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ON THE NONCOMMUTING GRAPH ASSOCIATED WITH A FINITE GROUP

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Abstract: Let G be a finite group. We define the noncommuting graph $\nabla(G)$ as follows: the vertex set of $\nabla(G)$ is $G \setminus Z(G)$ with two vertices x and y joined by an edge whenever the commutator of x and y is not the identity. We study some properties of $\nabla(G)$ and prove that, for many groups G, if H is a group with $\nabla(G)$ isomorphic to $\nabla(H)$ then $|G| = |H|$.

Keywords: group, noncommuting graph, regular graph

1. Introduction

Generally, there is an intimate relation between Group Theory and Graph Theory, and in many occasions the properties of graphs give rise to some properties of groups and vice versa. For instance, Gruenberg and Kegel introduced the *prime graph* $\Gamma(G)$ associated with a finite group G (cp. [1]). Also, the concept of solvable graph $\Gamma_S(G)$, for a finite group G, was defined recently by Abe and Iiyori in [2].

One of the graphs that has attracted the attention of many authors is the *commuting graph* associated with a finite group. For a finite group G and X a subset of G, the commuting graph on X denoted $\mathscr{C}(G, X)$ has X as its vertex set with $x, y \in X$ joined by an edge whenever $[x, y] = 1$ (x and y commute). Many authors have studied $\mathscr{C}(G, X)$ for different choices of G and X. In [3] and [4], Segev and Seitz apply the commuting graph $\Delta(G) := \mathscr{C}(G, X)$, with G a nonabelian simple group and $X = G \setminus \{1\}$, in order to prove the Margulis–Platonov conjecture on arithmetic groups.

In this article we consider the complementary graph of the commuting graph $\Delta(G) = \mathscr{C}(G, X)$ where $X := G \setminus Z(G)$, and for convenience we denote this graph by $\nabla(G)$. We call this graph the *noncommuting graph of* G, and in this graph the vertex set $V(G) := G \setminus Z(G)$ and $x, y \in V(G)$ form an edge if and only if $[x, y] \neq 1$ (denoted by $x \sim y$). Note that G is abelian if and only if $V(G) = \varnothing$. So, throughout this paper let G denote a nonabelian finite group.

In this article we first obtain some interesting properties of the noncommuting graph $\nabla(G)$. For example, we will show that $\nabla(G)$ is always connected for every finite group G (see Proposition 1). Next, we will determine the structure of the groups G such that $\nabla(G)$ is k-regular, for certain values of k; that is, the vertices of the graph are of the same degree k. Finally, we will verify for some finite groups G and H that if $\nabla(G) \cong \nabla(H)$ then $|G| = |H|$. For example, we consider the dihedral groups D_{2m} with m odd, the alternating groups A_n ($n \geq 4$), all sporadic simple groups, the simple groups of Lie type with nonconnected prime graph, the symmetric groups S_n ($n \geq 3$), etc.; and we show that the above statement holds for them. So far, we have not found any counterexample to the above statement. Hence, it is quite natural to put forward the following

Conjecture. *Let G and H be two arbitrary finite groups such that $\nabla(G) \cong \nabla(H)$. Then $|G| = |H|$.*

Note that there exist some groups H and K such that $\nabla(H) \cong \nabla(K)$ and $H \ncong K$. For example, we assume that $H = D_8$ and $K = Q_8$. Certainly $D_8 \ncong Q_8$, but $\nabla(D_8) \cong \nabla(Q_8)$.

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2. Notations and Definitions

To state our results we need some notation. Given a group G, we denote by $\pi_e(G)$ the set of all element orders of G and by $\pi(G)$, the set of all prime factors of $|G|$. It is clear that the set $\pi_e(G)$ is closed and partially ordered by divisibility. Hence, it is determined uniquely by $\mu(G)$, the subset of its maximal elements.

DEFINITION 1. The *prime graph* $\Gamma(G)$ of a group G is defined as follows. The set of vertices of $\Gamma(G)$ is $\pi(G)$ and two distinct primes p and q are joined by an edge if $pq \in \pi_e(G)$.

The number of connected components of $\Gamma(G)$ is denoted by $t(G)$, and the vertex sets of the connected components are denoted by $\pi_i = \pi_i(G)$, $i = 1, 2, \dots, t(G)$. If $2 \in \pi(G)$ we always assume $2 \in \pi_1$. Denote by $\mu_i(G)$ the set of $n \in \mu(G)$ such that $\pi(n) \subseteq \pi_i(G)$. For simple groups S, the connected components of $\Gamma(S)$ were found by Williams and Kondrat′ev (see [5] and [1]). Moreover, it was proved that in [6] if S is a simple group with disconnected prime graph $\Gamma(S)$ then $|\mu_i(S)| = 1$ for $2 \leq i \leq t(S)$. Denote by $n_i = n_i(S)$ the unique element in $\mu_i(S)$. The values for S, $\pi_1(S)$ and $n_i(S)$ for $2 \leq i \leq t(S)$ are the same as in Tables 1*a*–*c* of [7].

The number of edges incident with a vertex v, in a graph is called the *degree* of v and denoted by $\deg(v)$. For the noncommuting graph $\nabla(G)$, we put

$$\rho(G) := \sum_{g \in V(G)} \deg(g).$$

DEFINITION 2. Two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are said to be *isomorphic* (in writing: $\Gamma_1 \cong \Gamma_2$) if there exists a one-to-one onto mapping

$$\phi : V_1 \to V_2,$$

such that

$$u \sim v \Leftrightarrow \phi(u) \sim \phi(v) \quad \text{for all } u, v \in V_1.$$

Such a mapping ϕ is called a *graph isomorphism*. Note that, for isomorphic graphs Γ_1 and Γ_2 we have $|V_1| = |V_2|$, $|E_1| = |E_2|$ and $\deg(v) = \deg(\phi(v))$ for every $v \in V_1$.

Every group in this paper is a finite group. Also, we assume that $G \backslash Z(G) := \{g_1, g_2, \dots, g_n\}$. Evidently $|Z(G)| \big| n$. Moreover, if we know $\nabla(G)$ then we know the centralizer of g_i in G. Put $t_i := |C_G(g_i) \setminus Z(G)|$. It is easy to see that $|Z(G)| \big| t_i$. Suppose that the conjugacy class length of g_i is l_i. Then we also have

$$l_i := \frac{n + |Z(G)|}{t_i + |Z(G)|}. \tag{1}$$

To every group G we can associate the set of divisors of the order of G; namely, the *class size* of G defined as follows:

$$\mathrm{cs}(G) := \{|g^G| : g \in G\}.$$

Clearly, $1 \in \mathrm{cs}(G)$.

Given a group G, we denote its class number by $k(G)$. The rest of notation is standard.

3. Some Properties of $\nabla(G)$

In this section, we state a few results on the noncommuting graph. In what follows we show that the noncommuting graph of every group is always connected.

Proposition 1. *For every group G, the noncommuting graph $\nabla(G)$ is connected.*

PROOF. Assume $\nabla(G)$ is disconnected. So there exists a pair of vertices x and y in $V(G)$ such that $x \nsim y$. It is easy to see now that $C_G(x) \cup C_G(y) \subsetneq G$, which means that there exists $z \in G \backslash (C_G(x) \cup C_G(y))$. It is evidently by definition that $z \sim x$ and $z \sim y$, which means that $\nabla(G)$ is connected. This is a contradiction. $\square$

Proposition 2. *There is no group G with a normal subgroup $N \neq 1$ such that $\nabla(G) \cong \nabla(G/N)$.*

Proof. Let the order of G, N, $Z(G)$, and $N \cap Z(G)$ be l, n, m, and t, respectively. If $\nabla(G) \cong \nabla(G/N)$ then we have $|G - Z(G)| = |G/N - Z(G/N)|$. It is clear that $Z(G/N) \geq Z(G)N/N$ and $|Z(G)N/N| = m/t$. So we infer $|G - Z(G)| = |G/N - Z(G/N)| \leq |G/N - Z(G)N/N)|$, that is, $l - m \leq l/n - m/t$. Since $n \geq 2$, we have

$$l/m \leq \frac{1 - 1/t}{1 - 1/n} \leq \frac{1 - 1/t}{1 - 1/2} \leq 2.$$

It means that $|G/Z(G)| \leq 2$, and so G is abelian, which is a contradiction. □

Proposition 3. *There is no group G with $H < G$ such that $\nabla(G) \cong \nabla(H)$.*

Proof. If a group like G with a proper subgroup H exists satisfying $\nabla(G) \cong \nabla(H)$ then $|G| - |Z(G)| = |H| - |Z(H)|$. Clearly, $|H| \leq (1/2)|G|$ and $|Z(G)| < (1/2)|G|$. We now have

$$|G| = |H| - |Z(H)| + |Z(G)| < |G|,$$

which is a contradiction. □

Proposition 4. *Let G be a group. If $\nabla(G)$ is k-regular then $|\operatorname{cs}(G)| = 2$ and $G = P \times A$ with P a nonabelian p-group and A, abelian.*

Proof. Since, for all $g \in V(G)$, we have $\deg(g) = |G| - |C_G(g)| = k$; it follows that all $C_G(g)$ ($g \in V(G)$) and as a consequence all conjugacy classes g^G ($g \in V(G)$) have the same order. Therefore, since G is nonabelian we obtain $|\operatorname{cs}(G)| = 2$. Now, by a result of N. Ito (see [8]), we infer that G is a nilpotent group and $G = P \times A$ with P a p-group and A, abelian. Moreover, since G is a nonabelian group, it follows that P is nonabelian. □

4. Some Useful Results

In this section we list some useful results about $\nabla(G)$.

Proposition 5. *For a group G, we have $\rho(G) = |G|(|G| - k(G))$. In particular, $|E(G)| = \frac{|G|(|G|-k(G))}{2}$.*

Proof. By easy calculation, we obtain

$$\rho(G) = \sum_{x \in G} |G \backslash C_G(x)| = |G|^2 - \sum_{x \in G} |C_G(x)| = |G|^2 - |G| \sum_{x \in G} \frac{1}{|x^G|} = |G|^2 - |G|k(G),$$

as required. The rest of the proof is obvious. □

Corollary 1. *The class number of an odd order group is odd.*

This is immediate from the previous proposition because $\rho(G)$ is always even.

A graph $\Gamma = (V, E)$ is *k-partite*, $k > 1$, if it is possible to partition V into k subsets: $V_1, V_2, \ldots, V_k$ (called *partite sets*) such that every edge of E joins a vertex of V_i to a vertex in V_j, $i \neq j$. Now, if $[G : Z(G)] = m$ then $\nabla(G)$ is $(m-1)$-partite, with $m \geq 4$. In fact, if $T = \{x_0 = 1, x_1, x_2, \ldots, x_{m-1}\}$ is a transversal of $Z(G)$ in G then $V_i = x_i Z(G)$, $i = 1, 2, \ldots, m-1$.

Proposition 6. *Let G be a group with $[G : Z(G)] = m$. Let $T = \{x_0 = 1, x_1, \ldots, x_{m-1}\}$ be a transversal of $Z(G)$ in G, and $\Lambda(x_i) = \{x_j \mid [x_i, x_j] \neq 1\}$. Then for all $g \in x_i Z(G)$, $i \geq 1$, we have $\deg(g) = |\Lambda(x_i)||Z(G)|$, $|\Lambda(x_i)| \geq |x_i^G| - 1$ and $|\Lambda(x_i)| \geq 2$. In particular,*

$$\rho(G) = |Z(G)|^2 \sum_{i=1}^{m-1} |\Lambda(x_i)| \leq |Z(G)|^2 (m-1)(m-2).$$

Proof. Clearly, for all $u, v \in x_i Z(G)$, $[u, v] = 1$ and so $u \nsim v$. Also, if $[x_i, x_j] = 1$, $j \geq 1$, then for every $u \in x_i Z(G)$ and $v \in x_j Z(G)$, $[u, v] = 1$ and so $u \nsim v$. On the other hand, if $[x_i, x_j] \neq 1$ then for every $u \in x_i Z(G)$ and $v \in x_j Z(G)$, $[u, v] \neq 1$ and hence $u \sim v$. Because $|x_j Z(G)| = |Z(G)|$, for every j, it is easy to see that if $g \in x_i Z(G)$, $i \geq 1$, then $\deg(g) = |\Lambda(x_i)||Z(G)|$.

For all $i \geq 1$, we have $\deg(x_i) = |G \setminus C_G(x_i)| = |\Lambda(x_i)||Z(G)|$, and so $|C_G(x_i)|(|x_i^G| - 1) = |\Lambda(x_i)||Z(G)|$. Since $|Z(G)|$ divides $|C_G(x_i)|$, we now have

$$\frac{|\Lambda(x_i)|}{|x_i^G| - 1} = \frac{|C_G(x_i)|}{|Z(G)|} \in \mathbb{N}.$$

Thus $|x_i^G| - 1$ divides $|\Lambda(x_i)|$, and so $|\Lambda(x_i)| \geq |x_i^G| - 1$. Moreover, if there exists $i \geq 1$, such that $|\Lambda(x_i)| = 1$, then we have $|C_G(x_i)| = |Z(G)|$, which is a contradiction. The rest of the proof is clear. $\square$

Proposition 7. *Let G be a group. Then the following hold:*

(a) *$\nabla(G)$ has no vertex of degree 2.*

(b) *If $\nabla(G)$ has a vertex of degree p where p is an odd prime then $G \cong D_{2p}$.*

Proof. Assume there is $g \in V(G)$ with $\deg(g) = p$, where p is a prime. Then we have $p = \deg(g) = |C_G(g)|([G : C_G(g)] - 1)$, from which it follows $|C_G(g)| = p$, $[G : C_G(g)] = 2$ and $|G| = 2p$. Since G is nonabelian, p must be an odd prime and $G \cong D_{2p}$. Therefore, the proof of (a) and (b) is complete. $\square$

An immediate consequence of this result is the following

Corollary 2. *If G and H are two groups such that $\nabla(G) \cong \nabla(H)$ and $\nabla(G)$ possesses a vertex of prime degree p then $G \cong H$.*

Proposition 8. *Let G be a group such that there exists a vertex $g \in V(G)$ of degree p^2. Then one of the following occurs.*

(a) *G is a Frobenius group of order $p(p+1)$, whose complement is cyclic of order p.*

(b) *G is a group of order $2p^2$, where p is a prime.*

Proof. Since $\deg(g) = p^2$, we have $p^2 = |C_G(g)|([G : C_G(g)] - 1)$, and we consider the following two cases:

Case 1. $|C_G(g)| = p$.

In this case, $C_G(g) = \langle g \rangle$ and we have $|G| = p(p+1)$. Since the conjugacy class g^G has $p+1$ elements of order p, there exists an element x of order p with $x \notin \langle g \rangle$. Hence $\langle g \rangle$ cannot be a normal Sylow p-subgroup of G, and so the number of Sylow p-subgroups of G is $p+1$. Now we have $N_G(\langle g \rangle) = C_G(\langle g \rangle) = \langle g \rangle$, and therefore G is p-nilpotent. Thus,

$$G = M\langle g \rangle, \quad M \lhd G, \quad M \cap \langle g \rangle = 1.$$

Clearly, g acts fixed-point-freely on M. So M is nilpotent, and G is a Frobenius group with kernel M and complement $\langle g \rangle$.

Case 2. $|C_G(g)| = p^2$.

In this case G is a group of order $2p^2$. We show now that for every group of order $2p^2$, $\nabla(G)$ possess a vertex of degree p^2. We first assume that $p = 2$. In this case $G \cong D_8$ or Q_8, and in $\nabla(G)$ every vertex is of degree 4. So we may assume that p is an odd prime. Let t be an involution in G and let P be a Sylow p-subgroup of G which is clearly normal in G. Now $Z(G) = C_P(t) \leq P$ which is of order 1 or p. So, there always exists an element $x \in P \setminus Z(G)$, and obviously $\deg(x) = p^2$. $\square$

Corollary 3. *Let G and H be two groups such that $\nabla(G) \cong \nabla(H)$. If $\nabla(G)$ has a vertex of degree p^2 then $|G| = |H|$.*

Proof. By Proposition 8, $|G| = p(p+1)$ or $2p^2$. Also, from the graph isomorphism $\nabla(G) \cong \nabla(H)$, $\nabla(H)$ has a vertex of degree p^2, and again by Proposition 8, $|H| = p(p+1)$ or $2p^2$. Assume now that $|G| \neq |H|$. Without loss of generality, we may assume that $|G| = p(p+1)$ and $|H| = 2p^2$. In this case, G is a Frobenius group and $Z(G) = 1$. From $|V(G)| = |V(H)|$ it follows now that $|G| - 1 = |H| - |Z(H)|$, that is, $|Z(H)| = p^2 - p + 1$, which is a contradiction. $\square$

Corollary 4. *If $\nabla(G)$ is p^2-regular then $p = 2$ and $G \cong D_8$ or Q_8.*

Proof. We note first that if G is a Frobenius group then $\nabla(G)$ cannot be k-regular for any k. Now, if $\nabla(G)$ is p^2-regular then by Proposition 8, G is a nonabelian group of order $2p^2$ which is not Frobenius. In this case, let t be an involution in G. Then $|C_G(t)| = 2p$, and so $p^2 = \deg(t) = 2p^2 - 2p$, which implies $p = 2$ and $G \cong D_8$ or Q_8. $\square$

Proposition 9. *Let g be a vertex of the graph $\nabla(G)$ such that $\deg(g) = |G| - 2$. Then g is an involution and G is isomorphic to the Frobenius group of order $2m$, where m is odd.*

Proof. From $|G| - 2 = \deg(g) = |G| - |C_G(g)|$ it follows that $|C_G(g)| = 2$, and so g is an involution. Also, $C_G(g)$ is a Sylow 2-subgroup of G. Hence $|G| = 2m$ where m is an odd integer. Now, G is 2-nilpotent, and there exists a normal subgroup of order m, upon which the involution g acts fixed-point-freely. Therefore, G is a Frobenius group with abelian kernel of order m and cyclic complement of order 2. $\square$

Proposition 10. *Let G and H be two groups such that $\nabla(G) \cong \nabla(H)$. Suppose $y = \frac{|Z(G)|}{|Z(H)|} \in \mathbb{N}$. If $\{2, 3\} \cap \mathrm{cs}(G) \neq \varnothing$ then $|H| = |G|$.*

Proof. Let $x = t/|Z(G)|$ where $t = |C_G(g) \setminus Z(G)|$ for some $g \in G$. Suppose that the conjugacy class length of g is l. Then by (1), we have

$$n + |Z(H)| = (l - 1 + lx)y|Z(H)| + |Z(H)|$$

and

$$t + |Z(H)| = (xy + 1)|Z(H)|.$$

Again by (1), since $(n + |Z(H)|)/(t + |Z(H)|) \in \mathbb{N}$, we have

$$\frac{lxy + (l - 1)y + 1}{xy + 1} \in \mathbb{N}.$$

Hence, by a simple manipulation we obtain

$$\frac{(l - 1)(y - 1)}{xy + 1} \in \mathbb{N} \cup \{0\}.$$

We now consider the following two cases:

Case 1: $l = 2 \in \mathrm{cs}(G)$. In this case, it is easy to see that $y = 1$, and so $|Z(H)| = |Z(G)|$, which implies that $|H| = |G|$.

Case 2: $l = 3 \in \mathrm{cs}(G)$. Assume that $y > 1$. Since $xy + 1 \geq y + 1 > y - 1$, we find

$$0 \leq \frac{(3 - 1)(y - 1)}{xy + 1} < \frac{2(y - 1)}{y - 1} = 2,$$

which is a contradiction. Hence, $y = 1$. Therefore, $|Z(H)| = |Z(G)|$, and so $|G| = |H|$. $\square$

5. On Centerless Groups

Throughout this section we assume that $G \neq 1$ is a centerless group; i.e., $Z(G) = 1$. The purpose of this section is to verify the Conjecture for some centerless groups in order to strengthen the Conjecture.

Lemma 1. *Let $|G| = p + 1$, for some prime p. If H is a group with $\nabla(H) \cong \nabla(G)$ then $|H| = |G|$.*

Proof. From $\nabla(H) \cong \nabla(G)$ it follows that $|V(H)| = |V(G)|$. Therefore, $|H| - |Z(H)| = |G| - 1 = p$, and so

$$|Z(H)|(|\frac{H}{Z(H)}| - 1) = p.$$

We claim now that $|Z(H)| = 1$. If not then $|H/Z(H)| = 2$, which implies H is abelian, i.e., $H = Z(H)$. But then $|G| = 1$, which is a contradiction. It is evident now that $|H| = |G|$. $\square$

Proposition 11. *Let G and H be two groups such that $\nabla(G) \cong \nabla(H)$. Let g be a vertex in $V(G)$ such that $\deg(g) = |G| - 2$. Then $|H| = |G|$.*

Proof. From $\nabla(H) \cong \nabla(G)$ it follows that $|H| - |Z(H)| = |G| - 1$, and there exists $h \in V(H)$ such that $\deg(h) = |H| - |Z(H)| - 1$. We claim that $Z(H) = 1$. If not, we assume that $1 \neq z \in Z(H)$. Evidently, $hz \in V(H)$ and $[h, hz] = 1$. Hence $h \nsim hz$, and so $\deg(h) \leq |H \setminus Z(H)| - 2$, which is a contradiction. Therefore, $Z(H) = 1$ and $|H| = |G|$, as required. $\square$

EXAMPLE 1. Take $G = D_{2m} = \langle x, y : x^m = y^2 = 1,\ yxy = x^{-1}\rangle$, the dihedral group of order $2m$. When m is odd, $Z(G) = 1$. On the other hand, $[y, x^i y^j] \neq 1$, where $1 \leq i$ and $j = 0$ or 1, and so $\deg(y) = 2m - 2$. Now, if H is a group such that $\nabla(H) \cong \nabla(G)$, by Proposition 11 we deduce that $|H| = |G|$.

Lemma 2. *Let G and H be two groups such that $\nabla(H) \cong \nabla(G)$. The following hold:*

(a) *$|Z(H)|$ divides $|C_G(g_i)| - 1$ and $|g_i^G| - 1$ for every $g_i \in G^{\#}$. In particular, if one of the following two conditions holds:*

$$g.c.d.\ \{|C_G(g_1)| - 1, |C_G(g_2)| - 1, \ldots, |C_G(g_n)| - 1\} = 1,$$

or

$$g.c.d.\ \{|g_1^G| - 1, |g_2^G| - 1, \ldots, |g_n^G| - 1\} = 1$$

then $|H| = |G|$.

(b) *$|Z(H)|$ divides $k(G) - 1$.*

PROOF. (a) Let $\phi : V(G) \to V(H)$ be a graph isomorphism. First of all, we have $|H| - |Z(H)| = |G| - 1$, which implies that $|Z(H)| \mid |G| - 1$ and $(|Z(H)|, |G|) = 1$. Moreover, for all $g_i \in V(G) = G^{\#}$, we have $|Z(H)| \mid \deg(\phi(g_i)) = \deg(g_i)$, and since

$$\deg(g_i) = |G| - |C_G(g_i)| = (|G| - 1) - (|C_G(g_i)| - 1),$$

it follows that $|Z(H)| \mid |C_G(g_i)| - 1$.

We also have

$$\deg(g_i) = |G \setminus C_G(g_i)| = |C_G(g_i)|(|g_i^G| - 1)$$

for every $g_i \in V(G)$. From $(|Z(H)|, |C_G(g_i)|) = 1$ it follows now that $|Z(H)|$ divides $|g_i^G| - 1$ for every $g_i \in V(G)$. The rest of the proof is obvious.

(b) Since

$$|H| - |Z(H)| = |G| - 1 = \sum_{i=1}^{k(G)} (|g_i^G| - 1) + (k(G) - 1),$$

it follows by (a) that $|Z(H)|$ divides $k(G) - 1$. $\square$

REMARK 1. As a consequence of Lemma 2, it is easy to see that if $\nabla(H) \cong \nabla(G)$ and $2 \in \mathrm{cs}(G)$ then $|G| = |H|$. For instance, if $G = D_{2m}$, where m is odd, we always have $2 \in \mathrm{cs}(G)$, and hence from $\nabla(H) \cong \nabla(G)$ it follows that $|G| = |H|$ (which was previously addressed in Example 1).

Using Lemma 2(a), we can verify the Conjecture for alternating groups A_n ($n \geq 4$), all sporadic simple groups, simple groups of Lie type with nonconnected prime graph, symmetric groups S_n ($n \geq 3$). We deal with the above cases separately.

Theorem 1. *Let $S = A_n$ ($n \geq 4$). If H is a group such that $\nabla(H) \cong \nabla(S)$ then $|H| = |S|$.*

PROOF. By Lemma 1, the result holds for the alternating groups A_4, A_5, and A_6. If $S = A_7$ then there exists an element of order 7, say x, such that $|C_S(x)| = 7$. Hence, $|Z(H)|$ divides 6, and since $(|Z(H)|, |S|) = 1$ we deduce that $|Z(H)| = 1$, i.e., $|H| = |S|$. Now, we may assume $n \geq 8$. Consider the permutations $x = (12)(3456)$ and $y = (123)(456)$. Evidently, the length of the conjugacy classes of x and y is the same in S_n and A_n. Hence,

$$|x^S| = \frac{a}{8} \quad \text{and} \quad |y^S| = \frac{a}{18},$$

where $a = \frac{n!}{(n-6)!}$. It is easy to see now that $(|x^S| - 1, |y^S| - 1) = 1$, and by Lemma 2(a), we obtain $|H| = |S|$. $\square$

Theorem 2. *Let S be a sporadic simple group. If H is a group such that $\nabla(H) \cong \nabla(S)$ then $|H| = |S|$.*

Proof. It is easy to check the existence of a pair of elements $x, y \in S \setminus \{1\}$ such that $(|C_S(x)| - 1, |C_S(y)| - 1) = 1$ (see [9]), and hence by Lemma 2(a), we conclude that $|H| = |S|$. $\square$

Theorem 3. *Let S be a simple group of Lie type with $t(S) \geq 2$. If H is a group such that $\nabla(H) \cong \nabla(S)$ then $|H| = |S|$.*

Proof. In every finite simple group of Lie type S with $t(S) \geq 2$ (except $S = A_1(q)$, q odd), there are some maximal tours such as T which is a cyclic Hall subgroup of S and evidently $C_S(T) = T$ (see [6]). Since $n_i = |T|$ for some $i \geq 2$ we deduce that $|C_S(g)| = n_i$ for every $g \in T^{\#}$. Therefore, by Lemma 2(a), if H is a group such that $\nabla(H) \cong \nabla(S)$ then $|Z(H)| \,\big|\, n_i - 1$, for every $i \geq 2$. We recall that $|Z(H)|$ must also be a divisor of $|S| - 1$. Our observations through the results summarized in Tables 1*a*–*c* in [7] show that for each disconnected simple graph of Lie type S with exception of $A_1(q)$, q odd, the following is always true:

$$\text{g.c.d. } \{n_2 - 1, \ldots, n_{t(S)} - 1, |S| - 1\} = 1. \tag{2}$$

For instance, in the following we will investigate the property 2, for some simple groups.

(1) $S = B_n(q)$, $n = 2^m \geq 4$, q odd.

In this case we have $n_2 = \frac{q^n+1}{2}$ and

$$|S| = \frac{1}{(2, q-1)} q^{n^2} \sum_{i=1}^{n} (q^{2i} - 1).$$

Evidently, $n_2 - 1 = \frac{q^n-1}{2}$ divides $|S|$, and so $(n_2 - 1, |S| - 1) = 1$.

(2) $S = {}^3D_4(q)$.

In this case we have $n_2 = q^4 - q^2 + 1$ and

$$|S| = q^{12}(q^2 - 1)(q^8 + q^4 + 1)(q^6 - 1).$$

Again, since $n_2 - 1 = q^2(q^2 - 1)$ divides $|S|$, we obtain $(n_2 - 1, |S| - 1) = 1$.

(3) $S = G_2(q)$, $q \equiv 0 \bmod 3$.

For this group, we have $n_2 = q^2 - q + 1$, $n_3 = q^2 + q + 1$ and

$$|S| = q^6(q^2 - 1)(q^6 - 1).$$

By easy calculations we see now that g.c.d. $\{n_2 - 1, n_3 - 1, |S| - 1\} = 1$.

(4) $S = {}^2B_2(q)$, $q = 2^{2f+1}$.

Here $n_2 = q - 1$, $n_3 = q - \sqrt{2q} + 1$, $n_4 = q + \sqrt{2q} + 1$, and $|S| = q^2(q - 1)(q^2 + 1)$. Easy calculations show now that g.c.d. $\{n_2 - 1, n_3 - 1, n_4 - 1, |S| - 1\} = 1$.

(5) $S = E_8(q)$.

In this case $t(S) \geq 4$, and we have $n_4 = q^8 - q^4 + 1$, and

$$|S| = q^{120}(q^2 - 1)(q^8 - 1)(q^{12} - 1)(q^{14} - 1)(q^{18} - 1)(q^{20} - 1)(q^{24} - 1)(q^{30} - 1).$$

Since $n_4 - 1 = q^4(q^4 - 1)$ divides $|S|$, it is clear that $(n_4 - 1, |S| - 1) = 1$, and so the property 2 holds.

If $S = A_1(q)$, $3 < q \equiv \varepsilon \bmod 4$, $\varepsilon = \pm 1$ then S contains an element of order $(q - \varepsilon)/2$, and since $\pi_1(S) = \pi(q - \varepsilon)$, we conclude that S contains a selfcentralizing cyclic subgroup C_1 of order $(q - \varepsilon)/2$. On the other hand, S contains the selfcentralizing cyclic subgroup C_2 of order $n_3 = \frac{q+\varepsilon}{2}$. Since $|Z(H)|$ divides $(|C_1| - 1, |C_2| - 1) = 1$, we now obtain $|Z(H)| = 1$ or $|H| = |S|$, as required. $\square$

Theorem 4. *Let $G = S_n$ $(n \geq 3)$. If H is a group such that $\nabla(H) \cong \nabla(G)$ then $|H| = |G|$.*

Proof. By Lemma 2(a), it is sufficient to find a pair of elements in G, say x and y, such that $(|x^G| - 1, |y^G| - 1) = 1$. Suppose first that $n \geq 4$. Take $x = (12)(34)$ and $y = (1234)$. Clearly, $2|x^G| = |y^G| = a/4$, where $a = \frac{n!}{(n-4)!}$. It is easy to see now that $(|x^G| - 1, |y^G| - 1) = 1$, as required. Suppose next that $n = 3$. In this case we consider $x = (12)$ and $y = (123)$. $\square$

References

1. *Williams J. S.*, "Prime graph components of finite groups," J. Algebra, **69**, 487–513 (1981).
2. *Abe S. and Iiyori N.*, "A generalization of prime graphs of finite groups," Hokkaido Math. J., **29**, No. 2, 391–407 (2000).
3. *Segev Y.*, "On finite homomorphic images of the multiplicative group of a division algebra," Ann. of Math., **149**, 219–251 (1999).
4. *Segev Y. and Seitz G.*, "Anisotropic groups of type A_n and the commuting graph of finite simple groups," Pacific J. Math., **202**, 125–225 (2002).
5. *Kondrat′ev A. S.*, "On prime graph components of finite simple groups," Mat. Sb., **180**, No. 6, 787–797 (1989).
6. *Kondrat′ev A. S. and Mazurov V. D.*, "Recognition of alternating groups of prime degree from their element orders," Siberian Mat. J., **41**, No. 2, 294–302 (2000).
7. *Mazurov V. D.*, "Recognition of finite simple groups $S_4(q)$ by their element orders," Algebra and Logic, **41**, No. 2, 93–110 (2002).
8. *Ito N.*, "On finite groups with given conjugate types I," Nagoya Math. J., **6**, 17–28 (1953).
9. *Conway J. H., Curtis R. T., Norton S. P., Parker R. A., and Wilson R. A.*, Atlas of Finite Groups, Clarendon Press, Oxford (1985).