subsets of P, where β is some infinite ordinal. Given an ordinal $\alpha < \beta$ and $i \in \omega$, define the group $A_\alpha^{(i)}$ by the presentation

$$A_\alpha^{(i)} = \mathrm{gr}\big(a_\alpha^i, \{b_{p,i}^\alpha \mid p \in S_\alpha\} : pb_{p,i}^\alpha = a_\alpha^i\big),$$

and put

$$A_\alpha = \bigoplus_{i\in\omega} A_\alpha^{(i)}, \quad A_S = \bigoplus_{\alpha<\beta} A_\alpha.$$

Lemma 4. *The group $A \leftrightharpoons A_S$ is computably $S\Sigma$-generated by $A_0 = \{a_\alpha^i \mid \alpha < \beta,\ i \in \omega\}$.*

Proof amounts to checking conditions 1, 2′, 3 and 5 in the definition of a computable $S\Sigma$-generated model.

1. Given $\bar a = \langle a_{\alpha_0}^{t_0}, \dots, a_{\alpha_{n-1}}^{t_{n-1}}\rangle \in \big(A_0^{<\omega}\big)_{\neq}$, define the set

$$S_{\bar a} = S_{\alpha_0} \oplus \dots \oplus S_{\alpha_{n-1}} \oplus \omega.$$

It is easy to check that condition 1 is fulfilled.

2′. Let $[\ ] : \omega^{<\omega} \to \omega$ be an effective enumeration of finite tuples of naturals. Given numbers n and $r = [u_0, \dots, u_{n-1}, m]$ such that $D_{u_i} \subseteq P$, $i < n$, $m \in \omega$, define for each $i < n$ the group

$$B_i^r = \mathrm{gr}\big(b_i, \{c_p^i \mid p \in D_{u_i}\} : pc_p^i = b_i\big),$$

and put

$$M_r^n = \bigoplus_{i<n} B_i^r \bigoplus Z^m, \quad \bar b = \langle b_0, \dots, b_{n-1}\rangle, \quad S_r^n = D_{u_0} \oplus \dots \oplus D_{u_{n-1}} \oplus \{m\},$$

where Z^m is the direct sum of m copies of the infinite cyclic group.

It is easy to verify that the mapping $f : b_i \to a_{\alpha_i}^{t_i}$, $i < n$, can be extended to an isomorphic embedding of the model $\langle M_r^n, \bar b\rangle$ into $\langle A, \bar a\rangle$ if and only if $S_r^n \subseteq S_{\bar a}$.

3. Given $\bar a = \langle a_{\alpha_0}^{t_0}, \dots, a_{\alpha_{n-1}}^{t_{n-1}}\rangle \in \big(A_0^{<\omega}\big)_{\neq}$ and a finitely generated subgroup $A' \subseteq A$ that includes $a_i \leftrightharpoons a_{\alpha_i}^{t_i}$, define for each $i < n$ the set

$$R^i \leftrightharpoons R_{A'}^i = \{p \in P \mid A' \models \exists y\ (py = a_i)\}.$$

Suppose that $R^i = \{p_0^i, \dots, p_{l_i-1}^i\}$, and for each $j < l_i$ the number u_i and the element b_j^i are such that $D_{u_i} = R^i$ and $p_j^i b_j^i = a_i$. Denote by A'' the subgroup generated by $\{b_j^i, a_i \mid j < l_i,\ i < n\}$. It is easy to see that A'' is servant in A', and so $A' = A'' \oplus B$ for some finitely generated subgroup B.

By the fundamental theorem of finitely generated abelian groups there is a number m such that the groups B and Z^m are isomorphic. Put $r = [u_0, \dots, u_{n-1}, m]$. Then $S_r^n \subseteq S_{\bar a}$, and the groups $\langle A', \bar a\rangle$ and $\langle M_r^n, \bar b\rangle$ are isomorphic.

5. Each element of A depends linearly on $\langle a_\alpha^i \mid \alpha < \beta,\ i \in \omega\rangle$. This implies that condition 5 is fulfilled. □

Lemma 4 and Corollary 1 yield

Corollary 7. *A set $M \subseteq \omega$ is Σ-definable in the hereditarily finite admissible set $\mathbb{HF}(A_S)$ over the group A_S if and only if there exists a tuple $\bar a \in \big(A_0^{<\omega}\big)_{\neq}$ such that $M \leq_e S_{\bar a}$.*

Theorem 2. *Given an e-ideal I, there exists a torsion-free abelian group A such that I^* coincides with the family of all Σ-subsets of the set of naturals in $\mathbb{HF}(A)$. Moreover, this group can be chosen so that* $\mathrm{card}(A) = \mathrm{card}(I^*)$.

PROOF. Take an e-ideal I and $I^* = \{S_\alpha \subseteq \omega \mid \alpha < \beta\}$. For each $\alpha < \beta$ define the set $S'_\alpha = \{p_x \mid x \in S_\alpha\}$, where p_x is the xth prime, and put $I' = \{S'_\alpha \mid \alpha < \beta\}$, $A = A_{I'}$, where the group A is constructed from I' as before Lemma 4. Let us show that A is the required group. The construction of A implies that for each $\alpha < \beta$ and $i \in \omega$ we have

$$S'_\alpha = \{p \mid A \models \exists y \, (py = a^i_\alpha)\}.$$

Hence, S'_α, and so S_α as well, are Σ-definable in $\mathbb{HF}(A)$. Given a Σ-subset $M \subseteq \omega$ in $\mathbb{HF}(A)$, by Corollary 7 there exist n and $\bar{a} \in (A^n_0)_{\neq}$ such that

$$M \leq_e S_{\bar{a}} = S_{\alpha_0} \oplus \cdots \oplus S_{\alpha_{n-1}} \oplus \omega.$$

Since $S'_{\alpha_i} \in I^*$, we deduce $S_{\bar{a}} \in I^*$, which implies that $M \in I^*$. $\square$

Corollary 8. *Given an e-ideal I and the associated group A as in Theorem 2, a family S of subsets of the set of naturals is computable in $\mathbb{HF}(A)$ if and only if $S \cup \{\varnothing\}$ can be represented in the form*

$$\{\Theta_i(R, B) \mid i \in \omega, \ R \in I^*\}$$

for some computable sequence of enumeration operators Θ_i and some $B \in I^*$.

PROOF. As in Corollary 6, we prove the *necessity.* Suppose that S is computable. By Theorem 1

$$S \cup \{\varnothing\} = \{\Theta^1_i(S_{\langle \bar{a}, \bar{b}\rangle}) \mid i \in \omega, \ \bar{a} \in (A^{m_i}_0)_{\neq}, \ a_i \neq b_k\}$$

for some computable sequence of enumeration operators Θ^1_i and some $\bar{b} \in (A^t_0)_{\neq}$, and the function $m(i) = m_i$ is computable. By the proof of Lemma 4 there exists a computable sequence of enumeration operators Θ^2_i such that

$$S \cup \{\varnothing\} = \{\Theta^2_i(S'_{\alpha_0} \oplus \cdots \oplus S'_{\alpha_{m_i-1}}, B'_0 \oplus \cdots \oplus B'_{t-1} \otimes \omega) \mid S'_{\alpha_j} \in I', \ j < m_i, \ i \in \omega\}.$$

Since $S'_{\alpha_0} \oplus \cdots \oplus S'_{\alpha_{m_i-1}} \equiv_m S_{\alpha_0} \oplus \cdots \oplus S_{\alpha_{m_i-1}}$, where $S_{\alpha_j} \rightleftharpoons \{x \mid p_x \in S'_{\alpha_j}\}$, $j < m_i$, and $I^* = \{S_{\alpha_0} \oplus \cdots \oplus S_{\alpha_{m_i-1}} \mid S_{\alpha_j} \in I^*\}$, there exists a computable sequence of enumeration operators Θ_i such that

$$S \cup \{\varnothing\} = \{\Theta_i(R, B'_0 \oplus \cdots \oplus B'_{t-1} \otimes \omega) \mid R \in I^*, \ i \in \omega\}. \qquad \square$$

Suppose that an e-ideal I is generated by total e-degrees and is not principal, $I^* = \{S_\alpha \mid S_\alpha \neq \varnothing, \alpha < \beta\}$, and $A = A_{I'}$ is constructed as in the proof of Theorem 2. As in [1], for A we can verify

Corollary 9. *There exists a completely reducible torsion-free abelian group A such that in the hereditarily finite admissible set $\mathbb{HF}(A)$ there exists no universal Σ-function.*

Given a set $S \subseteq P$ of primes, define the group $G \rightleftharpoons G_S = \bigoplus\{Z_p \mid p \in S\}$. In each of the groups Z_p fix an element $a_p \neq 0$, and put $G_0 = \{a_p \mid p \in S\}$.

Lemma 5. *The group G is computably $S\Sigma$-generated by G_0.*

PROOF amounts to checking conditions 1, 2′, 3, and 5 in the definition of a computably $S\Sigma$-generated model.

1. Given $\bar{a} = \langle a_{p_0}, \ldots, a_{p_{n-1}} \rangle \in (G^{<\omega}_0)_{\neq}$, put

$$S_{\bar{a}} = \{[p_0, \ldots, p_{n-1}, q_0, \ldots, q_{m-1}] \mid \bar{q} \in (S^{<\omega}_0)_{\neq}, \ p_i \neq q_j, \ i < n, \ j < m\}.$$

It is easy to check that S, and therefore $S_{\bar{a}}$ as well, are Σ-definable in $\mathbb{HF}(G)$.

2′. Given n and r such that $r = [p_0, \dots, p_{n-1}, q_0, \dots, q_{m-1}]$, where $\langle p_0, \dots, p_{n-1}, q_0, \dots, q_{m-1}\rangle$ is a tuple of pairwise distinct primes, put

$$G_r^n = \bigoplus\{Z_{p_i} \mid i < n-1\} \oplus \bigoplus\{Z_{q_j} \mid j < m-1\},$$
$$S_r^n = \{[p_0, \dots, p_{n-1}, q_0, \dots, q_{m-1}]\}.$$

Put $Z_{p_i} = (b_{p_i})$. It is clear that the model $\langle G_r^n, \bar{b}\rangle$, with $\bar{b} = \langle b_{p_0}, \dots, b_{p_{n-1}}\rangle$, can be embedded into $\langle G, \bar{a}\rangle$ if and only if $S_r^n \subseteq S_{\bar{a}}$.

3. Take in $\langle G, \bar{a}\rangle$ some finitely generated subgroup $\langle G', \bar{a}\rangle$, $\bar{a} = \langle a_{p_0}, \dots, a_{p_{n-1}}\rangle \in \left(G_0^n\right)_{\neq}$. Denote by $H_0 \subseteq G'$ the subgroup generated by $a_i \leftrightharpoons a_{p_i}$ for $i < n$. Then there exists a tuple $q_0, \dots, q_{m-1}$ of primes such that $G' = H_0 \oplus Z_{q_0} \oplus \dots \oplus Z_{q_{m-1}}$.

Put $r = [p_0, \dots, p_{n-1}, q_0, \dots, q_{m-1}]$. Then the model $\langle G_r^n, \bar{b}\rangle$ defined in part 2 is isomorphic to $\langle G', \bar{a}\rangle$, and $S_r^n \subseteq S_{\bar{a}}$; thus, the validity of condition 3 is ascertained.

5. For each $r = [m_0, \dots, m_{n-1}, p_0, \dots, p_{n-1}]$, $m_i \in \omega$, $p_i \in P$, $m_i < p_i$, $p_i \neq p_j$, $i < j < n$, define the formula

$$\varphi_r^n(x, x_0, \dots, x_{n-1}) \rightleftharpoons (x = m_0x_0 + \dots + m_{n-1}x_{n-1}) \ \&\ \bigwedge_{i<n}(x_i \neq 0 \ \&\ p_ix_i = 0).$$

It is easy to check that for each $x \in G$ there exists a number r such that the formula $\varphi_r(x, a_{p_0}, \dots, a_{p_{n-1}})$ defines x in $\mathbb{HF}(G)$, and for each $\bar{a} \in G_0$ the set $\varphi_r^{\langle G,\bar{a}\rangle}[x, \bar{a}]$ contains at most one element.

All requirements are met of the definition of a computably $S\Sigma$-generated model for G and G_0. □

Corollary 10. *Take $S \subseteq P$ and $G = \bigoplus\{Z_p \mid p \in S\}$. A set $A \subseteq \omega$ is Σ-definable in $HF(G)$ if and only if $A \leq_e S$. The ideal $I_e(G)$ is principal.*

Indeed, by Lemma 5 and Corollary 1 there exists $\bar{a} \in \left(G_0^{<\omega}\right)_{\neq}$ such that $A \leq S_{\bar{a}}$. Because $S_{\bar{a}} \leq_e S$, it follows that $A \leq_e S$. The set S is Σ-definable in $\mathbb{HF}(G)$. Consequently, $d_e(S) \in I_e(G)$; thus, $I_e(G)$ is a principal ideal.

Corollary 11. *For each principal e-ideal I there exists a periodic abelian group G with $I_e(G) = I$.*

Indeed, if $S \subseteq P$ is such that $I = \widehat{d_e(S)}$ then Corollary 10 shows that $G \rightleftharpoons G_S$ is the required group.

Definition. Call $U_\omega(x_0, x_1)$ a *strictly universal numerical function* in the hereditarily finite admissible set $\mathbb{HF}(\mathfrak{M})$ whenever the family $\mathbb{HF}(\mathfrak{M})$ of all one-place numerical Σ-functions can be represented in the form

$$\{U_\omega(x_0, x_1) \mid x_0 \in \omega\}.$$

Lemma 6 [10]. *There exists a principal e-ideal I that has no universal function for the class of one-place functions on I.*

By Corollary 11, this implies

Corollary 12. *There exists a periodic abelian group G such that $\mathbb{HF}(G)$ has no strictly universal numerical Σ-function.*

Remark 6. There exists a model $\mathfrak{M}$ such that $\mathbb{HF}(\mathfrak{M})$ has no universal function, but does have a strictly universal numerical Σ-function.

Indeed, take the strongly constructivizable model $\mathfrak{M}$, constructed in [11], such that $\mathbb{HF}(\mathfrak{M})$ has no universal Σ-function. Corollary 3 implies that $I_e(\mathfrak{M}) = 0$. Thus, $I_e(\mathfrak{M})$ has a universal numerical function. Hence, $\mathbb{HF}(\mathfrak{M})$ has a strictly universal numerical function; i.e., $\mathfrak{M}$ is the required model.

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