On correct definition and use of normal heights in geodesy

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ABSTRACT

Physical heights is one of the most important topics in physical geodesy. Their original concept, introduced in the 19-th century, defined physical heights as lengths of plumblines of the Earth's gravity field between the geoid and points of interest. There are orthometric heights of surface points, that have been traditionally estimated by spirit levelling and measured gravity; however, the knowledge of the density distribution of topographic masses (masses between the geoid and Earth's surface) is required that significantly affects their determinability. This was also the main reason why a new type of physical heights was proposed in the mid of the 20-th century. Normal heights approximate orthometric heights in a sense that the Earth's gravity field is replaced by the normal gravity field, an analytic model based on the theory of an equipotential ellipsoid. This height system has been introduced since that time in different countries in Europe and beyond. Contrary to the classical height system based on orthometric heights, its counterpart based on normal heights may have slightly different definitions. Moreover, normal heights are often defined as heights of points above the quasigeoid. This contribution reviews alternative definitions of normal heights and respective height systems. It is argued that both orthometric and normal heights refer to the geoid. In the case physical heights are estimated by satellite positioning, normal heights must be computed through the height anomaly estimated at each point of interest, whether it is below, at or above the Earth's surface. On the contrary, orthometric heights of all points along the same plumbline, be it below, at or above the Earth's surface, are estimated by introducing one value of the geoid height. Normal heights of surface points can be estimated by spirit levelling easier than orthometric heights as no topographic mass density hypothesis is required; however, one has to keep in mind the gravity field approximation used both for their definition and realization.

Keywords: height systems, normal heights, orthometric heights, geoid, levelling, GNSS positioning

1. INTRODUCTION

Heights used in geodesy and surveying can be divided into: a) geometric heights defined only with respect to the geometric properties of the 3-D space where heights are usually estimated and applied (at the surface of the Earth and approximately several km below it or up to 20 km above it); and b) physical heights, which also take into the account its physical properties (gravity field) and which are important in many applications, e.g., in construction engineering. While the geometric height relative to the adopted reference ellipsoid can be directly obtained by satellite positioning with a cm-level accuracy, the physical height depends on the knowledge of the Earth's gravity field above the geoid, i.e., also within the topographic masses. Such physical height is called orthometric. Because the Earth's gravity field inside topographic masses was unknown in the past, an alternative physical height defined by a normal gravity field (an analytic model approximating the real Earth's gravity field) was proposed. This physical height, called normal, has been used in some countries in Europe and beyond since the 1960s.

The topic of physical heights has been recently revisited in geodetic literature, e.g., *Foroughi et al. (2017)*, *Sjöberg (2018)*, *Sansò et al. (2019)* or *Vaníček and Santos (2019)*, and at some scientific meetings such as the Hotine-Marussi Symposia on Mathematical Geodesy 2018 and 2022, see, e.g., *Novák et al. (2021)*. In particular, normal heights have been discussed, as their definition may vary slightly, their height reference surface is sometimes incorrectly defined, and their superiority in terms of better determinability with respect to orthometric heights has been decreasing as knowledge of the Earth's internal structure continues to improve. The main purpose of this text is to summarize the latest findings and show that normal heights are sometimes defined and used incorrectly, especially for vertical positioning of points below or above the Earth's surface.

The text is organized as follows: in Section 2, basic geometric and gravity field concepts required for definition of heights and height systems used in geodesy are introduced; height systems are defined in Section 3; properties of physical heights are discussed in Section 4; the determinability of physical heights is outlined in Section 5; several issues related to normal heights and their potentially incorrect applications are discussed in Section 6; finally, Section 7 summarizes the contribution.

2. DEFINITIONS AND CONCEPTS

Since heights and height systems discussed in this text relate to geometric and physical properties of 3-D space up to 20 km above and several kilometres below the Earth's surface, basic concepts from the Earth's geometry and gravity field are recalled below. These basic definitions are used to define heights and height systems, and to discuss their properties, determinability, and relationships.

2.1. Coordinate systems and the reference ellipsoid

The position of any point in 3-D space is described, for example, by means of a vector $\boldsymbol{x}$ (vectors are typed bold-italics) defined in the geocentric Earth-fixed Cartesian coordinate system. Its origin is in the centre of the Earth's mass, the x axis is realized as intersection of the mean Greenwich meridian plane with the Earth's equator, and the z axis coincides with the mean axis of Earth's rotation. The right-handed coordinate system rotates with the Earth. Two curvilinear coordinates are defined in terms of their transformations into the geocentric Cartesian coordinates as follows (*Heiskanen and Moritz, 1967, Eq. 1-103*; *Vaníček and Krakiwsky, 1986, Eq. 15.63*):

$$\begin{aligned} x &= r\cos\varphi\cos\lambda = \left(N_e + h\right)\cos\phi\cos\lambda\,, \\ y &= r\cos\varphi\sin\lambda = \left(N_e + h\right)\cos\phi\sin\lambda\,, \\ z &= r\sin\varphi = \left(N_e\frac{b^2}{a^2} + h\right)\sin\phi\,. \end{aligned} \tag{1}$$

Non-parametric spherical coordinates $\left(r, \varphi, \lambda\right)$ are defined by the geocentric radius r, geocentric latitude φ, and geocentric longitude λ. Two-parametric geodetic (Gauss ellipsoidal) coordinates are represented by a triplet $\left(h, \phi, \lambda\right)$, with the geodetic height h and geodetic latitude ϕ. The ellipsoidal prime vertical radius of curvature N_e can be found, e.g., in *Vaníček and Krakiwsky (1986, Eq. 15.58)*.

The two parameters in the case of geodetic coordinates are the major and minor semi-axes a and b of the biaxial ellipsoid, i.e., geodetic coordinates depend on the chosen reference ellipsoid $\mathcal{E}$. Currently, the reference ellipsoid is defined by the Geodetic Reference System 1980 (*Moritz, 1984*). The geodetic height h is then defined as the Euclidean distance of a point P from the reference ellipsoid, see Fig. 1. As the geocentric Cartesian coordinates are routinely estimated by global navigation satellite systems (GNSS) with the cm-level accuracy, geodetic heights can be estimated with the same level of accuracy, as they are functions of the former through simple algebraic relations.

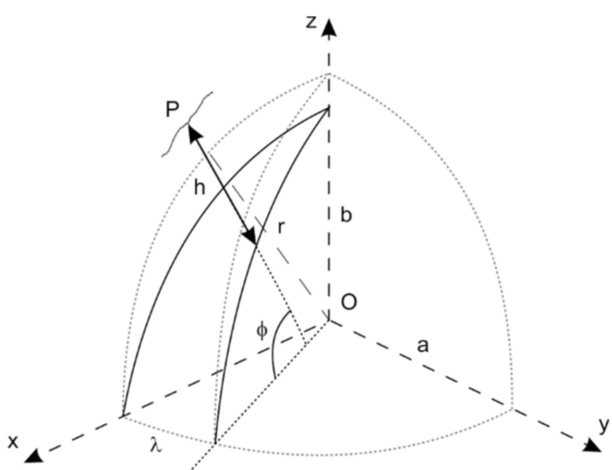

Fig. 1. Scheme of the geocentric Cartesian coordinates (x, y, z), major and minor semi-axes (a, b) of the international reference ellipsoid, and geodetic coordinates (ϕ, λ, h).

2.2. Earth's gravity field

The static Earth's gravity field corresponds to the idealized rigid Earth model with no changes in mass distribution, uniform rotation and absent time-dependent gravitational effects of all external masses. Being a vector field in 3-D space, the gravity acceleration vector $\boldsymbol{g}$ (in m s^{-2}) at a point positioned by the geocentric vector $\boldsymbol{x}$ reads

$$\boldsymbol{g}(\boldsymbol{x}) = G \int_{\mathcal{B}} \rho(\boldsymbol{x}') \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \mathrm{d}\mathcal{B}(\boldsymbol{x}') + \omega^2 \boldsymbol{p}(\boldsymbol{x}) \tag{2}$$

as it is composed of the gravitational acceleration generated by the Earth's volume $\mathcal{B}$ (*Newton, 1687*) with the infinitesimal volume element $\mathrm{d}\mathcal{B}$ positioned by the geocentric vector $\boldsymbol{x}'$ and the centrifugal acceleration acting upon each particle attached to the rotating Earth (*Huygens, 1659*). Input parameters include the universal gravitational constant G, mass density function ρ within the Earth's body $\mathcal{B}$ and angular velocity of the Earth's rotation ω. The vector $\boldsymbol{p} = (x, y, 0)$ is perpendicular to the rotation axis.

Since the static gravity field is irrotational, then it can be represented by a scalar-valued function of 3-D position called gravity potential W (in m^2 s^{-2})

$$W(\boldsymbol{x}) = G \int_{\mathcal{B}} \rho(\boldsymbol{x}') \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \mathrm{d}\mathcal{B}(\boldsymbol{x}') + \frac{1}{2} \omega^2 p^2 , \tag{3}$$

from which the gravity acceleration vector is constructed through the gradient operator ∇ (*Heiskanen and Moritz, 1967, Sec. 2-1*)

$$\nabla W(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}) , \tag{4}$$

and p is the modulus of the vector $\boldsymbol{p}$. Laplacian of the gravity potential then yields the well-known Poisson differential equation (*Heiskanen and Moritz, 1967, Sec. 1-6*)

$$\nabla^2 W(\boldsymbol{x}) = -4\pi G \rho(\boldsymbol{x}) + 2\omega^2 . \tag{5}$$

Surfaces of constant gravity potential are called *equipotential*. Their geometry inside and in the vicinity of the Earth's surface is complicated as it reflects irregularities of the Earth's gravity field. They are approximately realized by surfaces of liquids at rest.

One equipotential surface with the reference value of the gravity potential W_0 (*Sánchez et al., 2016*), that coincides with the global ocean surface at rest (*Gauss, 1828*), is of particular importance for heights used in geodesy. It is called the *geoid* (*Listing, 1872*) and serves as a height reference surface for physical heights. Considering two different points in space, the difference of their respective values of the gravity potential defines the shortest "physical distance" of the two points, see Appendix A. If one of the two points is at the geoid, the so-called *geopotential number C* is defined as follows:

$$C(\boldsymbol{x}) = W_0 - W(\boldsymbol{x}) . \tag{6}$$

Unfortunately, the geoid $\mathcal{G}$ is unobservable under continents (where heights matter at most); however, it can be estimated using gravity data observed at or outside the irregular Earth's surface $\mathcal{S}$ (*Stokes, 1849*). Curved lines perpendicular at each point in space to the equipotential surface are called plumblines *t*. Taken a particular point with the modulus of the gravity vector $|\boldsymbol{g}| = g$, one can define a unit tangent vector $\boldsymbol{t}$ (always pointing up) as follows:

$$\boldsymbol{t}(\boldsymbol{x}) = -\frac{\boldsymbol{g}(\boldsymbol{x})}{g(\boldsymbol{x})}. \tag{7}$$

The real Earth's gravity field is approximated in geodesy by an analytic model called the normal gravity field. It is generated by the reference ellipsoid $\mathcal{E}$ that rotates with the same angular velocity ω as the real Earth, and its mass coincides with the mass M of the real Earth. According to the Stokes-Poincaré theorem, the normal gravity field can be described using only four parameters, a, b, GM, and ω. The normal gravity potential can be conveniently expressed in spherical coordinates as follows (*Pizzetti, 1911*; *Somigliana, 1929*):

$$U(r,\varphi) = \frac{GM}{r}\left[1 - \sum_{i=1}^{\infty}\left(\frac{a}{r}\right)^{2i} U_{2i}(\varphi)\right] + \frac{\omega^2 r^2}{3}\left(1 - P_2(\varphi)\right) \tag{8}$$

with Legendre's polynomial P_2 and zonal spherical harmonics U_{2i} defined analytically (*Heiskanen and Moritz, 1967, Sec. 2-8*). Applying the gradient operator, the normal gravity vector γ is obtained

$$\nabla U(\boldsymbol{x}) = \gamma(\boldsymbol{x}), \tag{9}$$

and applying the Laplacian outside the reference ellipsoid (ibid)

$$\nabla^2 U(\boldsymbol{x}) = 2\omega^2. \tag{10}$$

The reference ellipsoid is the equipotential surface of the normal gravity field with the value of the normal gravity potential U_0 equal by definition to the real gravity potential W_0 at the geoid. Similarly to the real Earth's gravity field above, there is a normal plumbline t' defined for the normal gravity field. It is perpendicular to all equipotential surfaces of the normal gravity field. As the normal gravity field is rotationally symmetric, its geometry is much simpler compared to the real plumbline *t*. A unit tangent vector $\boldsymbol{t'}$ with γ pointing upwards at any point in space is then defined as follows:

$$\boldsymbol{t}'(\boldsymbol{x}) = -\frac{\boldsymbol{\gamma}(\boldsymbol{x})}{\gamma(\boldsymbol{x})}. \tag{11}$$

The spatial angle ε between the real and normal plumblines is the deflection of the verticals.

The *disturbing potential* is defined in any point as a difference of the real gravity potential W and the normal gravity potential U (*Heiskanen and Moritz, 1967, Sec. 2-13*)

$$T(\boldsymbol{x}) = W(\boldsymbol{x}) - U(\boldsymbol{x}). \tag{12}$$

Applying the gradient operator, one gets the disturbing gravity vector

$$\nabla T(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{\gamma}(\boldsymbol{x}) = \boldsymbol{\delta g}(\boldsymbol{x}). \tag{13}$$

Obviously, in a mass-free point, i.e., outside the Earth's masses, see Eqs (5) and (10),

$$\nabla^2 T(\boldsymbol{x}) = 0 . \tag{14}$$

In mass-free space, the disturbing potential is a harmonic function of 3-D position and the apparatus of the potential theory can be used for its representation, transformation, or continuation.

The reference ellipsoid and the geoid are related by the *geoid height N* measured along the ellipsoidal normal n that approximates the real plumbline t with sufficient accuracy. It can be estimated from the disturbing potential T (note the analogy of these quantities in gravity field and geometric spaces). The normal gravity potential U at the geoid can be represented by a power series with an expansion point $\boldsymbol{x}_e$ at the reference ellipsoid that is subsequently truncated to a linear term, i.e.,

$$U(\boldsymbol{x}_g) = U_0 + \left.\frac{\partial U(\boldsymbol{x})}{\partial n}\right|_{\boldsymbol{x}_e} N + \frac{1}{2}\left.\frac{\partial^2 U(\boldsymbol{x})}{\partial n^2}\right|_{\boldsymbol{x}_e} N^2 + \mathcal{O}\left(N^3\right) \approx U_0 - \gamma(\boldsymbol{x}_e) N , \tag{15}$$

where Landau's symbol $\mathcal{O}$ represents the order of magnitude of neglected higher-order terms of the power series. Based on the definition of the disturbing potential T, and applying the condition $W_0 = U_0$, the well-known spherical Bruns formula can be derived (*Bruns, 1878*)

$$T(\boldsymbol{x}_g) = \gamma(\boldsymbol{x}_e) N . \tag{16}$$

Thus, the problem of the geoid determination is based on solving for the disturbing potential T at the geoid. Note that Bruns's formula in Eq. (16) applies to two particular points in space, one at the geoid and the other at the reference ellipsoid along the same ellipsoidal normal n. However, this equation can be used anywhere in 3-D space to relate any two points P and P' along the same ellipsoidal normal n for which the condition $W_P = U_{P'}$ holds.

2.3. Summary and orders of magnitude

To summarize this section, we have described the static Earth's gravity field and defined its analytic model called normal gravity field. For both gravity fields, we introduced their scalar representations through the gravity potentials W and U, defined their equipotential surfaces (including the geoid and reference ellipsoid) and plumblines, and derived basic relationship for their ellipsoidal normal separation N under the condition values of the gravity potentials W and U at respective reference equipotential surfaces coincide.

To relate the real and normal gravity fields and to apply some approximations, orders of magnitude of selected parameters are provided next. The order of magnitude of the disturbing potential T, of the gravity disturbance δg, and of the geoid height N is $\pm 10^3$ m^2 s^{-2}, $\pm 10^{-3}$ m s^{-2} (10^2 mGal) and $\pm 10^2$ m, respectively. Defining the deflection of the verticals ε as the spatial angle between the unit vectors $\boldsymbol{t}$ and $\boldsymbol{n}$, their values range within $\pm 10^{-4}$ rad (1 arc-min). Note that the deflection of the verticals at the geoid is smaller than at the Earth's surface.

Magnitudes of other parameters shown in Fig. 2 can be conservatively estimated for heights up to 20 km above the Earth's surface as follows:

$$\left| EE'' \right| \approx \varepsilon \times 20 \text{ km} = 6 \text{ m} , \tag{17}$$

$$\left| PG - PP'' \right| < \frac{1}{2} \varepsilon^2 \times 20 \text{ km} \approx 1 \text{ mm} , \tag{18}$$

$$\left| P''G'' \right| \approx \varepsilon \left| EE'' \right| < 1.8 \text{ mm} , \tag{19}$$

$$\left| PE - PE'' \right| \ll 2.8 \text{ mm} . \tag{20}$$

Thus, one can conclude that the plumblines t and t' can conveniently be interchanged, and eventually even replaced by the ellipsoidal normal n at the mm-level accuracy that is usually required for heights in geodesy. These approximations are occasionally applied in derivations presented below.

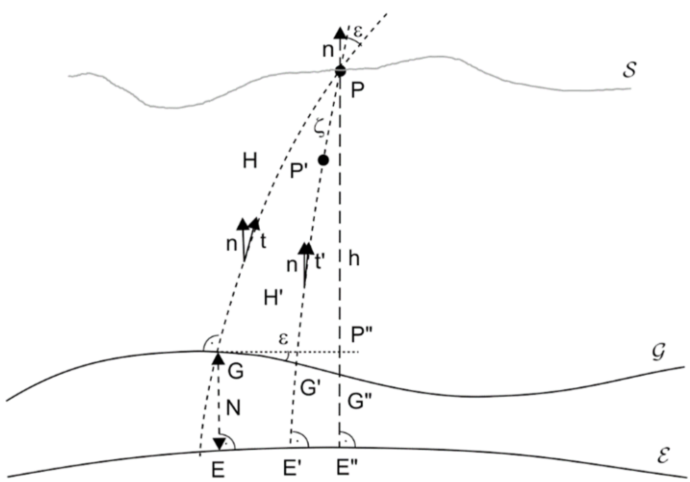

Fig. 2. Geodetic height h, orthometric height H and normal height H', actual and normal plumblines t and t', ellipsoidal normal n, and the deflection of the verticals ε.

3. HEIGHT SYSTEMS

Geodesy defines and uses several height systems to locate vertically points of interest at, above, or below the Earth's surface. Each height system consists of a height reference surface, upon which values of associated heights are equal to zero, and a height defined as a length (not necessarily Euclidean) of a point with respect to the height reference surface in 3-D space.

If geometric properties of 3-D space are only considered, then a geometric height system is used. Respective heights are called geodetic but other adjectives are also used in geodesy and elsewhere (e.g., ellipsoidal or GNSS); for more details, see Section 3.1. Geodetic heights are unique for a specific reference ellipsoid, i.e., a particular geometric model approximating globally the geoid. If physical properties of 3-D space are considered, then one refers to physical height systems. Physical heights are defined considering either the real Earth's gravity field or the normal gravity field. In this contribution, we consider as physical height systems either the orthometric height system, see Section 3.2, or the normal height system, see Section 3.3. It is acknowledged that other physical heights exist, such as dynamic or normal orthometric, but they are not considered in this text.

3.1. Geodetic height system

The geodetic height system, see Fig. 1, is defined as follows: the height reference surface is the reference ellipsoid. The geodetic height h of a point P at, below or above the Earth's surface is then measured along the ellipsoidal normal n passing through the point P, i.e., the geodetic height represents the shortest Euclidean distance (geodesic in the geometric sense) of the point P from the reference ellipsoid $\mathcal{E}$ in 3-D space. Numerical values of the geodetic height then differ rather significantly from any physical height; the differences of up to 10^2 m are caused by different reference surfaces and different height definitions.

Nowadays, geodetic heights are routinely estimated anywhere around the world by satellite positioning techniques, out of them GNSS is most widely used in practice. Geodetic heights can be estimated with an accuracy of up to 1–2 cm, given a geodetic GNSS receiver, good observation conditions and proper data analysis are applied. However, as many applications in geodesy and surveying must reflect physical properties of 3-D space, geodetic heights need to be often transformed into physical heights.

3.2. Orthometric height system

The classical height system based on physical heights, that respects real physical properties of the 3-D space at the Earth's surface and in its close vicinity, is the orthometric height system. This system, forged during the 19-th century, is usually connected with the names of Gauss, Helmert and Stokes.

The orthometric height system uses as height reference surface the geoid that can be estimated with the accuracy of 1 cm if good-quality topographic and gravity data are available, see, e.g., *Wang et al. (2021)*. The orthometric height H of the point P at, above or below the Earth's surface $\mathcal{S}$ is defined as the length of a segment of the real plumbline t between the point G at the geoid $\mathcal{G}$ and the point P, see Fig. 2, with a minus sign for points below. The orthometric height is not the shortest distance between the geoid and the point in 3-D geometric space but rather in 3-D physical space. One can prove in fact that defining the "physical distance" between the point G at the geoid and the point P along the plumbline t as the geopotential number, see Eq. (6),

$$C_P = \int_G^P g(t)\, \mathrm{d}t \,, \tag{21}$$

then the length of the segment of the plumbline t between the points G and P becomes the minimum distance (geodesic in the physical sense) between the point P and the geoid. The proof is given in the Appendix A.

The orthometric height system is particularly important for geodesy and surveying. Orthometric heights are used in many countries around the world despite certain limitations in terms of their determinability. These limits are related to the topographic mass density required for determination of orthometric corrections to levelled height differences, see Section 5.1., and the geoid height, see Section 5.2.

3.3. Normal height system

As the orthometric heights are determined using observations corrected for values that depend on the topographic mass density, an alternative height system was proposed around mid of the 20-th century independently by *Molodensky (1945)* and *Vignal (1954)*. This alternative physical height system does not require the knowledge of the topographic mass density which is its most significant advantage compared to the classical height system based on orthometric heights.

The normal height system is defined similarly as the orthometric height system, just the normal gravity field replaces the real gravity field. The height reference surface for normal heights H', defined as the surface where H' goes to zero, is also the geoid $\mathcal{G}$. The normal height is then defined as the length of a segment of the normal plumbline t' between the point P and the geoid. One can use Eq. (21) and replace real gravity g by normal gravity γ. Thus, the metric of normal heights differs from the metric used for orthometric heights. For more details, see Sections 4.1 and 4.2.

Normal heights can be estimated from geodetic heights h by applying the so-called *height anomaly* ζ (*Molodensky, 1945*) that represents a vertical separation of the point P and the point P' with the condition $W_P = U_{P'}$, see Fig. 2. Connecting points P', a surface called the *telluroid* (*Hirvonen, 1960*) is obtained. Both the geoid height N and the height anomaly ζ are two particular examples of a general function of 3-D position that defines a vertical separation of the real and normal equipotential surfaces with the same value of the respective real and normal gravity potentials. The height anomaly ζ can be determined from the value of the disturbing potential T at the Earth's surface by Bruns's formula exactly as the geoid height N is estimated from the value of the disturbing potential at the geoid. *Molodensky (1945)* proposed to use gravity data as observed at the Earth's surface, apply a suitable integral transform to recover a surface value of the disturbing potential and then, through Bruns's formula, recover the height anomaly, see Section 5.2 below. Moreover, normal heights can be also estimated by spirit levelling corrected for normal gravity field effects, see Section 5.1.

4. PHYSICAL HEIGHTS

In this section, orthometric and normal heights defined using the real and normal gravity fields, respectively, are reviewed in more details. As orthometric heights refer to the real Earth's gravity field, they would be natural candidates for practical applications, pending they can be estimated from observed data with the sufficient accuracy. Meaning of the adjective "sufficient" depends on their particular applications; in geodesy and surveying, a cm-level accuracy is typically required. However, specific applications, such as geodynamics, monitoring large structures (e.g., dams) or construction engineering, may have even more stringent accuracy requirements. Independently of their definitions, physical heights must obey certain requirements, in particular they must represent a vertical coordinate of a point of interest in 3-D space, see *Sansò et al. (2019)*.

4.1. Orthometric heights

Orthometric heights are defined using the gravity potential W of the Earth's gravity field. As it is defined above, the geoid coincides with the mean sea level at rest. Defining the value of the reference gravity potential at the geoid as W_0, then for the point P at, below or above the Earth's surface

$$W_P = W_0 + \int_G^P \frac{\partial W}{\partial t}\,\mathrm{d}t = W_0 - \int_G^P g(t)\,\mathrm{d}t\,. \tag{22}$$

Here, the symbol G represents the point at the geoid $\mathcal{G}$ along the real plumbline t passing through the point P, g is the modulus of the gradient of the real gravity potential along a tangent to the real plumbline t. Defining the integral mean value of real gravity along the segment of the real plumbline, i.e.,

$$\bar{g} = \frac{1}{H_P}\int_G^P g(t)\,\mathrm{d}t\,, \tag{23}$$

then the orthometric height H of the point P is defined as follows:

$$H_P = \frac{W_0 - W_P}{\bar{g}} = \frac{C_P}{\bar{g}} \tag{24}$$

Equation (24) means that in order to estimate the orthometric height H_P, one must estimate the geopotential number C_P and the integral mean of real gravity $\bar{g}$ along the segment of the real plumbline t between the points G and P. As real gravity can be neither directly measured nor computed inside the topographic masses, the definition in Eq. (24) seems to be of a little practical use. Fortunately, there are at least two approaches (both based on certain assumptions) for the estimation of orthometric heights from levelled height differences and GNSS-based geodetic heights, see Section 5.

4.2. Normal heights

Let us turn now our attention to the normal heights defined using the normal gravity field. As real gravity cannot be accurately estimated or directly measured inside the topographic masses, *Molodensky et al. (1960)* suggested to use normal gravity instead. The advantage is that normal gravity is defined analytically above the reference ellipsoid $\mathcal{E}$; thus, its mean value can be easily estimated there. The same advantage would apply also to estimation of normal corrections to levelled height differences and height anomalies.

Following the definition of the orthometric heights in Eq. (22) and replacing the actual gravity field by its normal counterpart, then

$$W_P = W_0 + \int_{G'}^{P} \frac{\partial U}{\partial t'} \mathrm{d}t' = W_0 - \int_{G'}^{P} \gamma\left(t'\right) \mathrm{d}t' . \tag{25}$$

The normal plumbline t' passes through the surface point P and the respective point G' at the geoid, γ is the gradient of the normal gravity potential along a tangent to the normal plumbline t', see Fig. 2. Using the integral mean value of normal gravity along the segment of the normal plumbline between the point G' at the geoid and the point P

$$\bar{\gamma} = \frac{1}{H'_P} \int_{G'}^{P} \gamma\left(t'\right) \mathrm{d}t' , \tag{26}$$

the normal height H' of the point P reads

$$H'_P = \frac{C_P}{\bar{\gamma}} . \tag{27}$$

Comparing Eqs (24) and (27), it is obvious that the orthometric height H and the normal height H' do not use the same metric. Actually, both physical heights are related as follows:

$$H_P = \frac{\bar{\gamma}}{\bar{g}} H'_P \tag{28}$$

Their differences may reach the magnitude at the order of 1 m over areas with high topographies and complex gravity fields. If the real and normal gravity fields would coincide, then $\bar{\gamma} = \bar{g}$ and the two heights would be equal for any point below or above the geoid.

Equation (26) is based on the integral mean value of normal gravity evaluated between the geoid and the point P along the normal plumbline t'. As the following two conditions $U_0 = W_0$ and $U_{P'} = W_P$ apply, then Eq. (25) can be written completely using the normal gravity potential

$$U_{P'} = U_0 + \int_{E'}^{P'} \frac{\partial U}{\partial t'} \mathrm{d}t' = U_0 - \int_{E'}^{P'} \gamma\left(t'\right) \mathrm{d}t' . \tag{29}$$

The symbol E' represents a point at the reference ellipsoid E along the normal tangent t' passing through the point P, see Fig. 2. This equation implies that the normal height of the point P approximates the normal height of the point P' above the reference ellipsoid as mean values of normal gravity are evaluated over two different segments of the normal plumbline, i.e., between the points G' and P in Eq. (26), and between the points E' and P' in Eq. (29). However, their differences are very small and Eq. (29) is significant from a practical point of view as the mean value of normal gravity along the segment of the ellipsoidal normal between the points E' and P' can be evaluated analytically (obviously, some iterations are needed) neglecting very small differences between the ellipsoidal normal n and normal plumbline t', see Section 2.3.

4.3. Conclusions on physical heights

First of all, both orthometric and normal heights refer to the geoid as their common height reference surface, in fact both coordinates attain the zero value on such a surface. Orthometric heights are considered along the real plumblines and normal heights are considered along the normal plumblines; however, the effect caused by using two different plumblines is negligibly small at the cm-level accuracy, see Section 2.3. As both physical heights use the same height reference surface and curvature effects of two different plumblines are negligible, yet their numerical values differ at the level of a few dm, there must be some intrinsic difference. Indeed, the two physical heights use different metrics as it is obvious from their defining equations based on the geopotential number, see Eqs (24) and (27). In case of the orthometric height, the geopotential number is metricized by the mean value of real gravity g. In case of the normal height, the geopotential number is metricized by the mean value of normal gravity γ. Thus, normal heights only approximate orthometric heights as the normal gravity field approximates the real Earth's gravity field. This approximation is sufficiently accurate for many applications, particularly when local height differences matter.

5. ESTIMATION OF PHYSICAL HEIGHTS

To be practically useful, physical heights must be estimable from observable data. In general, there are two major observation techniques that are used recently for estimation of physical heights in geodesy and surveying: spirit levelling and satellite positioning (namely GNSS).

Spirit levelling provides height differences measured with a sub-cm accuracy at the Earth's surface along segments of levelling lines. These levelling increments, integrated along the levelling line, define a so-called unholonomic coordinate, i.e., they do not sum up to zero along a closed line and respective corrections reflecting gravity field effects must be applied. While corrections to orthometric heights depend on real gravity at the Earth's surface as well as inside topography, corrections to normal heights depend only on real gravity at the Earth's surface, see, e.g., *Heiskanen and Moritz (1967, Sec. 4-5)* or *Sansò et al. (2013)*.

Satellite positioning then provides geocentric Cartesian coordinates that can be directly transformed into geodetic coordinates, see Eq. (1). The geodetic height h can be estimated with the accuracy of 1–2 cm. It can be transformed into physical heights when respective transformation parameters (geoid height N or height anomaly ζ) are known, ideally with the cm-level accuracy.

5.1. Spirit levelling

For a long time spirit levelling had been the only observation technique that could provide physical heights. Starting with a reference point (gauge station) along a sea coast (defining zero height), heights of points inland are estimated through height difference measured over segments of a levelling line. As levelled increments $\delta L = -\delta W / g$ are not exact differentials, they do not sum up exactly to the physical height difference of a particular point P with respect to the reference point. Corrections, that take into the account curvatures of the plumblines (equipotential surfaces are not parallel), must be applied.

To derive corrections to the levelled height difference δL (in order to obtain respective differences of orthometric or normal heights), one can start with its observation equation

$$\delta L = \boldsymbol{t} \cdot \delta \boldsymbol{r} , \tag{30}$$

which represents an orthogonal projection of the inter-station vector $\delta\boldsymbol{r}$ between two surface points into the tangent of the local plumbline t in their mid-point. In the notation of differential calculus

$$\mathrm{d}L = \boldsymbol{t} \cdot \mathrm{d}\boldsymbol{r} . \tag{31}$$

The unit vector $\boldsymbol{t}$ tangent to the actual plumbline defined in Eq. (7) can be approximated by the unit vector $\boldsymbol{t}'$ tangent to the normal plumbline, see Eq. (11), and a small correction term defined as follows (*Sansò et al., 2019, Sec. 5.3*):

$$\boldsymbol{t} = \boldsymbol{t}' - \frac{1}{\gamma} \nabla T + \frac{1}{\gamma} \left(\boldsymbol{t}' \cdot \nabla T \right) \boldsymbol{t}' . \tag{32}$$

Substituting Eq. (32) into Eq. (31) yields

$$\mathrm{d}L = \boldsymbol{t}' \cdot \mathrm{d}\boldsymbol{r} - \frac{1}{\gamma} \mathrm{d}\boldsymbol{r} \cdot \nabla T + \frac{1}{\gamma} \left(\boldsymbol{t}' \cdot \nabla T \right) \boldsymbol{t}' \cdot \mathrm{d}\boldsymbol{r} . \tag{33}$$

Using the definition of the geodetic height

$$t' \cdot \mathrm{d}\boldsymbol{r} = \mathrm{d}h , \tag{34}$$

and recognizing that

$$\mathrm{d}\boldsymbol{r} \cdot \nabla T = \mathrm{d}T , \tag{35}$$

the differential height increment reads

$$\mathrm{d}L = \mathrm{d}h - \frac{1}{\gamma} \mathrm{d}T + \frac{1}{\gamma} \left(\boldsymbol{t}' \cdot \nabla T \right) \mathrm{d}h . \tag{36}$$

Recognizing also that

$$\frac{1}{\gamma}\mathrm{d}T=\mathrm{d}\left(\frac{T}{\gamma}\right)+\frac{\gamma'}{\gamma^2}T\ \mathrm{d}h=\mathrm{d}\varsigma+\frac{\gamma'}{\gamma^2}T\ \mathrm{d}h\ , \tag{37}$$

the differential height increment in Eq. (36) can be written using the gravity anomaly Δg in spherical approximation

$$\mathrm{d}L=\mathrm{d}h-\mathrm{d}\varsigma-\frac{1}{\gamma}\left(\frac{\boldsymbol{t}'\cdot\boldsymbol{\gamma}}{\gamma}T-\boldsymbol{t}'\cdot\nabla T\right)\mathrm{d}h=\mathrm{d}H'-\frac{\Delta g}{\gamma}\mathrm{d}h\ . \tag{38}$$

Integrating the differential height difference along a levelling line between the points A and B yields

$$\int_A^B \mathrm{d}L=H'_B-H'_A-\int_A^B\frac{\Delta g}{\gamma}\mathrm{d}h\ , \tag{39}$$

which results in

$$\Delta H'_{AB}=H'_B-H'_A=\int_A^B \mathrm{d}L+\int_A^B\frac{\Delta g}{\gamma}\mathrm{d}h=\Delta L_{AB}+NC_{AB}\ , \tag{40}$$

with the so-called normal correction NC given by the second integral on the right-hand side. Thus, the normal correction to the levelled height differences can be computed using actual gravity data measured at the Earth's surface along the levelling line and normal gravity. No hypothesis on the topographic mass density is needed that is the main advantage of normal heights.

To derive the orthometric correction OC, one can use as a starting point

$$H'_P=H_P+N-\varsigma=H_P+\int_G^P\frac{\Delta g}{\gamma}\mathrm{d}h\ , \tag{41}$$

that results in

$$\Delta H_{AB}=H_B-H_A=\int_A^B \mathrm{d}L+\int_A^B\frac{\Delta g}{\gamma}\mathrm{d}h+\int_{A_G}^A\frac{\Delta g}{\gamma}\mathrm{d}h-\int_{B_G}^B\frac{\Delta g}{\gamma}\mathrm{d}h=\Delta H'_{AB}+OC_{AB}\ . \tag{42}$$

In this case, one needs to know values of gravity inside the topography along the plumblines (respective points A_G and B_G are at the geoid) passing through the points A and B. They cannot be estimated without adopting some topographic mass density model.

5.2. Satellite positioning

Let us now focus on estimation of physical heights from geodetic heights computed from geocentric Cartesian coordinates estimated by satellite positioning. The relationships between the geodetic height h, orthometric height H and normal height H' are as follows:

$$h_P \approx H_P + N \approx H'_P + \varsigma \, . \tag{43}$$

The approximation signs used in Eq. (43) reflect the fact that the geodetic height h is considered along the ellipsoidal normal n, while the physical heights H and H' are considered along the real plumbline t and normal plumbline t', respectively. However, extreme differences implied by their geometry are at the mm level even for very high mountains so that they can be neglected, see Section 2.3.

How can the two parameters N and ζ in Eq. (43) be then estimated? Let us start with the geoid height N defined in the real Earth's gravity field. It can be estimated by Bruns's formula (Eq. (15)). As values of normal gravity γ at the reference ellipsoid can be computed analytically, see Section 2.2, the main problem is to estimate the disturbing potential T at the geoid. This problem, that arises particularly over continents where the geoid is unknown and cannot be directly measured, can be solved using different methods and input data. The classical solution, published already by *Stokes (1849)*, is based on the potential theory and surface anomalous gravity continued down to the mean sea level. As new gravity data types have become available in due time, new solutions have been introduced recently. A review of integral transforms and integral equations for estimating the disturbing potential from measured values of its gradients of up to the third order can be found in *Novák et al. (2021)*. An interested reader can consult also other references focused on this topic, e.g., *Sansò and Sideris (2013)*. Note that one value of the geoid height can be used in transformation of Eq. (43) for all points at the particular plumbline, i.e., at the Earth's surface as well as below or above it.

In the case of the height anomaly ζ, the situation is similar, but not equal. Similar are methods and data that can be used for its estimation, different is where the computations take place (*Molodensky et al., 1960*). As in the case of the orthometric heights, one can write

$$U_P = U_{P'} + \left.\frac{\partial U}{\partial n}\right|_{P'} \varsigma + \left.\frac{\partial^2 U}{\partial n^2}\right|_{P'} \varsigma^2 + \mathcal{O}\left(\varsigma^3\right) . \tag{44}$$

Keeping again only the linear term and using the condition $U_{P'} = W_P$, i.e.,

$$U_P = W_P - \gamma_{P'} \varsigma \, , \tag{45}$$

yield again the Bruns's formula

$$\varsigma = \frac{T_P}{\gamma_{P'}} \, . \tag{46}$$

In this case, the disturbing potential must be computed at the point P the normal height of which is being estimated. If the point is located at the Earth's surface, then normal gravity refers to the telluroid. Normal gravity at the telluroid can be computed by iterations as the normal height is required. If points are located either below or above the Earth's surface, the disturbing potential must be estimated at the each point and normal gravity at its corresponding "telluroid".

Evaluating the value of the disturbing potential at the Earth's surface or in its close vicinity is then the main advantage of this approach. If surface gravity is used, then it is reduced (only) to the level defined by the height of the computation point, not all the way down to the geoid. However, there is a certain reduction involved which must be partially applied to gravity data above the computation level. On the other hand, irregularities of the Earth's surface cause problems to numerical solutions. A regularized Earth's surface must be used that implies a certain ambiguity associated with a particular surface model. Again, an interested reader can consult many available references.

6. INCORRECT USE OF NORMAL HEIGHTS

The height transformation of Eq. (43) may imply an incorrect use of the height anomaly and normal heights in general. The height anomaly, when referred to the reference ellipsoid, defines a surface called the *quasigeoid*. And the quaigeoid is then sometimes considered as the height reference surface for normal heights. Then the height H^M of the surface point is defined as the length of the respective ellipsoidal normal from the quasigeoid. These heights are sometimes referred to as Molodensky's heights as it was *Molodensky (1960)* who introduced them in geodesy.

The alternative heights H^M are often confused with the normal heights H' defined in Section 5.2. The reason is that for the point P at the Earth's surface, it holds

$$H'_P = H_P^M \ . \tag{47}$$

However, once the point P is located above or inside the Earth's surface, then the height transformation formula in Eq. (43) is not valid, i.e.,

$$h_P \neq H_P^M + \varsigma \ , \tag{48}$$

with ζ representing now the vertical separation of the quasigeoid with respect to the reference ellipsoid. For example, if the geodetic height h of a tall structure is estimated by GNSS, its respective normal height H' cannot be determined by this transformation. The same would apply if the normal height of the point P above the Earth's surface should be estimated. The reason is that ζ is used as the separation of the quasigeoid with respect to the reference ellipsoid. The correct transformation can be performed only through the height anomaly estimated at the observation point P. This fact complicates the general use of normal heights in practice. However, the Earth's gravity field attenuates with elevation. Decomposing the gravitational potential into the harmonic series, its signal degree variances are attenuated exponentially, i.e., the gravity field becomes smooth with elevation. Current high resolution global geopotential models approximate quite well the external Earth's gravity field; thus, they can be used for estimation of the disturbing potential at points outside the Earth's masses. Note that the same problem does not apply to orthometric heights as one value of the geoid height can be used for any point below, at or above the Earth's surface along the same plumbline.

The use of the quasigeoid as the height reference surface distinguishes the alternative heights H^M from other physical heights (including dynamic heights not discussed herein). The quasigeoid can be computed using surface gravity data by solving a geodetic boundary-value problem, thus avoiding some complications attributed to determination of the geoid (namely harmonization and downward continuation of surface gravity data). However, its definition results in several issues that effectively disqualify it from being a reasonable height reference surface. Thus, the entire concept of the alternative height system is compromised.

Problems of the quasigeoid have been recognized for some time; however, it was assumed that they did not affect significantly the respective height system. They were recently reported in several studies, e.g., *Kingdon at al. (2022)*; only main arguments are summarized herein. First of all, the quasigeoid is not an equipotential surface of any gravity field. Thus, it lacks the physical meaning of the geoid for the real Earth's gravity field (or the reference ellipsoid for the normal gravity field). Fluids can flow along the quasigeoid, a feature that contradicts the very definition of the height reference surface for physical heights. Second, the quasigeoid is not a smooth surface as it mirrors geometry of the Earth's surface. As the Earth's surface includes folds and sharp edges, the quasigeoid can be even a multi-valued function of angular coordinates. A regularized (smoothed) Earth's surface can be used; however, a certain ambiguity issue arises as to which Earth's surface model should be used to approximate reality.

In the following, we demonstrate that even with regularized topography the quasigeoid is a much rougher surface (measured by its horizontal inclination) than the geoid, especially over areas with rough topographies and complex gravity fields. To prove this proposition, the inclination of the geoid is compared with that of the quasigeoid, see Fig. 3. The height anomaly ζ changes along the slant distance d$\boldsymbol{l}$ as follows:

$$\mathrm{d}\varsigma = \nabla\varsigma \cdot \mathrm{d}\boldsymbol{l} \,. \tag{49}$$

The height anomaly defined at the surface point P can be represented by Bruns's formula

$$\varsigma_P = \frac{T_P}{\gamma_{P'}} \,, \tag{50}$$

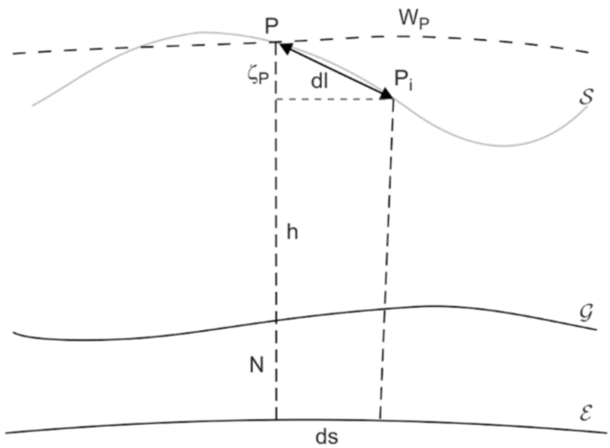

Fig. 3. Comparing "roughnesses" of the geoid and the quasigeoid – formulation of the inclination terms.

and the slant distance can be decomposed into two orthogonal components

$$\mathrm{d}\boldsymbol{l} = \boldsymbol{t}'\mathrm{d}h + \boldsymbol{e}\,\mathrm{d}s\,, \tag{51}$$

with the unit horizontal vector $\boldsymbol{e}$ and the horizontal shift ds. Then (all functions refer to the points as defined above)

$$\mathrm{d}\varsigma = \frac{1}{\gamma}\mathrm{d}T + T\,\mathrm{d}\frac{1}{\gamma}\,. \tag{52}$$

Using the spherical approximation and the deflection of the verticals ε

$$\mathrm{d}T = \nabla T \cdot \mathrm{d}\boldsymbol{l} = \frac{\partial T}{\partial r}\mathrm{d}h - \gamma\varepsilon\cdot\boldsymbol{e}\,\mathrm{d}s\,, \tag{53}$$

then results in

$$\mathrm{d}\varsigma = \frac{1}{\gamma}\left(\frac{\partial T}{\partial r} + \frac{2}{r}T\right)dh - \varepsilon\cdot\boldsymbol{e}\,ds\,. \tag{54}$$

Note that the term $\varepsilon\cdot\boldsymbol{e}$ represents the deflection of the verticals projected into the direction $\boldsymbol{e}$. As for the inclination of the Earth's surface I_{top} in the direction $\boldsymbol{e}$, it reads

$$\tan I_{top} = \frac{\mathrm{d}h}{\mathrm{d}s}\,, \tag{55}$$

and Eq. (54) can be re-written using the gravity anomaly Δg as follows:

$$\mathrm{d}\varsigma = \left(\frac{\Delta g}{\gamma}\tan I_{top} - \varepsilon\cdot\boldsymbol{e}\right)\mathrm{d}s\,. \tag{56}$$

The inclination of the quasigeoid then reads

$$\tan I_{\varsigma} = \frac{\mathrm{d}\varsigma}{\mathrm{d}s} = -\frac{\Delta g}{\gamma}\tan I_{top} - \varepsilon\cdot\boldsymbol{e}\,. \tag{57}$$

The geoid height N is given by Bruns's formula

$$N = \frac{T_G}{\gamma_E}\,, \tag{58}$$

and its inclination in the direction of $\boldsymbol{e}$ reads

$$\tan I_N = \frac{\mathrm{d}N}{\mathrm{d}s} = -\frac{\Delta g}{\gamma}\tan I_N - \varepsilon\cdot\boldsymbol{e}\,. \tag{59}$$

Solving for tan I_N and retaining only the first-order terms yield

$$\tan I_N \approx -\varepsilon\cdot\boldsymbol{e}\,. \tag{60}$$

Comparing Eqs (57) and (60), it is obvious that the quasigeoid is much rougher than the geoid as it reflects the roughness of the Earth's surface represented by the term tan I_{top}. The difference is especially significant in mountainous areas where the term tan I_{top} attains large values (close to 1 or even larger). Tests performed over a 300×300 km^2 area in Northern

Italy and Switzerland quantify the differences. Figures 4 and 5 depict values of the deflection term $\varepsilon \cdot \boldsymbol{e}$ and the inclination term $\Delta g \, \gamma^{-1} \tan I_{top}$ in Eq. (57). Figures 6 and 7 then zoom the same parameters over a small $10 \times 10 \, \mathrm{km}^2$ region just in the Alps. In particular, values shown in Fig. 7 nicely represent the differences in roughness of the geoid and the quasigeoid. Moreover, the quasigeoid can fold as it reflects all features of the Earth's surface; thus, it may even attain more values over a specific ellipsoidal normal.

The practice of using the quasigeoid as the height reference surface for normal heights can be encountered in some countries, namely in Europe. Surveying offices often compute regional ("national or state") quasigeoid models to be used for transformation of geodetic heights into normal heights. Based on the arguments presented herein (and elsewhere), it would be advisable not to use this practice any further. If one wants to use the normal heights, then the height anomaly must be computed at each point of interest at, above or below the Earth's surface before it is applied in the height transformation formula (alternatively, one would have to clarify every time that the procedure gives correct results only for points at the Earth's surface).

7. CONCLUSIONS

In this contribution, heights and height systems used in geodesy are discussed. Any of the two physical height systems considered in this contribution is defined using a height reference surface represented by the geoid, upon which specific heights are equal to zero, and a height defined as a geodesic in a particular gravity field space.

If the real Earth's gravity field is used, then it is the orthometric height system based on the geoid as the height reference surface and the orthometric height defined as a length of the real plumbline between the point of interest at, below or above the Earth's surface, and the geoid. This physical height system has the main advantage of being related to the real Earth's gravity field; moreover, the geoid has a physical meaning as the equipotential surface coinciding with the mean sea level. However, the estimation of orthometric heights is affected by the necessity to hypothesise the topographic mass density.

To avoid the mass density issue, the normal height system was introduced in geodesy. It uses also the geoid as the height reference surface but the normal height is defined as a length of the normal plumbline between the reference ellipsoid and the point at the telluroid where the value of the normal gravity potential coincides with the value of the actual gravity potential at the point of interest. This applies to any point at, below or above the Earth's surface, i.e., the height anomaly must be estimated independently for every point. This also means that the practice of relating the height anomaly to the reference ellipsoid, that results in realization of the quasigeoid, is not correct for points below or above the Earth's surface.

But the quasigeoid should not be used as the height reference surface in combination with a certain distance in the vertical direction even for the surface points. They are at least two reasons why it is not suitable for taking the role as the height reference surface: 1) it is not a physically-meaningful surface which is not convenient for a height reference surface of a physical height system, and 2) it can be a very rough surface as it follows geometry of the Earth's surface that is complex and contains sharp edges and folds.

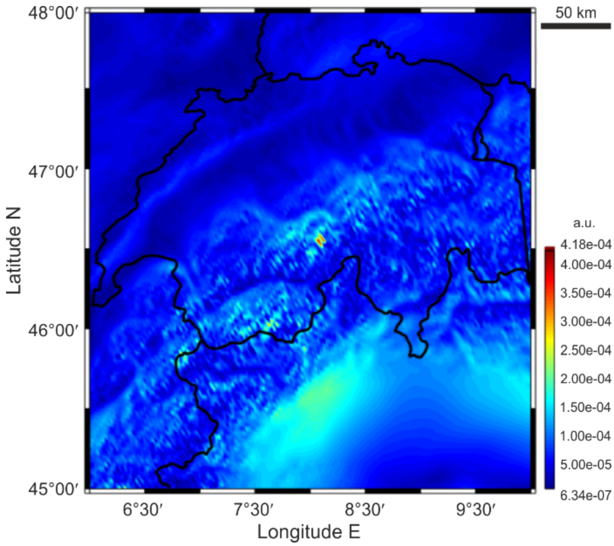

Fig. 4. The deflection of the verticals term $|\varepsilon|$ in Eq. (57) over Northern Italy and Switzerland (300×300 km^2).

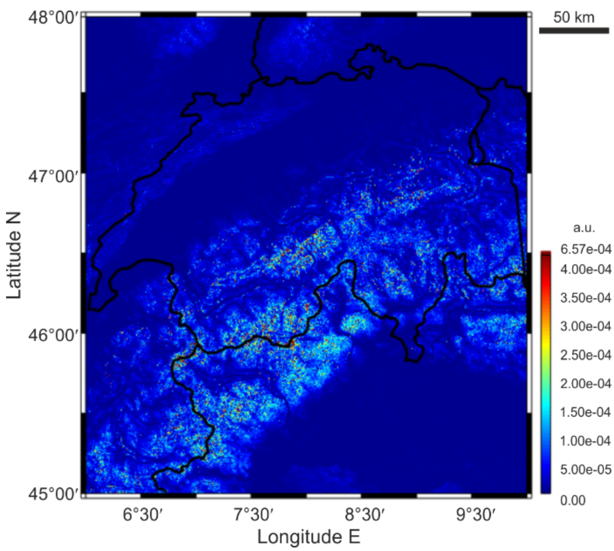

Fig. 5. The term $\Delta g\, \gamma^{-1} \tan I_{top}$ in Eq. (57) over Northern Italy and Switzerland (300×300 km^2).

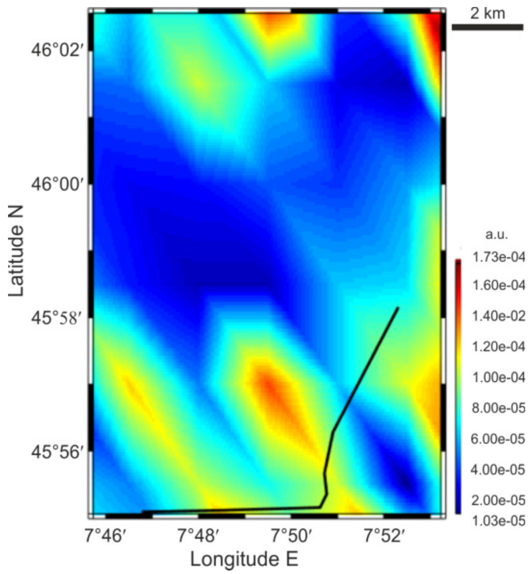

Fig. 6. The same as in Fig. 4, but for the Alps (10 × 10 km^2).

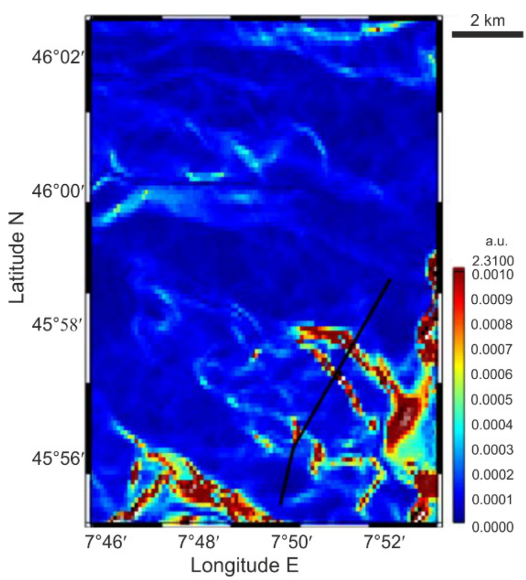

Fig. 7. The same as in Fig. 5, but for the Alps (10 × 10 km^2).

The conclusion on the quasigeoid was also proposed as a motion discussed at the 10-th Hotine-Marussi Symposium on Mathematical Geodesy held in Milan, June 2022. According to this motion, "the quasigeoid is not fit as a height reference surface". The quasigeoid must be considered only as "a concession to conventional conceptions that call for a geoid-like surface" (*Heiskanen and Moritz, 1967, p. 294*). Note that values of normal heights of surface points are not affected (in contrary to points below or above the Earth's surface).

APPENDIX A
DEFINITION OF THE SHORTEST DISTANCE IN THE PHYSICAL SPACE

Theorem: Let $g = |\boldsymbol{g}| = |\nabla W|$ be the modulus of gravity in a region including the point P and the plumbline t passing through the point P down to the geoid $\mathcal{G}$ crossing it at the point G, see Fig. A.1. Let l be any other line starting from the point P and crossing the geoid at the point G'; then

$$\int_{G}^{P} g(t)\,\mathrm{d}t \leq \int_{G'}^{P} g(l)\,\mathrm{d}l \,. \tag{A.1}$$

In other words, if we define a non-Euclidean distance between the point P and the geoid along a general line l, i.e.,

$$D_{G'}^{P} = \int_{G'}^{P} g(l)\,\mathrm{d}l \,, \tag{A.2}$$

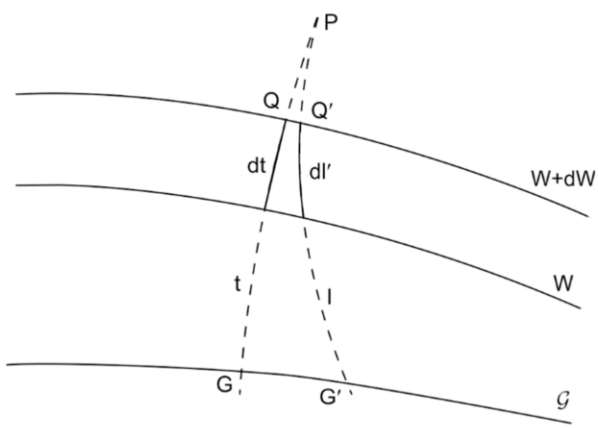

Fig. A.1. Definition of the shortest distance (height) of the point P from the geoid $\mathcal{G}$ in the physical space with the plumbline t and alternative length l.

then as a functional of l it is minimized when the line l becomes the plumbline t crossing the point P. Using other words, the plumbline t passing through the point P is the geodesic of the metric

$$ds^2 = g^2 dt^2 , \quad dt^2 = dx^2 + dy^2 + dz^2 . \tag{A.3}$$

Proof: Let d$\boldsymbol{t}$ and d$\boldsymbol{l}$ be the displacement vectors of two points Q and Q' along the lines t and l, respectively, such that

$$\left| d\boldsymbol{t} \right| = dt , \quad \left| d\boldsymbol{l} \right| = dl . \tag{A.4}$$

Moreover, let $\boldsymbol{t}_Q$ be the unit tangent vector of the plumbline t downward looking at the point Q and $\boldsymbol{l}_{Q'}$ be the unit tangent vector of the line l at the point Q'. Then one obviously gets

$$dW = \boldsymbol{g}_Q \cdot d\boldsymbol{t} = g_Q dt = \boldsymbol{g}_{Q'} \cdot d\boldsymbol{l} = g_{Q'} dl \cos\theta , \tag{A.5}$$

where θ is the angle between $\boldsymbol{l}_{Q'}$, i.e., the line l, and the direction of the gravity vector $\boldsymbol{g}_{Q'}$, i.e., the direction of the vertical at the point Q'. It follows that

$$\int_{G'}^{P} g(l)\, dl = \int_{G}^{P} \frac{g(t)}{\cos\theta}\, dt \geq \int_{G}^{P} g(t)\, dt . \tag{A.6}$$

Equality of Eq. (A.6) is satisfied only if $\theta = 0$, namely for $t_Q = l_{Q'}$. But if this is the case, since both lines t and l have the same origin in the point P, this implies $t \equiv l$.

Remark: This is also the elementary proof of Fermat's principle.

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