Siberian Mathematical Journal, Vol. 46, *No.* 4, *pp.* 637–651, 2005
Original Russian Text Copyright © 2005 *Levenshtam V. B.*

ASYMPTOTIC INTEGRATION OF PARABOLIC PROBLEMS WITH LARGE HIGH-FREQUENCY SUMMANDS

V. B. Levenshtam UDC 517.956

Abstract: We develop the averaging method theory for parabolic problems with rapidly oscillating summands some of which are large, i.e., proportional to the square root of the frequency of oscillations. In this case the corresponding averaged problems do not coincide in general with those obtained by the traditional averaging, i.e., by formally averaging the summands of the initial problem (since the principal term of the asymptotic expansion of a solution to the latter problem is not in general a solution to the so-obtained problem). In this article we consider the question of time periodic solutions to the first boundary value problem for a semilinear parabolic equation of an arbitrary order $2k$ whose nonlinear terms, including the large, depend on the derivatives of the unknown up to the order $k-1$. We construct the averaged problem and the formal asymptotic expansion of a solution. When the large summands depend on the unknown rather than its derivatives we justify the averaging method and the complete asymptotic expansion of a solution.

Keywords: parabolic equations, asymptotic behavior, boundary layer method, averaging method

§ 0. Introduction

Under consideration in the classical theory of the averaging method by N. N. Bogolyubov [1] (also see [2]) are the systems of ordinary differential equations representable as

$$\frac{dx}{dt} = f(x, \omega t), \tag{0.1}$$

where the vector-function $f(x, \tau)$ possesses the means in τ:

$$\langle f(x,\tau)\rangle = \lim_{N\to\infty} N^{-1} \int_0^N f(x,\tau)\, d\tau$$

and ω is a large parameter. At present, the main results of this theory are translated (for example, see [3] and the bibliography therein) to the parabolic problems and abstract parabolic equations of the form

$$\frac{dy}{dt} = Ay + g(y, \omega t), \tag{0.2}$$

where A is an unbounded linear operator (in the case of a parabolic problem A is generated by the corresponding elliptic differential expression and boundary conditions) and the vector-function $g(y,\tau)$ is subordinated to A in a sense and possesses the means in τ. In particular, the author constructed and justified in [4, 5] the complete asymptotic expansions for solutions to such parabolic problems (their partial sums can be considered as the leading approximations).

Together with the well-studied broad classes of equations (0.1) and (0.2), the various forms of the averaging method have been applied to study of some other classes of ordinary differential and parabolic

The author was supported by the Russian Foundation for Basic Research (Grant 01–01–00678) and the Program "Universities of Russia" (UR.04.01.029).

Rostov-on-Don. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 46, No. 4, pp. 805–821, July–August, 2005. Original article submitted April 29, 2004. Revision submitted October 27, 2004.

equations (other classes are not considered here) and several other problems. They in particular include the ordinary differential equations of the form

$$\frac{dx}{dt} = f(x, \omega t) + \omega^{\alpha} f_1(x, \omega t) \tag{0.3}$$

and the (abstract) parabolic problems of the form

$$\frac{dy}{dt} = Ay + g(y, \omega t) + \omega^{\alpha} g_1(y, \omega t), \tag{0.4}$$

where $\alpha > 0$, which differ from (0.1) and (0.2) by the presence of respective high-frequency summands proportional to some positive powers of the frequency ω and possessing the zero means in τ: $\langle f_1(x,\tau)\rangle = 0$ and $\langle g_1(x,\tau)\rangle = 0$. The problems with such large summands include some well-known problems: the problem on stability of the upper position of a mathematical pendulum under high-frequency vibrations of the pivot point [6, 7]; the problem on stabilization of the rectilinear rod under high-frequency contractions and dilatations [8]; the problem on the influence of high-frequency vibrations on appearance of convection [9]; the problem on motion of a material point under the action of vibration forces and connections; and the problem on motion of an ideal fluid in high-frequency vibration fields [10–13]. The important physical phenomena in these problems are connected with the presence of large high-frequency summands. This is one of the stimuli for construction of a systematic averaging theory for broad classes of equations with large high-frequency terms.

The author's interest in the averaging theory for equations of the form (0.3) and (0.4) was inspired by V. I. Yudovich's lectures on the averaging method (1991). In particular, he observed at these lectures that the averaging theory for equations (0.3) and (0.4) is interesting in the case of exponents $\alpha \geq 1/2$, since for $\alpha < 1/2$ the averaged equations for (0.3) and (0.4) are the same as for (0.1) and (0.2); therefore, this situation is of little interest from the viewpoint of applications and its mathematical study is plain. V. I. Yudovich named the exponent $\alpha = 1/2$ the *first bifurcation exponent* [12]. In this article we consider the parabolic problem (2.1), (2.2) of the form (0.4) with exponent $\alpha = 1/2$.

We should note that the averaging theory for (0.4) with $\alpha = 1/2$ is naturally constructed under essentially more stringent requirements on the nonlinear summands of the equations as compared with the averaging theory for equations (0.2). Since this article is devoted to questions of the averaging method theory for the semilinear parabolic problem (2.1), (2.2) of type (0.4), we will give explanations by example of these problems. In this case the indicated strengthening of the requirements consists in reduction of the maximal order of the derivatives of the unknown in the nonlinear part of the equation. For example, in the article [4] on construction of the complete asymptotic expansion for a semilinear parabolic problem of order $2k$ of type (0.2) we supposed that its nonlinear part may depend on the spatial derivatives of the solution up to the order $2k-1$. In this article, where we consider the parabolic problem (2.1), (2.2) of order $2k$ of type (0.4), we can only consider nonlinearities depending on the derivatives up to the order $k-1$. The point is that, in [4], the principal boundary layer term of the asymptotic expansion of a solution (in the norm of C) is of order $O(\omega^{-1})$, while here it is of order $O(\omega^{-1/2})$ as $\omega \to \infty$. In the latter case the derivatives of the order k have the asymptotic order $O(1)$. Therefore, in the case of dependence of the nonlinear part of a semilinear parabolic problem of type (0.4) on the derivatives of order higher than $k-1$ the usual arguments lead to a problem for the principal boundary layer term of the asymptotic expansion which is unsolvable in general.

In § 2 we construct the averaged problem for (2.1), (2.2) and the complete formal asymptotic expansion of a time periodic solution. However, we failed with justification of the averaging principle and the asymptotic expansion for the whole class of such problems (the obstacle is the dependence of the large summand $\sqrt{\omega}\varphi$ on the derivatives of the solution). In § 1 we justify the averaging method and the complete asymptotic expansion of a periodic solution constructed in § 2 for a class of problems narrower than (2.1), (2.2). Namely, we do this for problems (1.1), (1.2) in which the large summand can depend only on the unknown rather than its derivatives. Apparently, the last requirement is not necessary in general and dictated only by the methods for justification of asymptotic expansions available here. By the way, observe that in the case of functions $f_s(x,e)$ and $\varphi_s(x,e)$ compactly-supported in x we can justify the complete asymptotic expansion of a solution to problem (2.1), (2.2).

In conclusion, note that the terms of the equations considered in this article are represented by finite sums of harmonics (in the time variable) with multiple frequencies $s\omega$, $s \in \mathbb{Z}$, and we study time periodic solutions. However, replacing $s\omega$ in these equations with the expressions $\lambda_s\omega$, where λ_s is an arbitrary real, we can similarly consider the problem on almost periodic (in time) solutions.

The author expresses his gratitude to V. I. Yudovich for his interest in this research and useful discussion.

§ 1. Construction of the Complete Asymptotic Expansion, and Justification for One Class of Semilinear Parabolic Problems

1°. Let k, m, and n be natural numbers and let Ω be a bounded domain of the space $\mathbb{R}^n$ with a C^∞-smooth boundary $\partial\Omega$. We consider the problem of finding real $2\pi\omega^{-1}$-periodic solutions (in time t) to the following semilinear parabolic equation depending on a large parameter ω:

$$\frac{\partial u}{\partial t} - \sum_{|\alpha|\le 2k} a_\alpha(x) D^\alpha u - \sum_{0\le|s|\le m} f_s(x, \delta^{k-1}u)\exp(is\omega t)$$
$$-\sqrt{\omega}\sum_{0<|s|\le m} \varphi_s(x,u)\exp(is\omega t) = 0, \quad (x,t)\in\Omega\times\mathbb{R}^1 \equiv Q, \tag{1.1}$$

with the Dirichlet boundary conditions

$$u|_\Gamma = \frac{\partial u}{\partial \nu}\Big|_\Gamma = \dots = \frac{\partial^{k-1}u}{\partial\nu^{k-1}}\Big|_\Gamma = 0. \tag{1.2}$$

Here $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_n)$ is a multi-index of length $|\alpha| = \sum_{i=1}^n \alpha_i$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}$, $\Gamma = \partial\Omega\times\mathbb{R}^1$, ν is the inward normal to $\partial\Omega$, and $\delta^{k-1}u$ is the vector-function constituted by the function $u(x,t)$ and all its derivatives with respect to x up to the order $k-1$. Denote the number of components of such vector-function by p. Suppose that the real-valued functions $a_\alpha(x)$ have derivatives of arbitrary order, are continuous in the closure $\overline{\Omega}$ of Ω, and (1.1) enjoys the parabolicity condition; i.e., the following inequality is valid for all vectors $\xi = (\xi_1,\dots,\xi_n)\in\mathbb{R}^n$ and $x\in\overline{\Omega}$:

$$(-1)^{k+1}\sum_{|\alpha|=2k} a_\alpha(x)\xi_1^{\alpha_1}\dots\xi_n^{\alpha_n} \ge 0.$$

We also suppose that the complex-valued functions $f_s(x,e)$, where $e = (e_0,e_1,\dots,e_{p-1})\in\mathbb{R}^p$, and $\varphi_s(x,e_0)$ are defined and continuous on[1)] $\overline{\Omega}\times\mathbb{R}^p$ and $\overline{\Omega}\times\mathbb{R}^1$ and have there continuous derivatives with respect to all arguments of arbitrary order. We assume that the values $f_s(x,e)$ and $f_{-s}(x,e)$ as well as $\varphi_s(x,e_0)$ and $\varphi_{-s}(x,e_0)$ are complex conjugate. Alongside the perturbed problem (1.1), (1.2), we consider the so-called averaged problem:

$$\frac{\partial v}{\partial t} - \sum_{|\alpha|\le 2k} a_\alpha(x)D^\alpha v - f_0(x,\delta^{k-1}v)$$
$$-\sum_{0<|s|\le m} is^{-1}\frac{\partial\varphi_s}{\partial v}(x,v)\varphi_{-s}(x,v) = 0, \quad (x,t)\in Q, \tag{1.3}$$

$$v|_\Gamma = \frac{\partial v}{\partial \nu}\Big|_\Gamma = \dots = \frac{\partial^{k-1}v}{\partial\nu^{k-1}}\Big|_\Gamma = 0. \tag{1.4}$$

[1)] We can assume that f_s are defined only on $\overline{\Omega}\times U_p$ and φ_s are defined on $\overline{\Omega}\times U_0$, where U_p and U_0 are balls in $\mathbb{R}^p$ and $\mathbb{R}^1$ containing $\{\delta^{k-1}u_0(x),\ x\in\overline{\Omega}\}$ and $\{u_0(x),\ x\in\overline{\Omega}\}$, where $u_0(x)$ is a solution to the averaged problem (1.3), (1.4).

Suppose that this problem has a nondegenerate stationary solution $u_0(x)$. Nondegeneracy of u_0 means that the problem

$$Lv \equiv \sum_{|\alpha|\le 2k} a_\alpha(x)D^\alpha v + \sum_{j=0}^{p-1} \frac{\partial f_0[x, \delta^{k-1}u_0(x)]}{\partial e_j}(\delta^{k-1}v)_j$$
$$+ \sum_{0<|s|\le m} is^{-1}\frac{\partial^2 \varphi_s}{\partial u^2}[x, u_0(x)]\varphi_{-s}[x, u_0(x)]v = 0, \quad x \in \Omega, \tag{1.5}$$

$$v|_{\partial\Omega} = \frac{\partial v}{\partial \nu}\bigg|_{\partial\Omega} = \dots = \frac{\partial^{k-1} v}{\partial \nu^{k-1}}\bigg|_{\partial\Omega} = 0$$

has only the zero solution. Here $(\delta^{k-1}v)_j$ is the jth component of the vector-function $\delta^{k-1}v$.

Below (see Items 1 and 2 of the theorem) we establish existence of a solution u_ω to (1.1), (1.2) which is unique in some neighborhood of u_0 and is $2\pi\omega^{-1}$-periodic in t. In the next subsection we construct asymptotic expansion for this solution.

2°. Since we construct the asymptotic expansion of u_ω, using the boundary layer method [14]; as a preliminary we introduce local curvilinear coordinates in some boundary strip of Ω.

Let $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_{n-1})$ be curvilinear orthogonal coordinates on $\partial\Omega$. For every point $x \in \Omega$ lying in a neighborhood of the boundary $\partial\Omega$ small enough for the normals to $\partial\Omega$ to be disjoint in this neighborhood, denote by $r(x)$ the distance from x to $\partial\Omega$ measured along the normal. Now, for such x's define the orthogonal curvilinear coordinates $(\varphi, r) \equiv (\varphi(x), r(x))$, where $\varphi(x)$ is the point of $\partial\Omega$ lying on the same normal with x. Given a sufficiently small number $r_0 > 0$, put $\Omega_0 = \{x \in \Omega : r(x) < r_0\}$. For brevity, we consider the boundary layer terms of the asymptotic expansion of the solution $u_\omega(x,t)$ only for $x \in \Omega_0$. Recall that, according to the boundary layer method [14], having constructed these terms in the boundary strip Ω_0, we have to extend them by zero to $\Omega \setminus \Omega_0$ and then multiply by a smooth truncator $\mu(x)$ in $\overline{\Omega}$ such that $\mu(x) = 1$ for $r(x) < r_0/2$ and $\mu(x) = 0$ for $x \in \Omega\setminus\Omega_0$. Observe also that for application of the boundary layer method to (1.1), (1.2) we have to pass to the curvilinear coordinates (φ, ρ), $\rho = r\omega^{1/2k}$ in it. Moreover, we will use the representation

$$\sum_{|\alpha|\le 2k} a_\alpha(x)D^\alpha u = \omega\left[b(\varphi)\frac{\partial^{2k}u}{\partial\rho^{2k}} + \sum_{j=1}^{N} \omega^{-\frac{j}{2k}}M_j(\varphi,\rho)u + \omega^{-\frac{N+1}{2k}}M_{N+1}(\varphi,\rho,r)u\right]. \tag{1.6}$$

Here $b \in C^\infty(\partial\Omega)$, $(-1)^{k+1}b > 0$, N is an arbitrary natural number, M_j, $1 \le j \le N$, is a differential expression in φ and ρ whose coefficients are polynomials in ρ with coefficients infinitely differentiable with respect to φ, M_{N+1} is a differential expression in φ and ρ with coefficients infinitely differentiable with respect to φ and r.

We construct the asymptotic expansion of u_ω in the form

$$u_\omega(x,t) = u_0(x) + \omega^{-1/2}[u_k(x) + v_k(x,\tau) + z_k(\varphi,\rho,\tau)]$$
$$+ \sum_{j=k+1}^{\infty} \omega^{-\frac{j}{2k}}[u_j(x) + z_j(\varphi,\rho,\tau)] + \sum_{j=2k}^{\infty} \omega^{-\frac{j}{2k}}[v_j(x,\tau) + w_j(\varphi,\rho)], \tag{1.7}$$

where $\rho = \omega^{1/(2k)}r$, $\tau = \omega t$. Here $u_i(x)$ and $v_i(x,\tau)$ are the so-called regular functions, while $z_i(\varphi,\rho,\tau)$ and $w_i(\varphi,\rho)$ are the boundary layer functions[2)] such that $\lim_{\rho\to\infty} z_i(\varphi,\rho,\tau) = 0$ and $\lim_{\rho\to\infty} w_i(\varphi,\rho) = 0$.

[2)] Taking some liberty, we use (φ,ρ) rather than x as the spatial arguments of the boundary layer functions, thereby emphasizing that these functions are only interesting for $x \in \Omega_0$, vanishing at $x \in \Omega \setminus \Omega_0$.

Moreover, the functions $v_i(x,\tau)$ and $z_i(\varphi,\rho,\tau)$ are 2π-periodic in τ with the zero mean:

$$\langle v_i(x,\tau)\rangle = (2\pi)^{-1}\int\limits_0^{2\pi} v_i(x,\tau)\,d\tau = 0, \quad \langle z_i(\varphi,\rho,\tau)\rangle = 0.$$

To find the coefficients of (1.7), substitute the series for u in (1.1), (1.2), expand formally f_s and φ_s in the Taylor series about the respective points $(x,\delta^{k-1}u_0)$ and (x,u_0) and, in the so-obtained equalities, using (1.6), collect and equate the coefficients of the same powers of ω separately for the regular and boundary layer functions. Now, divide the resulting equations into the equations for stationary coefficients and those for oscillating (in τ) coefficients with the zero mean (we can interpret this procedure as a consequence of the solvability condition for the equations). Eventually, we arrive at a recurrent sequence of problems which, as demonstrated below, determines uniquely the coefficients of (1.7).

We thus obtain the following problem for the coefficient $v_k(x,\tau)$ of (1.7):

$$\frac{\partial v_k}{\partial \tau} = \sum_{0<|s|\le m} \varphi_s[x,u_0(x)]\exp(is\tau), \quad \langle v_k(x,\tau)\rangle = 0,$$

where x plays the role of a parameter. Hence, we find

$$v_k(x,\tau) = -\sum_{0<|s|\le m} is^{-1}\varphi_s[x,u_0(x)]\exp(is\tau). \tag{1.8}$$

For the coefficient u_0 we obtain problem (1.3), (1.4) which is solvable by the assumption of 1°.

The coefficient v_{2k} satisfies the conditions

$$\begin{aligned}\frac{\partial v_{2k}}{\partial \tau} &= \sum_{0<|s|\le m} f_s(x,\delta^{k-1}u_0)\exp(is\tau)\\ &- \sum_{\substack{0<|s_1|,|s_2|\le m\\ s_1+s_2\ne 0}} is_2^{-1}\frac{\partial \varphi_{s_1}[x,u_0(x)]}{\partial u}\varphi_{s_2}[x,u_0(x)]\exp[i(s_1+s_2)\tau]\\ &+ \sum_{0<|s|\le m}\frac{\partial \varphi_s}{\partial u}[x,u_0(x)]u_k(x)\exp(is\tau), \quad \langle v_{2k}\rangle = 0.\end{aligned}$$

Hence,

$$v_{2k}(x,\tau) = \sum_{0<|\ell|\le 2m}\gamma_\ell(x)\exp(i\ell\tau) + \sum_{0<|s|\le m} is^{-1}\frac{\partial\varphi_s}{\partial u}[x,u_0(x)]\exp(is\tau)u_k, \tag{1.9}$$

where $\gamma_\ell(x)$ is a known function.

The coefficient $u_k(x)$ is a solution to the problem

$$Lu_k = \psi_k(x), \quad u_k\big|_{\partial\Omega} = 0, \quad \frac{\partial u_k}{\partial \nu}\bigg|_{\partial\Omega} = 0, \dots, \frac{\partial^{k-1}u_k}{\partial \nu^{k-1}}\bigg|_{\partial\Omega} = 0, \tag{1.10}$$

where ψ_k is a known function. Determining u_k from (1.10) and inserting it in (1.9), we find v_{2k}.

By (1.6), the coefficient $z_k(\varphi,p,\tau)$ of (1.7) is a solution to the problem

$$\frac{\partial z_k}{\partial \tau} = b(\varphi)\frac{\partial^{2k}z_k}{\partial\rho^{2k}}, \quad z_k|_{\rho=0} = -v_k|_{r=0}, \quad \frac{\partial z_k}{\partial\rho}\bigg|_{\rho=0} = 0, \dots, \frac{\partial^{k-1}z_k}{\partial\rho^{k-1}}\bigg|_{\rho=0} = 0. \tag{1.11}$$

By (1.8), a solution to (1.11) has the form

$$z_k(\varphi,\rho,\tau) = \sum_{0<|s|\le m} c_s(\varphi,\rho)\exp(is\tau), \tag{1.12}$$

where the coefficients c_s in turn are solutions to the problems

$$
\begin{gathered}
isc_s - b(\varphi)\frac{\partial^{2k} c_s}{\partial \rho^{2k}} = 0, \\
c_s|_{\rho=0} = is^{-1}\varphi_s[x, u_0(x)]|_{r=0} \equiv a_s(\varphi), \quad \left.\frac{\partial c_s}{\partial \rho}\right|_{\rho=0} = 0, \dots, \left.\frac{\partial^{k-1} c_s}{\partial \rho^{k-1}}\right|_{\rho=0} = 0, \\
c_s|_{\rho=\infty} = 0, \quad s = \pm 1, \pm 2, \dots, \pm m.
\end{gathered}
\tag{1.13}
$$

Denoting by $\lambda_{js}(\varphi)$, $j = 1, 2, \dots, k$, the roots of the equation $b(\varphi)\lambda^{2k} = is$ satisfying the conditions $\operatorname{Re}\lambda_{js} < 0$, we find the unique solution to (1.13) in the form

$$
c_s = \sum_{j=1}^{k} d_{js}(\varphi)\exp(\lambda_{js}\rho). \tag{1.14}
$$

From (1.12) and (1.14) we find a solution to (1.11).

The coefficient w_{2k} of (1.7) is a solution to the problem

$$
b(\varphi)\frac{\partial^{2k} w_{2k}}{\partial \rho^{2k}} = \left\langle \frac{\partial \varphi_s}{\partial u}[x, u_0(x)]\big|_{r=0} z_k \right\rangle \equiv \sum_{\substack{0<|s|\le m \\ 1\le j\le k}} e_{js}(\varphi)\exp(\lambda_{js}\rho), \quad w_{2k}|_{\rho=\infty} = 0.
$$

Hence,

$$
w_{2k}(\varphi, \rho) = \sum_{\substack{0<|s|\le m \\ 1\le j\le k}} f_{js}(\varphi)\exp(\lambda_{js}\rho), \quad f_{js}(\varphi) = \frac{e_{js}(\varphi)}{b(\varphi)\lambda_{js}^{2k}(\varphi)}.
$$

It is easy to show that, using the procedure described above, we can find all coefficients of (1.7). We do not dwell upon the matter now, since this fact will be established in § 2 for a broader class of problems.

Introduce the function

$$
\begin{gathered}
\overset{N}{u}_\omega(x, t) = u_0(x) + \omega^{-1/2}[u_k(x) + v_k(x, \omega t) + z_k(\varphi, \rho, \omega t)] \\
+ \sum_{j=k+1}^{N} \omega^{-\frac{j}{2k}}[u_j(x) + z_j(\varphi, \rho, \omega t)] + \sum_{j=2k}^{N} \omega^{-\frac{j}{2k}}[v_j(x, \omega t) + w_j(\varphi, \rho)],
\end{gathered}
\tag{1.15}
$$

where we assume $N \ge 2k$ for simplicity of notation. If $N < 2k$ then (1.15) is understood in the natural sense: for $N < k$ it is $\overset{N}{u}_\omega(x, t) = u_0(x)$ and for $k \le N < 2k$ we have to delete the last summand $\sum_{j=2k}^{N} \dots$ from (1.15).

We now state the theorem on the estimate for the difference of a $2\pi\omega^{-1}$-periodic (in t) solution to (1.1), (1.2) and the approximation $\overset{N}{u}$ in the Hölder norm of the space $C^{\ell,\ell/2k}(Q) \equiv C^{\ell,\ell/2k}$, $\ell \ge 0$. Recall [15, p. 81] that the space $C^{\ell,\ell/2k}(Q)$ is constituted by the functions $u(x, t)$, $(x, t) \in Q$, satisfying the condition

$$
\begin{gathered}
\|u\|_{C^{\ell,\ell/2k}} = \sum_{j=0}^{[\ell]} \sum_{2k\mu+|\nu|=j} \sup_{(x,t)\in Q} \left| \frac{\partial^\mu}{\partial t^\mu} D^\nu u(x, t) \right| \\
+\delta_\ell \Bigg(\frac{1}{|x-y|^{\ell-[\ell]}} \sum_{2k\mu+|\nu|=[\ell]} \sup_{\substack{(x,t),(y,t)\in Q \\ x\ne y}} \left| \frac{\partial^\mu}{\partial t^\mu} D^\nu u(x, t) - \frac{\partial^\mu}{\partial t^\mu} D^\nu u(y, t) \right| \\
+ \sum_{0<\ell-2k\mu-|\nu|<2k} \sup_{\substack{(x,t),(x,\tau)\in Q \\ t\ne\tau}} \left| \frac{\partial^\mu}{\partial t^\mu} D^\nu u(x, t) - \frac{\partial^\mu}{\partial \tau^\mu} D^\nu u(x, \tau) \right| |t-\tau|^{-\frac{\ell-2k\mu-|\nu|}{2k}} \Bigg) < \infty,
\end{gathered}
$$

where $[\ell]$ is the integral part of ℓ and $\delta_\ell = 0$ for an integer ℓ and $\delta_\ell = 1$ for a noninteger ℓ.

Similarly, we define the Banach space $C^{\ell_1,\ell_2}(Q) \equiv C^{\ell_1,\ell_2}_{x,t}(Q)$, $\ell_1 \geq 0$, $\ell_2 \geq 0$.

Denote by $C_\mu(R, E)$, where E is a Banach space and $\mu \in (0, 1]$, the Banach space of vector-functions $u : R \to E$ satisfying the condition

$$\|u\|_{C_\mu(R,E)} = \sup_{t\in\mathbb{R}} \|u(t)\|_E + \sup_{-\infty<t_1<t_2<\infty} \|u(t_2) - u(t_1)\|/(t_2 - t_1)^\mu < \infty.$$

We also use the similar space $C_\mu([T_1, T_2], E)$, $-\infty < T_1 < T_2 < \infty$, and the space $C(R, E)$ of continuous bounded vector-functions.

Theorem. *There is a number $\omega_0 > 0$ such that the following assertions are valid for $\omega > \omega_0$:*

1. *Problem* (1.1), (1.2) *has an infinitely differentiable $2\pi\omega^{-1}$-periodic (in time t) solution $u_\omega(x, t)$ satisfying the inequality*

$$\|u_\omega - u_0\|_{C^{\ell,\ell/2k}} \leq C(0, \ell)\omega^{-\frac{1}{2}+\frac{\ell}{2k}}, \quad C(0, \ell) = \text{const}.$$

2. *The solution u_ω is unique in the ball $\|u - u_0\|_{C^{k-1,0}} \leq r_0$ for some $r_0 > 0$.*

3. *The leading approximations $\overset{N}{u}$ defined by* (1.15) *satisfy the following estimates for all $\ell \geq 0$:*

$$\|u_\omega - \overset{N}{u}\|_{C^{\ell,\ell/2k}} \leq C(0, \ell)\omega^{-\frac{1}{2}+\frac{\ell}{2k}}, \quad N < k;$$

$$\|u_\omega - \overset{N}{u}\|_{C^{\ell,\ell/2k}} \leq C(N, \ell)\omega^{-\frac{N+1-\ell}{2k}}, \quad N \geq k,$$

where $C(N, \ell) = \text{const}$. Construction of the approximations $\overset{N}{u}$ reduces to solution of N uniquely solvable linear elliptic Dirichlet problems in the domain Ω with the same differential expression L of order $2k$ (see (1.5)) *and infinitely differentiable right-hand sides and boundary conditions.*

The proof is given in the following two subsections.

3°. Carry out the following change of variables in (1.1), (1.2):

$$u = w + y_\omega, \tag{1.16}$$

where

$$\begin{aligned} y_\omega = u_0 &+ \omega^{-1/2}[u_k(x) + v_k(x, \omega t) + z_k(\varphi, \rho, \omega t)] \\ &+ \sum_{j=k+1}^{M+2k+1} \omega^{-\frac{j}{2k}} u_j(x) + \sum_{j=k+1}^{2k-1} \omega^{-\frac{j}{2k}} z_j(\varphi, \rho, \omega t) \\ &+ \sum_{j=2k}^{M+3k} \omega^{-\frac{j}{2k}} [v_j(x, \omega t) + z_j(\varphi, \rho, \omega t) + w_j(\varphi, \rho)] \\ &+ \big(1 - \delta_k^1\big) \sum_{j=M+2k+2}^{M+3k} \omega^{-\frac{j}{2k}} \hat{u}_j(x) \equiv u_0 + \hat{y}_\omega. \end{aligned}$$

Here u_j, v_j, z_j, and w_j are the same as in 2° (see (1.7) and (1.15)) and the functions $\hat{u}_j$ are found from the same differential equations as the corresponding coefficients u_j of (1.7) but with the boundary conditions guaranteeing validity of (1.2) for y_ω. As a result of the change, we obtain the problem

$$\frac{\partial w}{\partial t} - Lw = \Phi_\omega(x, t, w) + \beta_\omega(x, t) + \sqrt{\omega} \sum_{0<|s|\leq m} \frac{\partial \varphi_s}{\partial u}(x, u_0) \exp(is\omega t) w, \tag{1.17}$$

$$w|_\Gamma = \frac{\partial w}{\partial \nu}\bigg|_\Gamma = \cdots = \frac{\partial^{k-1} w}{\partial \nu^{k-1}}\bigg|_\Gamma = 0. \tag{1.18}$$

Here

$$\beta_\omega(x,t) = -P_\omega(y_\omega), \tag{1.19}$$

where $P_\omega(u)$ is the left-hand side of (1.1) and $\Phi_\omega = \Phi_{\omega,1} + \Phi_{\omega,2}$, with

$$\Phi_{\omega,1}(x,t,w) = \sum_{0<|s|\le m}\sum_{j=0}^{p-1}\left\{\int_0^1 \frac{\partial f_s}{\partial e_j}[x,\delta^{k-1}(y_\omega+\theta w)]\,d\theta - \frac{\partial f_s}{\partial e_j}(x,\delta^{k-1}u_0)\right\}$$

$$\times(\delta^{k-1}w)_j\exp(is\omega t) + \sum_{j=0}^{p-1}\left\{\int_0^1 \frac{\partial f_0}{\partial e_j}[x,\delta^{k-1}(y_\omega+\theta w)]d\theta - \frac{\partial f_0}{\partial e_j}(x,\delta^{k-1}u_0)\right\}(\delta^{k-1}w)_j$$

$$+\sum_{0<|s|\le m}\left[\frac{\partial^2\varphi_s}{\partial u^2}(x,u_0+\theta_1\hat{y}_\omega) - \frac{\partial^2\varphi_s}{\partial u^2}(x,u_0)\right]u_k\exp(is\omega t)w$$

$$+\sum_{0<|s|\le m}\frac{\partial^2\varphi_s}{\partial u^2}(x,u_0+\theta_1\hat{y}_\omega)\left[\sum_{j=k+1}^{M+2k+1}\omega^{-\frac{j-k}{2k}}u_j + \sum_{j=2k}^{M+3k}\omega^{-\frac{j-k}{2k}}(v_j+z_j+w_j)\right.$$

$$\left.\sum_{j=k+1}^{2k-1}\omega^{-\frac{j-k}{2k}}z_j + \sum_{j=2k+2}^{M+3k}\omega^{-\frac{j-k}{2k}}\hat{u}_j + z_k\right]\exp(is\omega t)w$$

$$-\sum_{\substack{0<|s_1|,|s_2|\le m\\ s_1+s_2\ne 0}} is_2^{-1}\left[\frac{\partial^2\varphi_{s_1}}{\partial u^2}(x,u_0+\theta_1\hat{y}_\omega) - \frac{\partial^2\varphi_{s_1}}{\partial u^2}(x,u_0)\right]\varphi_{s_2}(x,u_0)\exp[i(s_1+s_2)\omega t]w$$

$$+\sqrt{\omega}\sum_{0<|s|\le m}\int_0^1(1-\theta)\frac{\partial^2\varphi_s}{\partial u^2}(x,y_\omega+\theta w)\,d\theta\,w^2, \quad 0\le\theta_1\le 1, \tag{1.20}$$

$$\Phi_{\omega,2} = \sum_{0<|s|\le m}\sum_{j=0}^{p-1}\frac{\partial f_s}{\partial e_j}(x,\delta^{k-1}u_0)(\delta^{k-1}w)_j\exp(is\omega t)$$

$$+\sum_{\substack{0<|s_1|,|s_2|\le m\\ s_1+s_2\ne 0}} is_2^{-1}\frac{\partial^2\varphi_{s_1}}{\partial u^2}(x,u_0)\varphi_{s_2}(x,u_0)\exp[i(s_1+s_2)\omega t]w$$

$$+\sum_{0<|s|\le m}\frac{\partial^2\varphi_s(x,u_0)}{\partial u^2}u_k\exp(is\omega t)w. \tag{1.21}$$

Carry out the following change of variables in (1.17), (1.18):

$$w = v\chi_\omega(x,t), \tag{1.22}$$

where

$$\chi_\omega(x,t) = \exp\left\{\omega^{-1/2}\sum_{0<|s|\le m}(is)^{-1}\frac{\partial\varphi_s}{\partial u}(x,u_0)\exp(is\omega t)\right\}.$$

Eventually, we arrive at the problem

$$\frac{\partial v}{\partial t} - Lv = F_\omega(x,t,v) + b_\omega(x,t), \tag{1.23}$$

$$v|_\Gamma = \left.\frac{\partial v}{\partial\nu}\right|_\Gamma = \cdots = \left.\frac{\partial^{k-1}v}{\partial\nu^{k-1}}\right|_\Gamma = 0. \tag{1.24}$$

Here

$$F_\omega(x,t,v) = \{\Phi_{\omega,1}(x,t,v\chi_\omega)\chi_\omega^{-1} + [L(v\chi_\omega)\chi_\omega^{-1} - Lv]\}$$
$$+\Phi_{\omega,2}(x,t,v\chi_\omega)\chi_\omega^{-1} \equiv F_{\omega,1}(x,t,v) + F_{\omega,2}(x,t,v), \tag{1.25}$$

$$b_\omega(x,t) = \beta_\omega(x,t)\chi_\omega^{-1}. \tag{1.26}$$

Denote by A the linear operator in the Banach space $B = L_r(\Omega)$, $r > 1$, with domain $D(A) = \{v \in W_r^{2k}(\Omega) : v \text{ satisfies (1.18)}\}$ acting by the rule $Av = Lv$. As is well known, the spectrum $\sigma(A)$ of A is discrete and there exist numbers c_0 and λ_0 (for example, see the reference in [16, p. 349]) such that the inequality

$$\|(A - \lambda_0 I)^{-1}\| \le c_0(1 + |\lambda|)^{-1}$$

is valid for every λ with $\operatorname{Re}\lambda \ge 0$. By this inequality, the operator A generates the analytic semigroup $\exp(tA)$ in B (for example, see [17]). This enables us to consider (1.23), (1.24) as an abstract parabolic equation in the Banach space B. Let $t_0 > 0$ be a number such that $\exp(\lambda t_0) \ne 1$ for all $\lambda \in \sigma(A)$, and let $T_\omega = 2\pi\omega^{-1}[t_0(2\pi)^{-1}\omega]$, where $[a]$ is the integral part of a. Then $\exp(\lambda T_\omega) \ne 1$, $\lambda \in \sigma(A)$, for ω large and hence, as is well known, the problem on T_ω-periodic (in time t) solutions to system (1.23), (1.24) for the indicated ω is equivalent to the integral equation

$$v(t) = \int_0^t \exp[(t-\tau)A]\{F_\omega[\,\cdot\,,\tau,v(\tau)] + b_\omega(\,\cdot\,,\tau)\}d\tau + [I - \exp(T_\omega L)]^{-1}$$
$$\times \int_0^{t_\omega} \exp[(T_\omega + t - \tau)A]\{F_\omega[\,\cdot\,,\tau,v(\tau)] + b(\,\cdot\,,\tau)\}d\tau \equiv [M_\omega(v)](t), \tag{1.27}$$

$v \in C([0,t_0], C^{2k-1}(\overline{\Omega}))$.[3)] Denote by B^α, $\alpha \ge 0$, the Banach space constituted by the functions $v \in D((-A_1)^\alpha)$, where $(-A_1)^\alpha$ is the fractional power of the operator $-A_1 = -A + \lambda_0 I$, with the norm $\|v\|_{B^\alpha} = \|(-A_1)^\alpha v\|_B$. As is well known [17, 3], for $q > n$, there is $\beta \in (0,1)$, for which we have the continuous embedding

$$B^\beta \subset \mathring{C}^{2k-1}(\overline{\Omega}), \tag{1.28}$$

where $\mathring{C}^{2k-1}(\overline{\Omega})$ is the subspace of $C^{2k-1}(\overline{\Omega})$ spanned by $v \in C^{2k-1}(\overline{\Omega})$ satisfying (1.18).

Lemma 1. *Let $\mu \in (0, 1-\beta)$. Then there exist numbers r and ω_0 such that, for $\omega > \omega_0$, the operator M_ω acting in the space $C_\mu([0,t_0], \mathring{C}^{2k-1}(\overline{\Omega}))$ and defined by the right-hand side of* (1.27) *is a contraction in the ball $S_{\omega,r}$:*

$$\|v\|_{C_\mu([0,t_0],\mathring{C}^{2k-1}(\overline{\Omega}))} \le r\omega^{-\frac{M+1+k}{2k}}.$$

Proof. Here we do not present the complete proof of the lemma, outlining only the basic moments. First of all we write down the two well-known inequalities [17, 3]:

$$\|(-A_1)^\gamma \exp(tA)\|_{\mathrm{Hom}(B,B)} \le c_1 t^{-\gamma}, \quad t \in (0,\tau_0], \tag{1.29}$$

$$\|(-A_1)^\gamma[\exp(t_2A) - \exp(t_1A)]\|_{\mathrm{Hom}(B,B)} \le c_2(t_2 - t_1)^\eta t_1^{-\gamma-\eta}, \tag{1.30}$$

$0 < t_1 < t_2 \le \tau_0$. Here $\gamma \ge 0$, $\eta \in (0,1]$, $\tau_0 > 0$, $c_1 = c_1(\gamma,\tau_0)$ and $c_2 = c_2(\gamma,\eta,\tau_0)$ are constants, and $\mathrm{Hom}(B,B)$ is the Banach space of bounded linear operators in B with the usual operator norm.

[3)] Speaking of equivalence, here we mean the corresponding connection between T_ω-periodic solutions to (1.23), (1.24) and T_ω-periodic (in t) extensions of solutions to (1.28) to the whole axis $t \in \mathbb{R}$.

Using (1.28)–(1.30), we establish that M_ω acts in $C_\mu([0,t_0],\overset{\circ}{C}{}^{2k-1}(\overline{\Omega}))$ for ω sufficiently large. Now, observe two simple properties of $\chi_\omega(x,t)$ to be used essentially up to the end of this section:

$$\left\|D^\alpha \chi_\omega^{\pm 1}\right\|_{C(R,C(\overline{\Omega}))} = O(\omega^{-1/2}) \quad \text{as } \omega \to \infty$$

if $|\alpha| \neq 0$;

$$\left\|\chi_\omega^{\pm 1}(x,t_2) - \chi_\omega^{\pm 1}(x,t_1)\right\|_{C(\overline{\Omega})} \le c_5 |t_2 - t_1|^{1/2}$$

for arbitrary values $t_1, t_2 \in \mathbb{R}$, $c_5 = \text{const}$. From (1.19), (1.26), and (1.25) and the results of 2° we obtain the relations

$$\|b_\omega\|_{C([0,t_0],B)} \le c_3 \omega^{-\frac{M+1+k}{2k}}, \; c_3 = \text{const}, \quad F_\omega(x,t,0) = 0,$$

where $B = L_q(\Omega)$, $q > n$. Hence, by (1.28)–(1.30), we derive the estimate

$$\|M_\omega(0)\|_{C_\mu([0,t_0],\overset{\circ}{C}{}^{2k-1}(\overline{\Omega}))} \le c_4 \omega^{-\frac{M+1+k}{2k}}. \tag{1.31}$$

From (1.20), (1.21), and (1.25) we find that, for each pair r_1, ω_1 of positive numbers, there is a value $c(r_1,\omega_1)$ such that the following estimate is valid for $\omega > \omega_1$ and $\||v_1| + |v_2|\|_{C([0,t_0],C^{2k-1}(\overline{\Omega}))} \le r_1 \omega^{-1/2}$:

$$\|F_{\omega,1}(x,t,v_2) - F_{\omega,1}(x,t,v_1)\|_{C([0,t_0],B)} \le c(r_1,\omega_1)\|v_2 - v_1\|_{C([0,t_0],C^{2k-1}(\overline{\Omega}))}; \tag{1.32}$$

moreover,

$$\lim_{r_1 \to 0, \omega_1 \to \infty} c(r_1,\omega_1) = 0. \tag{1.33}$$

Denote by $M_{\omega,1}(v)$ the expression obtained from $M_\omega(v)$ by replacing F_ω in (1.27) with $F_{\omega,1}(v)$ and put $M_{\omega,2} = M_\omega - M_{\omega,1}$. Consider the ball $S_{\omega,r}$ for $r = 2c_4$, where c_4 is the same as in (1.31). Relations (1.32), (1.33), and (1.28)–(1.30) imply existence of $\omega_2 > 0$ such that, for $\omega > \omega_2$, all functions $v_1, v_2 \in S_{\omega,r}$ satisfy the estimate

$$\|M_{\omega,1}(v_2) - M_{\omega,1}(v_1)\|_{C_\mu([0,t_0],C^{2k-1}(\overline{\Omega}))} < \frac{1}{4}\|v_2 - v_1\|_{C([0,t_0],C^{2k-1}(\overline{\Omega}))}. \tag{1.34}$$

Following the method of [3, § 1], we can establish existence of $\omega_3 > 0$ such that, for $\omega > \omega_3$, the functions $v_1, v_2 \in S_{\omega,r}$ satisfy the estimate

$$\|M_{\omega,2}(v_2) - M_{\omega,2}(v_1)\|_{C_\mu([0,t_0],C^{2k-1}(\overline{\Omega}))} < \frac{1}{4}\|v_2 - v_1\|_{C_\mu([0,t_0],C^{2k-1}(\overline{\Omega}))}. \tag{1.35}$$

From (1.32), (1.34), and (1.35) we obtain the conclusion of the lemma.

4°. We now prove Item 1 of the theorem. In 3° we carried out the change of variables (1.16) in (1.1), (1.2): $u = w + y_\omega$, where the function $y_\omega \equiv \overset{M}{y}_\omega$ depends on M. In particular,

$$\begin{aligned} \overset{0}{y}_\omega &= u_0 + \omega^{-1/2}[v_k(x,\omega t) + z_k(\varphi,\rho,\omega t)] \\ &+ \sum_{j=k}^{2k+1} \omega^{-\frac{j}{2k}} u_j(x) + \sum_{j=k+1}^{2k-1} \omega^{-\frac{j}{2k}} z_j(\varphi,\rho,\omega t) \\ + \sum_{j=2k}^{3k} \omega^{-\frac{j}{2k}} &[v_j(x,\omega t) + z_j(\varphi,\rho,\omega t) + w_j(\varphi,\rho)] + \left(1 - \delta_k^1\right) \sum_{j=2k+2}^{3k} \omega^{-\frac{j}{2k}} \hat{u}_j(x). \end{aligned} \tag{1.36}$$

By Lemma 1 with $M = 0$, Remark 2, and the changes of variables (1.16) and (1.22), the following assertion is valid. There are numbers c_0 and ω_0 such that, for $\omega > \omega_0$, (1.1), (1.2) has a unique T_ω-periodic (in t) solution $u_\omega(x,t)$ in the ball

$$\|u - \overset{0}{y}_\omega\|_{C_\mu(R,\overset{\circ}{C}{}^{2k-1}(\overline{\Omega}))} \le c_0 \omega^{-\frac{k+1}{2k}}. \tag{1.37}$$

Since, together with $u_\omega(x,t)$, the function $u_\omega(x,t+2\pi\omega^{-1})$ also lying in the ball (1.37) is a T_ω-periodic solution to (1.1), (1.2), we have $u_\omega(x,t) = u_\omega(x,t+2\pi\omega^{-1})$ and so $u_\omega(x,t)$ is a $2\pi\omega^{-1}$-periodic (in t) solution.

Infinite differentiability of u_ω follows from the containment $u_\omega \in C_\mu(R, C^{2k-1}(\overline{\Omega}))$ proven above and the well-known a priori estimates [15] for solutions to linear initial-boundary value problems.

Item 1 of the theorem is proven.

Here we do not dwell upon the proof of Item 2 of the theorem on relative uniqueness of u_ω. Observe only that the corresponding result was obtained in a more general situation, namely for abstract parabolic equations, by means of a change of variables like the Krylov–Bogolyubov substitution. From the arguments of this article we can easily conclude that we have existence and uniqueness of the solution u_ω for a sufficiently large ω in a ball of the form (1.37) with $\overset{0}{y}_\omega$ replaced with

$$a_\omega = u_0 + \omega^{-1/2}(u_k + v_k + z_k) + \omega^{-\frac{k+1}{2k}} u_{k+1} + \sum_{j=k+1}^{2k-1} \omega^{-\frac{j}{2k}} z_j + \sum_{j=2k}^{3k} \omega^{-\frac{j}{2k}} (w_j + z_j).$$

Prove Item 3. Using the structure of the approximations $\overset{N}{u}$ (see (1.15)), we infer this from the following assertion:

Lemma 2. *For arbitrary nonnegative ℓ and d, there is a number M_0 such that*

$$\|u_\omega - \overset{M}{y}_\omega\|_{C^{\ell,\ell/2k}} \leq C(M)\omega^{-d}, \quad C(M) = \text{const},$$

for $M > M_0$.

Proof. Put $r_M = (u_\omega - \overset{M}{y}_\omega)\lambda(t)$, where $\lambda(t)$ is an infinitely differentiable compactly-supported function on the real axis, $t \in \mathbb{R}$, supported in the interval $t \in (0;3)$ and such that $\lambda(t) = 1$ for $t \in [1,2]$. Using the structure of $y_\omega \equiv \overset{M}{y}_\omega$ and the fact that $u_\omega - \overset{M}{y}_\omega$ has a small period $2\pi\omega^{-1}$, we see that Lemma 2 will be proven, if we establish it with r_M instead of $u_\omega - y_\omega$. Lemma 1 implies existence of numbers $\gamma \in (0,1)$ and $c_0 > 0$ such that the following inequalities hold for $\omega > \omega_0$ and all multi-indices α with $|\alpha| \leq 2k-1$:

$$\|D^\alpha(u_\omega - y_\omega)\|_{C^{\gamma,\gamma/(2k)}}, \|D^\alpha r_M\|_{C^{\gamma,\gamma/(2k)}} \leq c_0\omega^{-\frac{M+1+k}{2k}}. \tag{1.38}$$

Consider the cylinder $Q_0 = \Omega \times [0,3]$ with the lateral surface $\Gamma_0 = \partial\Omega \times [0,3]$. It follows from (1.17) and (1.18) that the function r_M considered in the cylinder Q_0 is a solution to the problem

$$\frac{\partial r_M}{\partial t} - Lr_M = \Phi_\omega(x,t,u_\omega - y_\omega)\lambda(t) + \beta_\omega(x,t)\lambda(t)$$

$$+\sqrt{\omega} \sum_{0<|s|\leq m} \frac{\partial \varphi_s}{\partial u}(x,u_0)\exp(is\omega t) r_M - (u_\omega - y_\omega)\lambda'(t) \equiv \Psi_M(x,t,\omega), \tag{1.39}$$

$$r_M(x,0) = 0, \quad r_M|_{\Gamma_0} = \frac{\partial r_M}{\partial \nu}\bigg|_{\Gamma_0} = \dots = \frac{\partial^{k-1} r_M}{\partial \nu^{k-1}}\bigg|_{\Gamma_0} = 0. \tag{1.40}$$

From (1.35) we obtain the inequality

$$\|\Psi_M\|_{C^{\gamma,\gamma/(2k)}} \leq d_0\omega^{-\frac{M+1}{2k}}, \quad d_0 = \text{const}. \tag{1.41}$$

It follows from the definition of r_M that the boundary and initial data of (1.39), (1.40) satisfy the agreement conditions of arbitrarily high order [15]. By the well-known a priori estimates [15] and (1.41), the following estimate holds:

$$\|r_M\|_{C^{2k+\gamma,1+\gamma/(2k)}} \leq d_1\omega^{-\frac{M+1}{2k}}, \quad d_1 = \text{const}.$$

Using the last inequality, estimate again the right-hand side Ψ_M of (1.36) and write down the a priori estimate for r_M, etc.; i.e., use the so-called "bootstrap method." In finitely many steps (depending on ℓ), we obviously obtain the estimate of the form $\|r_M\|_{C^{\ell/\ell/2k}} \leq c_1(\ell)\omega^{-\frac{M+1}{2k}+c_2(\ell)}$, where $c_1(\ell)$, $c_2(\ell) = \text{const}$, which implies Lemma 2 and so Item 2 of the theorem. The theorem is proven.

§ 2. Formal Asymptotic Expansion for a Broader Class of Problems

In this section, the numbers k, m, and n and the domain Ω are the same as in § 1. We consider the problem on real $2\pi\omega^{-1}$-periodic solutions to the semilinear parabolic equations

$$\begin{gathered}\frac{\partial u}{\partial t} - \sum_{|\alpha|\le 2k} a_\alpha(x) D^\alpha u - \sum_{0\le |s|\le m} f_s(x, \delta^{k-1}u)\exp(is\omega t) \\ -\sqrt{\omega}\sum_{0<|s|\le m} \varphi_s(x,\delta^{k-1}u)\exp(is\omega t) = 0, \quad (x,t)\in Q, \end{gathered} \tag{2.1}$$

with the Dirichlet boundary conditions

$$u|_\Gamma = \frac{\partial u}{\partial \nu}\bigg|_\Gamma = \dots = \frac{\partial^{k-1} u}{\partial \nu^{k-1}}\bigg|_\Gamma = 0. \tag{2.2}$$

Here the functions a_α and $f_s(x,e)$, the boundary Γ of Q, and the symbol δ^{k-1} are the same as in § 1, while the functions $\varphi_s(x,e)$ are defined and continuous on $\Omega\times R^p$ and have there continuous derivatives with respect to all its arguments of arbitrarily high order. The values $\varphi_s(x,e)$ and $\varphi_{-s}(x,e)$ are assumed to be complex conjugate. Suppose that the averaged problem

$$\begin{gathered}\frac{\partial v}{\partial t} - \sum_{|\alpha|\le 2k} a_\alpha(x) D^\alpha v - f_0(x,\delta^{k-1}v) \\ -\sum_{0<s\le m} is^{-1}\sum_{j=0}^{p-1}\frac{\partial \varphi_s}{\partial e_j}(x,\delta^{k-1}v)(\delta^{k-1}\varphi_{-s}(x,\delta^{k-1}v))_j = 0, \quad (x,t)\in Q,\end{gathered} \tag{2.3}$$

$$v|_\Gamma = \frac{\partial v}{\partial \nu}\bigg|_\Gamma = \dots = \frac{\partial^{k-1} v}{\partial \nu^{k-1}}\bigg|_\Gamma = 0 \tag{2.4}$$

has a stationary solution $u_0(x)$.

This solution is supposed to be nondegenerate; i.e., the problem

$$\begin{gathered}Lv \equiv \sum_{|\alpha|\le 2k} a_\alpha(x)D^\alpha v + \sum_{j=0}^{p-1}\frac{\partial f_0[x,\delta^{k-1}u_0(x)]}{\partial e_j}(\delta^{k-1}v)_j \\ + \sum_{0\le s\le m} is^{-1}\sum_{j_1,j_2=0}^{p-1}\frac{\partial\varphi_s[x,\delta^{k-1}u_0(x)]}{\partial e_{j_1}}\left(\delta^{k-1}\frac{\partial\varphi_{-s}}{\partial e_{j_2}}[x,\delta^{k-1}u_0(x)](\delta^{k-1}v)_{j_2}\right)_{j_1} \\ + \sum_{0\le s\le m} is^{-1}\sum_{j_1,j_2=0}^{p-1}\frac{\partial^2\varphi_s[x,\delta^{k-1}u_0(x)]}{\partial e_{j_1}\partial e_{j_2}}(\delta^{k-1}\varphi_{-s}[x,u_0(x)])_{j_1}(\delta^{k-1}v)_{j_2} = 0, \quad x\in\Omega,\end{gathered} \tag{2.5}$$

$$v|_{\partial\Omega} = \frac{\partial v}{\partial \nu}\bigg|_{\partial\Omega} = \dots = \frac{\partial^{k-1} v}{\partial \nu^{k-1}}\bigg|_{\partial\Omega} = 0$$

has only the zero solution. Note that, in the case when φ_s depend only on x and u, i.e., under the conditions of § 1, the middle summand on the right-hand side of (2.5) vanishes.

We construct the formal asymptotic expansion of a solution to (2.1), (2.3) by the boundary layer method, following § 1 and using the notations introduced there. Moreover, in this section the asymptotic expansion of u_ω has the form

$$u_\omega(x,t) = u_0(x) + \sum_{j=2} \omega^{-\frac{j}{2k}} u_j(x) + \omega^{-1/2}\big[\delta_k^1 u_k(x) + v_k(x,\tau) + z_k(\varphi,\rho,\tau)\big]$$

$$+ \sum_{j=k+2} \omega^{-\frac{j}{2k}} v_j(x,\tau) + \sum_{j=k+1} \omega^{-\frac{j}{2k}} [w_j(\varphi,\rho) + z_j(\varphi,\rho,\tau)], \quad \tau = \omega t. \tag{2.6}$$

Show that, acting as in § 1, we can find all coefficients of (2.6).

For the coefficient v_k we obtain the problem

$$\frac{\partial v_k}{\partial \tau} = \sum_{0<|s|\le m} \varphi_s[x, \delta^{k-1} u_0(x)] \exp(is\tau), \quad \langle v_k(x,\tau)\rangle = 0,$$

where x and $\delta^{k-1}u_0(x)$ play the role of parameters, and therefore

$$v_k(x,\tau) = -\sum_{0<|s|\le m} is^{-1}\varphi_s[x, \delta^{k-1}u_0(x)] \exp(is\tau). \tag{2.7}$$

For the coefficient $u_0(x)$ we obtain (2.1), (2.2) which is the only nonlinear problem in the process of finding the coefficients of (2.6). By condition, it is solvable and we have already written down its nondegenerate solution $u_0(x)$ in (2.6). The function z_k is a solution to the following parabolic problem in the half-space $(\rho, t) \in [0,\infty) \times R$:

$$\frac{\partial z_k}{\partial \tau} = b(\varphi) \frac{\partial^{2k} z_k}{\partial \rho^{2k}}, \tag{2.8}$$

$$z_k|_{\rho=0} = -v_k|_{r=0}, \quad \frac{\partial z_k}{\partial \rho}\Big|_{\rho=0} = 0, \dots, \frac{\partial^{k-1} z_k}{\partial \rho^{k-1}}\Big|_{\rho=0} = 0, \quad z_k|_{\rho=\infty} = 0. \tag{2.9}$$

By (2.7),

$$z_k(\varphi,\rho,\tau) = \sum_{0<|s|\le m} d_s(\varphi,\rho)\exp(is\tau),$$

where the coefficients c_s satisfy the relations

$$isc_s - b(\varphi)\frac{\partial^{2k} c_s}{\partial \rho^{2k}} = 0, \tag{2.10}$$

$$c_s|_{\rho=0} = -\varphi_s[x, \delta^{k-1}u_0(x)]|_{r=0} \equiv d_s(\varphi), \quad \frac{\partial c_s}{\partial \rho}\Big|_{\rho=0} = 0, \dots, \frac{\partial^{k-1} c_s}{\partial \rho^{k-1}}\Big|_{\rho=0} = 0,$$

$$c_s|_{\rho=\infty} = 0, \quad s = \pm 1, \pm 2, \dots, \pm m.$$

Denoting by $\lambda_{js}(\varphi)$ the same roots of the characteristic equation corresponding to (2.10) as in § 1, we find a solution to (2.8), (2.9) in the form

$$z_k(\rho,\varphi,\tau) = \sum_{0<|s|\le m} \sum_{j=1}^{k} c_{js}(\varphi)\exp(\lambda_{js}\rho)\exp(is\tau).$$

The coefficient w_{k+1} of (2.8) satisfies the relations

$$b(\varphi)\frac{\partial^{2k}w_{k+1}}{\partial\rho^{2k}} = \sum_{\substack{0<|s|\le m\\1\le j\le k}} g_{js}(\varphi)\exp(\lambda_{js}\rho), \quad w_{k+1}|_{\rho=\infty} = 0.$$

Hence, we easily find it in the form

$$w_{k+1}(\varphi,\rho) = \sum_{\substack{0<|s|\le m\\1\le j\le k}} h_{js}(\varphi)\exp(\lambda_{js}\rho).$$

Suppose now that we know the coefficients $u_i(x)$, $v_{i+k}(x,\tau)$, $z_{i+k}(\varphi,\rho,\tau)$, and $w_{i+k+1}(\varphi,\rho)$ of (2.6) for $i < i_0$, $i_0 \ge 1$. Moreover, assume that the functions $v_j(x,\tau)$ and $z_j(\varphi,\rho,\tau)$ are trigonometric polynomials in τ and the coefficients of these polynomials representing z_j as well as the functions w_j in turn are quasipolynomials in ρ (with coefficients depending on φ) in which the exponents of the exponential functions have negative real parts. Show that then the coefficients u_{i_0}, v_{i_0+k}, z_{i_0+k}, and w_{i_0+k+1} having similar structure are determined uniquely. The function $v_{i_0+k}(x,\tau)$ satisfies the relations

$$\frac{\partial v_{i_0+k}}{\partial\tau} = \Phi_{i_0+k}(x,\tau) \;+ \sum_{0<|s|\le m}\sum_{j=0}^{p-1}\frac{\partial\varphi_s}{\partial e_j}[x,\delta^{k-1}u_0(x)](\delta^{k-1}u_{i_0}))_j\exp(is\tau),$$

$$\langle v_{i_0+k}\rangle = 0,$$

with $\Phi_{i_0+k}(x,\tau)$ a known function which is a finite sum of harmonics in τ with amplitudes depending on x. Considering x and u_{i_0} as parameters, we find the expressions v_{i_0+k}. The coefficient u_{i_0} is a solution to the problem

$$Lu_{i_0} = \psi_{i_0},$$

$$u_{i_0}|_{\partial\Omega} = -w_{i_0}|_{r=0}, \quad \left.\frac{\partial u_{i_0}}{\partial\nu}\right|_{\partial\Omega} = -\left.\frac{\partial w_{i_0+1}}{\partial r}\right|_{r=0}, \dots,$$

$$\left.\frac{\partial^{k-1}u_{i_0}}{\partial\nu^{k-1}}\right|_{\partial\Omega} = -\left.\frac{\partial^{k-1}w_{i_0+k-1}}{\partial r^{k-1}}\right|_{r=0},$$

where ψ_{i_0} is a known function. Determining u_{i_0} from here, we then find v_{i_0+k}. The coefficient z_{i_0+k} satisfies the relations

$$\frac{\partial z_{i_0+k}}{\partial\tau} = b(\varphi)\frac{\partial^{2k}z_{i_0+k}}{\partial\rho^{2k}} + \chi_{i_0+k}(\varphi,\rho,\tau),$$

$$z_{i_0+k}|_{\rho=0} = -v_{i_0+k}|_{\partial\Omega}, \quad \left.\frac{\partial z_{i_0+k}}{\partial\rho}\right|_{\rho=0} = -\left.\frac{\partial v_{i_0+k-1}}{\partial\nu}\right|_{\partial\Omega}, \dots,$$

$$\left.\frac{\partial^{k-1}z_{i_0+k}}{\partial\rho^{k-1}}\right|_{\rho=0} = -\left.\frac{\partial^{k-1}v_{i_0+1}}{\partial\nu^{k-1}}\right|_{\partial\Omega}, \quad z_{i_0+k}|_{\rho=\infty} = 0.$$

Here $\chi_{i_0+k}(\varphi,\rho,\tau)$ is a known function depending in particular on z_{i+k} and w_{i+k}, $i < i_k$, representable by a finite sum of harmonics in τ whose amplitudes in turn are finite sums of exponential functions of ρ with exponents having negative real parts with coefficients of the exponential functions depend on φ. The problem for finding w_{i_0+k+1} has the form

$$b(\varphi)\frac{\partial^{2k}w_{i_0+k+1}}{\partial\rho^{2k}} = \Lambda(\varphi,\rho), \quad w_{i_0+k+1}|_{\rho=\infty} = 0,$$

where $\Lambda(\varphi,\rho)$ is a known function (since the coefficient z_{i_0+k} is found) which is a finite sum of exponential functions of ρ with exponents having negative real parts and coefficients depending on φ. Thus, w_{i_0+k+1} is determined uniquely and has the sought structure.

The construction of the formal asymptotic expansion (2.6) is complete.

References

1. *Bogolyubov N. N.*, Some Statistical Methods in Mathematical Physics [in Russian], Izdat. Akad. Nauk USSR, Kiev (1945).
2. *Bogolyubov N. N. and Mitropol′skiĭ Yu. A.*, Asymptotic Methods in the Theory on Nonlinear Oscillations [in Russian], Nauka, Moscow (1974).
3. *Simonenko I. B.*, The Averaging Method in the Theory of Nonlinear Equations of Parabolic Type with Applications to the Problems of Hydrodynamic Stability [in Russian], Rostovsk. Univ., Rostov-on-Don (1989).
4. *Levenshtam V. B.*, "Asymptotic integration of quasilinear parabolic equations with coefficients rapidly oscillating with respect to time," Izv. Vyssh. Uchebn. Zaved. Mat., No. 5, 36–43 (2002).
5. *Levenshtam V. B.*, "Higher-order approximations of the averaging method for parabolic initial-boundary value problems with rapidly oscillating coefficients," Differentsial′nye Uravneniya, **39**, No. 10, 1395–1403 (2003).
6. *Bogolyubov N. N.*, "Perturbation theory in nonlinear mechanics," Sb. Inst. Stroit. Mekh. Akad. Nauk USSR, No. 4, 9–34 (1950).
7. *Kapitsa P. L.*, "Dynamic stability of a pendulum with oscillating suspension point," Prikl. Mat. i Mekh., **59**, No. 6, 922–929 (1995).
8. *Chelomeĭ V. N.*, "On the possibility of raising the stability of elastic systems by means of vibrations," Dokl. Akad. Nauk SSSR, **110**, No. 3, 345–347 (1956).
9. *Zen′kovskaya S. M. and Simonenko I. B.*, "On influence of high-frequency vibrations on the origin of convection," Izv. Akad. Nauk SSSR Mekh. Zhidk. i Gaza, No. 5, 51–55 (1966).
10. *Yudovich V. I.*, "Vibrodynamics of systems with constraints," Dokl. Ross. Akad. Nauk, **354**, No. 5, 622–624 (1997).
11. *Yudovich V. I.*, "The dynamics of a material particle on a smooth vibrating surface," Prikl. Mat. i Mekh., **62**, No. 6, 968–976 (1998).
12. *Yudovich V. I.*, "Vibrodynamics and vibrogeometry of systems with constraints," submitted to VINITI on July 17, 2003, No. 1407–B2003.
13. *Yudovich V. I.*, "Vibrodynamics and vibrogeometry of systems with constraints. II," submitted to VINITI on July 17, 2003, No. 1408–B2003.
14. *Vishik M. I. and Lyusternik L. A.*, "Regular degeneration and a boundary layer for linear differential equations with a small parameter," Uspekhi Mat. Nauk, **12**, No. 5, 3–122 (1957).
15. *Solonnikov V. A.*, "On boundary value problems for linear parabolic systems of general differential equations," Trudy Mat. Inst. Steklov., **83**, 3–162 (1965).
16. *Sobolevskiĭ P. E.*, "On parabolic type equations in Banach spaces," Trudy Moskov. Mat. Obshch., **10**, 207–350 (1961).
17. *Krasnosel′skiĭ M. A. et al.*, Integral Operators in Spaces of Summable Functions [in Russian], Nauka, Moscow (1966).