# Parameterization of anisotropic media by A-parameters

IVAN PŠENČÍK[1,2] AND VÉRONIQUE FARRA[3]

1 Institute of Geophysics, The Czech Academy of Sciences, Boční II/1401, 141 00 Praha 4, Czech Republic (ip@ig.cas.cz)

2 Charles University, Faculty of Mathematics and Physics, Department of Geophysics, Ke Karlovu 3, 121 16 Praha 2, Czech Republic

3 Université Paris Cité, Institut de Physique du Globe de Paris, UMR 7154 CNRS, Paris, France

*Received: July 23, 2023; Revised: October 17, 2023; Accepted: November 16, 2023*

## ABSTRACT

*Most common parameterization of anisotropic media is by twenty one independent elements $a_{ijkl}$ of the density-normalized stiffness tensor or by twenty one independent elements $A_{\alpha\beta}$ of the density-normalized matrix of elastic parameters in the Voigt notation. These parameters are commonly of significantly different sizes, are dimensional, in (km/s)$^2$, often appear in combinations. We are offering an alternative parameterization by twenty one A-parameters (anisotropic parameters), which removes the mentioned disadvantages and possesses some additional useful properties. For example, axes or planes of coordinate systems, in which A-parameters are defined, need not be related to symmetry axes or planes of the considered anisotropy symmetry as required in other similar parameterizations. In combination with the first-order weak-anisotropy approximation, in which anisotropy is considered as the first-order perturbation of reference isotropy, parameterization by A-parameters yields insight into the role of individual A-parameters in the wave propagation problems. For example, it turns out that in the first-order weak-anisotropy approximation, P- and S-wave velocities are each controlled by fifteen A-parameters. A set of six of them appears only in the expression for P-wave velocity, a set of other six A-parameters appears only in S-waves velocity expressions. Remaining set of nine A-parameters is common for both waves. We present transformation of A-parameters, analogue to Bond transformation, and useful formulae for the weak-anisotropy approximation for anisotropy of any symmetry and arbitrary tilt.*

Keywords: A-parameters, anisotropy, weak-anisotropy approximation

## 1. INTRODUCTION

The set of twenty one A-parameters represents an alternative to twenty one independent elements $a_{ijkl}$ of the density-normalized stiffness tensor or twenty one independent density-normalized elastic parameters $A_{\alpha\beta}$ in the Voigt notation, see, for example, *Fedorov (1968)*, *Červený (2001)*, *Chapman (2004)*, *Tsvankin and Grechka (2011)*, *Carcione (2014)*. In media of higher-anisotropy symmetry, the number of independent A-parameters reduces similarly as the number of independent $a_{ijkl}$ or $A_{\alpha\beta}$ elements. A-parameters can be used for the specification of anisotropy of any type and strength. The parameterization by A-parameters is a generalization of *Thomsen (1986)* parameterization proposed for transversely isotropic media with vertical axis of symmetry (VTI symmetry). A-parameters are closely related to weak-anisotropy (WA) parameters, whose detailed description can be found in *Farra et al. (2016)*. WA parameters are, in turn, slightly modified parameters introduced for P waves by *Sayers (1994)*, *Tsvankin (1997)* and *Pšenčík and Gajewski (1998)*, and generalized for all waves by *Mensch and Rasolofosaon (1997)* or *Farra and Pšenčík (2003)*.

A-parameters have a series of useful properties. A-parameters are related linearly to density-normalized elastic parameters $A_{\alpha\beta}$. This makes the transformation of A-parameters to elastic parameters and back an elementary task. Note that this is difficult if not impossible with Thomsen-style (*Thomsen, 1986*) parameters, in which relations of some of Thomsen's parameters to elastic parameters are non-linear. In contrast to elements of the stiffness tensor, to elastic parameters or to some of Thomsen's parameters, A-parameters are non-dimensional. Their size can be controlled by the choice of P- and S-wave velocities $\alpha$ and $\beta$ of the reference isotropic medium. By an appropriate choice of the reference velocities, it is possible to make all A-parameters of a comparable size. This property is especially useful in solving inverse problems, in which A-parameters are the sought unknowns. In contrast to Thomsen-style (*Thomsen, 1986*) or Tsvankin-style (*Tsvankin, 1997*) parameters, which are defined in coordinate systems whose coordinate axes or planes coincide with symmetry axes or planes of the corresponding anisotropy symmetry, for A-parameters such a coincidence is not required. A-parameters are especially useful when used in combination with the weak-anisotropy approximation, the approximation, in which anisotropy is considered to be a perturbation of isotropy. Use of A-parameters within the weak-anisotropy approximation leads to simple, easily manageable and transparent formulae for phase (or ray) velocities and polarization vectors in any type of anisotropy symmetry. In the first-order (in terms of deviations of anisotropy from isotropy) approximation, *Farra and Pšenčík (2003)* found that velocities and polarization vectors can be simply expressed in terms of elements of the Christoffel matrix specified in the coordinate system related to a specific direction (direction of wave propagation). Below we show that in these elements, A-parameters represent coefficients of directional cosines specifying this specific direction. Use of A-parameters in studies of tilted higher-symmetry anisotropies is easy because of the simplicity of the transformation of A-parameters from one coordinate system to another. Note that this transformation is an analogue to Bond transformation (*Bond, 1943*; *Chapman, 2004*).

We mostly use two right-handed Cartesian coordinate systems. One is the *global* coordinate system $z_i$. Its $z_3$ axis is vertical, positive down, axes $z_1$ and $z_2$ are situated in a horizontal plane. The other coordinate system, the *crystal* coordinate system, is the system, in which A-parameters are specified, and from which they are to be transformed to A-parameters specified in the global coordinate system. For higher-anisotropy symmetry, such as transverse isotropy (TI) or orthorhombic (OR) symmetry, the crystal coordinate system is chosen so that its coordinate axes and planes coincide with symmetry axes and planes of the considered anisotropic medium.

The structure of the paper is as follows. A short Section 2 introducing A-parameters is followed by Section 3, in which expressions for the elements of rotated Christoffel matrix, which plays a basic role in the first-order weak-anisotropy approximation, are expressed in terms of A-parameters and directional cosines. Section 4 contains formulae for the transformation of A-parameters of three important anisotropy symmetries, triclinic, orthorhombic and transversely isotropic, from the crystal to the global coordinate system. It also contains transformation formulae for the case that P- and S-wave velocities $\alpha$ and $\beta$ of the reference isotropic medium are changed. Short discussion and conclusions can be found in Section 5. The paper ends with Appendix presenting the rotation matrix used for the transformation of A-parameters from the crystal to global coordinate system. The matrix is expressed in terms of Euler angles.

## 2. DEFINITION OF A-PARAMETERS

All twenty one A-parameters can be expressed in terms of the twenty one independent density-normalized elastic parameters $A_{\alpha\beta}$ and the velocities $\alpha$ and $\beta$ of the reference isotropic medium in the following way:

$$\varepsilon_x = \frac{A_{11}-\alpha^2}{2\alpha^2}, \quad \varepsilon_y = \frac{A_{22}-\alpha^2}{2\alpha^2}, \quad \varepsilon_z = \frac{A_{33}-\alpha^2}{2\alpha^2},$$

$$\chi_x = \frac{A_{14}+2A_{56}}{\alpha^2}, \quad \chi_y = \frac{A_{25}+2A_{46}}{\alpha^2}, \quad \chi_z = \frac{A_{36}+2A_{45}}{\alpha^2},$$

$$\eta_x = \frac{2(A_{23}+2A_{44})-A_{22}-A_{33}}{2\alpha^2}, \quad \eta_y = \frac{2(A_{13}+2A_{55})-A_{33}-A_{11}}{2\alpha^2},$$

$$\eta_z = \frac{2(A_{12}+2A_{66})-A_{11}-A_{22}}{2\alpha^2},$$

$$\xi_{24} = \frac{A_{14}+2A_{56}-A_{24}}{\alpha^2}, \quad \xi_{34} = \frac{A_{14}+2A_{56}-A_{34}}{\alpha^2}, \quad \xi_{15} = \frac{A_{25}+2A_{46}-A_{15}}{\alpha^2}, \tag{1}$$

$$\xi_{35} = \frac{A_{25}+2A_{46}-A_{35}}{\alpha^2}, \quad \xi_{16} = \frac{A_{36}+2A_{45}-A_{16}}{\alpha^2}, \quad \xi_{26} = \frac{A_{36}+2A_{45}-A_{26}}{\alpha^2},$$

$$\gamma_x = \frac{A_{44}-\beta^2}{2\beta^2}, \quad \gamma_y = \frac{A_{55}-\beta^2}{2\beta^2}, \quad \gamma_z = \frac{A_{66}-\beta^2}{2\beta^2},$$

$$\varepsilon_{45} = \frac{A_{45}}{\beta^2}, \quad \varepsilon_{46} = \frac{A_{46}}{\beta^2}, \quad \varepsilon_{56} = \frac{A_{56}}{\beta^2}.$$

Because the relations between A-parameters and density-normalized elastic parameters $A_{\alpha\beta}$ are linear, it is easy to express the parameters $A_{\alpha\beta}$ in terms of A-parameters:

$$\begin{gathered}
A_{11} = \alpha^2 \left(1+2\varepsilon_x\right), \quad A_{12} = \alpha^2 \left(1+\varepsilon_x+\varepsilon_y+\eta_z\right) - 2\beta^2\left(1+2\gamma_z\right), \\
A_{13} = \alpha^2 \left(1+\varepsilon_x+\varepsilon_z+\eta_y\right) - 2\beta^2\left(1+2\gamma_y\right), \quad A_{14} = \alpha^2 \chi_x - 2\beta^2 \varepsilon_{56}, \\
A_{15} = \alpha^2 \left(\chi_y - \xi_{15}\right), \quad A_{16} = \alpha^2 \left(\chi_z - \xi_{16}\right), \quad A_{22} = \alpha^2 \left(1+2\varepsilon_y\right), \\
A_{23} = \alpha^2 \left(1+\varepsilon_y+\varepsilon_z+\eta_x\right) - 2\beta^2\left(1+2\gamma_x\right), \quad A_{24} = \alpha^2 \left(\chi_x - \xi_{24}\right), \\
A_{25} = \alpha^2 \chi_y - 2\beta^2 \varepsilon_{46}, \quad A_{26} = \alpha^2 \left(\chi_z - \xi_{26}\right), \quad A_{33} = \alpha^2 \left(1+2\varepsilon_z\right), \\
A_{34} = \alpha^2 \left(\chi_x - \xi_{34}\right), \quad A_{35} = \alpha^2 \left(\chi_y - \xi_{35}\right), \quad A_{36} = \alpha^2 \chi_z - 2\beta^2 \varepsilon_{45}, \\
A_{44} = \beta^2 \left(1+2\gamma_x\right), \quad A_{45} = \beta^2 \varepsilon_{45}, \quad A_{46} = \beta^2 \varepsilon_{46}, \\
A_{55} = \beta^2 \left(1+2\gamma_y\right), \quad A_{56} = \beta^2 \varepsilon_{56}, \quad A_{66} = \beta^2 \left(1+2\gamma_z\right).
\end{gathered} \tag{2}$$

## 3. FIRST-ORDER WEAK-ANISOTROPY APPROXIMATION

Weak-anisotropy approximation is a perturbation technique allowing a simple and sufficiently accurate approximation of various attributes of waves propagating in anisotropic media. These attributes may be, for example, phase velocities or polarizations. Basic role in the weak-anisotropy approximation is played by elements $B_{ij}$ of the Christoffel matrix **B** specified in the Cartesian coordinate system defined by three unit mutually perpendicular vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{n}$. The vector $\boldsymbol{n}$ is oriented along a direction, along which we wish to make an approximate estimate of the wave characteristics, remaining two vectors are perpendicular to it. *Farra and Pšenčík (2003)* showed that elements $B_{13}$ and $B_{23}$ of the matrix **B** play mostly a role in the construction of the first-order approximations of polarization vectors, and the elements $B_{11}$, $B_{12}$, $B_{22}$ and $B_{33}$ appear in the expressions for phase velocities.

The first-order approximation of the square of P-wave phase velocity, $\widetilde{v_P^2}(\boldsymbol{n})$ (tilde indicates approximation), in the direction $\boldsymbol{n}$ is given by the equation:

$$\widetilde{v_P^2}(\boldsymbol{n}) = B_{33}(\boldsymbol{n}), \tag{3}$$

see *Farra and Pšenčík (2016)*.

If we use the unit vector $\boldsymbol{n}$ to specify the direction, in which we seek approximate expressions for $S_1$- and $S_2$-wave phase velocities, we get slightly more complicated expressions for the first-order approximations of their squares:

$$\widetilde{v_{S_1}^2}(\boldsymbol{n}) = \frac{1}{2}\left[B_{11}(\boldsymbol{n}) + B_{22}(\boldsymbol{n}) + \sqrt{\left(B_{11}(\boldsymbol{n}) - B_{22}(\boldsymbol{n})\right)^2 + 4B_{12}^2(\boldsymbol{n})}\right],$$
$$\widetilde{v_{S_2}^2}(\boldsymbol{n}) = \frac{1}{2}\left[B_{11}(\boldsymbol{n}) + B_{22}(\boldsymbol{n}) - \sqrt{\left(B_{11}(\boldsymbol{n}) - B_{22}(\boldsymbol{n})\right)^2 + 4B_{12}^2(\boldsymbol{n})}\right], \tag{4}$$

see *Farra and Pšenčík (2008)*.

Very important concept in the coupling ray theory is the concept of the common S wave (*Bakker, 2002*; *Klimeš, 2006*). The first-order approximation of the square of the phase velocity of the common S wave is given as an average of squares of $S_1$- and $S_2$-wave phase velocities. It means that the first-order approximation of the square of the common S-wave phase velocity is given by the equation:

$$\widetilde{v_S^2}(\boldsymbol{n}) = \frac{1}{2}\left(B_{11}(\boldsymbol{n}) + B_{22}(\boldsymbol{n})\right), \tag{5}$$

see *Farra and Pšenčík (2008)*.

First-order expressions for the polarization vectors can be found in *Farra and Pšenčík (2003)*. The first-order polarization vector $\tilde{\boldsymbol{g}}_P(\boldsymbol{n})$ of the P wave propagating in the direction specified by the vector $\boldsymbol{n}$ has the form:

$$\tilde{\boldsymbol{g}}_P(\boldsymbol{n}) = \boldsymbol{n} + \left(\alpha^2 - \beta^2\right)^{-1}\left(B_{13}(\boldsymbol{n})\boldsymbol{e}_1 + B_{23}(\boldsymbol{n})\boldsymbol{e}_2\right). \tag{6}$$

Here $\alpha$ and $\beta$ are the reference velocities in the reference isotropic medium, the same as used in Eqs (1) and (2). Vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are the above-introduced two unit, mutually perpendicular vectors, arbitrarily chosen in the plane perpendicular to the vector $\boldsymbol{n}$.

The determination of the first-order polarization vectors of separate S waves is more involved, see *Farra and Pšenčík (2003)*. As in Eq. (6), principal role in the expressions for polarization vectors of separate S waves is played again by the elements $B_{13}$ and $B_{23}$ of the matrix **B**. But these elements must belong to the matrix **B** rotated so that $B_{12} = 0$ and $B_{11} > B_{22}$, see *Farra and Pšenčík (2003)*.

In case of the common S wave, its first-order polarization plane is specified by vectors $\boldsymbol{f}_1(\boldsymbol{n})$ and $\boldsymbol{f}_2(\boldsymbol{n})$ (*Farra and Pšenčík, 2008*):

$$\boldsymbol{f}_K(\boldsymbol{n}) = \boldsymbol{e}_K - \left(\alpha^2 - \beta^2\right)^{-1} B_{K3}(\boldsymbol{n})\boldsymbol{n}, \tag{7}$$

where $K = 1, 2$.

Expressions for the elements of the matrix **B** can be found in many publications. Mostly they are expressed in terms of the above-mentioned weak-anisotropy (WA) parameters (see detailed description of WA parameters in *Farra et al., 2016*). Selected elements of the matrix **B** expressed in terms of A-parameters can be found in *Farra and Pšenčík (2023)*. Here we present the complete set of elements of the matrix **B** expressed in terms of A-parameters:

$$
\begin{aligned}
B_{11}(\boldsymbol{n}) = \beta^2 \Big\{ & 1 + 2D^{-2}\left(\gamma_x n_2^2 + \gamma_y n_1^2 + \varepsilon_{45} n_1 n_2\right) - 2r^2 \Big[ \eta_x n_2^2 n_3^2 + \eta_y n_1^2 n_3^2 \\
& - \eta_z D^{-2} n_1^2 n_2^2 n_3^2 + \xi_{15} D^{-2}\left(1 - 2D^2\right) n_1^3 n_3 - \xi_{35}\left(1 - 2D^2\right) n_1 n_3 \\
& + \xi_{24} D^{-2}\left(1 - 2D^2\right) n_2^3 n_3 - \xi_{34}\left(1 - 2D^2\right) n_2 n_3 \\
& + 2\xi_{16} D^{-2} n_1^3 n_2 n_3^2 + 2\xi_{26} D^{-2} n_1 n_2^3 n_3^2 \Big] \Big\} , \\
B_{12}(\boldsymbol{n}) = \beta^2 D^{-2} \Big\{ & 2\gamma_x n_1 n_2 n_3 - 2\gamma_y n_1 n_2 n_3 + \varepsilon_{45} n_3\left(n_1^2 - n_2^2\right) + \varepsilon_{46} n_2 D^2 \\
& - \varepsilon_{56} n_1 D^2 + r^2 \Big[ \eta_y D^2 n_1 n_2 n_3 - \eta_x D^2 n_1 n_2 n_3 + \eta_z\left(n_1^2 - n_2^2\right) n_1 n_2 n_3 \\
& + \xi_{34} n_1 n_3^2 D^2 - \xi_{35} n_2 n_3^2 D^2 - \xi_{15} n_1^2 n_2\left(3D^2 - 2\right) + \xi_{24} n_1 n_2^2\left(3D^2 - 2\right) \\
& + \xi_{16} n_1^2 n_3\left(3n_2^2 - n_1^2\right) - \xi_{26} n_2^2 n_3\left(3n_1^2 - n_2^2\right) \Big] \Big\} , \\
B_{13}(\boldsymbol{n}) = \alpha^2 D^{-1} \Big[ & \varepsilon_x n_1^2 n_3 + \varepsilon_y n_2^2 n_3 - \varepsilon_z D^2 n_3 + \chi_x n_2\left(1 - 2D^2\right) + \chi_y n_1\left(1 - 2D^2\right) \\
& + 2\chi_z n_1 n_2 n_3 + \eta_x n_2^2 n_3\left(1 - 2D^2\right) + \eta_y n_1^2 n_3\left(1 - 2D^2\right) + 2\eta_z n_1^2 n_2^2 n_3 - 4\xi_{16} n_1^3 n_2 n_3 \\
& - 4\xi_{26} n_1 n_2^3 n_3 + \xi_{15} n_1^3\left(D^2 - 3n_3^2\right) + \xi_{24} n_2^3\left(D^2 - 3n_3^2\right) \\
& + \xi_{34} n_2 n_3^2\left(3D^2 - n_3^2\right) + \xi_{35} n_1 n_3^2\left(3D^2 - n_3^2\right) \Big] , \\
B_{22}(\boldsymbol{n}) = \beta^2 \Big\{ & 1 + 2D^{-2}\Big(\gamma_x n_1^2 n_3^2 + \gamma_y n_2^2 n_3^2 + \gamma_z D^4 - \varepsilon_{45} n_1 n_2 n_3^2 \\
& + \varepsilon_{46} D^2 n_1 n_3 + \varepsilon_{56} D^2 n_2 n_3\Big) - 2r^2 D^{-2} \Big[ \eta_z n_1^2 n_2^2 + \xi_{24} n_1^2 n_2 n_3 + \xi_{15} n_1 n_2^2 n_3 \\
& - \xi_{16} n_1 n_2\left(n_1^2 - n_2^2\right) + \xi_{26} n_1 n_2\left(n_1^2 - n_2^2\right) \Big] \Big\} , \\
B_{23}(\boldsymbol{n}) = \alpha^2 D^{-1} \Big[ & -\varepsilon_x n_1 n_2 + \varepsilon_y n_1 n_2 + \chi_x n_1 n_3 - \chi_y n_2 n_3 + \chi_z\left(n_1^2 - n_2^2\right) \\
& + \eta_x n_1 n_2 n_3^2 - \eta_y n_1 n_2 n_3^2 + \eta_z n_1 n_2\left(n_1^2 - n_2^2\right) + \xi_{16} n_1^2\left(3n_2^2 - n_1^2\right) \\
& - \xi_{26} n_2^2\left(3n_1^2 - n_2^2\right) + 3\xi_{15} n_1^2 n_2 n_3 - 3\xi_{24} n_1 n_2^2 n_3 - \xi_{34} n_1 n_3^3 + \xi_{35} n_2 n_3^3 \Big] , \\
B_{33}(\boldsymbol{n}) = \alpha^2 \Big( & 1 + 2\varepsilon_x n_1^2 + 2\varepsilon_y n_2^2 + 2\varepsilon_z n_3^2 + 4\chi_x n_2 n_3 + 4\chi_y n_1 n_3 + 4\chi_z n_1 n_2 \\
& + 2\eta_x n_2^2 n_3^2 + 2\eta_y n_1^2 n_3^2 + 2\eta_z n_1^2 n_2^2 - 4\xi_{16} n_1^3 n_2 - 4\xi_{26} n_1 n_2^3 - 4\xi_{15} n_1^3 n_3 \\
& - 4\xi_{24} n_2^3 n_3 - 4\xi_{34} n_2 n_3^3 - 4\xi_{35} n_1 n_3^3 \Big) .
\end{aligned}
\tag{8}
$$

In Eq. (8),

$$r = \frac{\alpha}{\beta}, \quad \text{and} \quad D^2 = n_1^2 + n_2^2 . \tag{9}$$

We can see that A-parameters represent coefficients of directional cosines in the expressions for individual elements of the matrix **B**. This makes them an ideal parameterization for various types of inversions, for example traveltime inversions (*Pšenčík et al., 2018, 2020*) or local determination of anisotropy from the polarization information (*Zheng and Pšenčík, 2002*; *Gomes at al., 2004*).

Let us concentrate first on the elements of the matrix **B** related to phase velocities, i.e., on elements $B_{11}$, $B_{22}$, $B_{12}$ and $B_{33}$. We can see that the element $B_{33}$ related to P-wave phase velocity is controlled by the set of fifteen A-parameters. Elements $B_{11}$, $B_{22}$ and $B_{12}$ related to S-waves phase velocities are also controlled by the set of fifteen A-parameters. Each set, however, contains different parameters. Some A-parameters appear only in $B_{33}$ controlling P-wave phase velocity. Therefore,we call them *P-wave A-parameters*. They are six, specifically, $\varepsilon_x$, $\varepsilon_y$, $\varepsilon_z$, $\chi_x$, $\chi_y$ and $\chi_z$. Other A-parameters appear only in the elements related to S-waves phase velocities. Those we call *S-wave A-parameters*. They are again six, specifically, $\gamma_x$, $\gamma_y$, $\gamma_z$, $\varepsilon_{45}$, $\varepsilon_{46}$ and $\varepsilon_{56}$. Remaining nine A-parameters, which we call *common A-parameters*, appear in all mentioned elements of the matrix **B**. They are: $\eta_x$, $\eta_y$, $\eta_z$, $\xi_{15}$, $\xi_{16}$, $\xi_{24}$, $\xi_{26}$, $\xi_{34}$ and $\xi_{35}$.

As can be seen from Eqs (6) and (7), the first-order approximations of P-wave polarization vector and common S-wave polarization plane are controlled by the elements $B_{13}$ and $B_{23}$ of the matrix **B**. These elements depend on only P-wave and common A-parameters, not on S-wave A-parameters. Because of the involvement of elements $B_{12}$, $B_{11}$ and $B_{22}$ in their specification, polarization vectors of separate S1 and S2 waves depend on all twenty one A-parameters.

## 4. TRANSFORMATION RULES FOR A-PARAMETERS

Let us consider the above-introduced global and crystal coordinate systems. We are going to specify A-parameters in the crystal coordinate system and seek their expressions in the global coordinate system, with respect to which the crystal coordinate system is rotated. This transformation is equivalent to the Bond transformation (*Bond, 1943*; *Chapman, 2004*), which transforms elastic parameters from one coordinate system to another. The transformation from a crystal coordinate system to the global coordinate system is performed with the use of an orthogonal rotation matrix **R**. For an example of such a matrix expressed in terms of Euler angles, see Appendix A.

In the following, we present three sets of equations for the transformation of A-parameters from the crystal coordinate system to the global one. Specifically, we transform A-parameters specifying triclinic, orthorhombic and transversely isotropic symmetries in the crystal coordinate systems arbitrarily rotated with respect to the global coordinate system to the A-parameters specified in the global coordinate system. We use upper indices TR, OR or TI for A-parameters specified in the crystal coordinate system to indicate that they specify triclinic (TR), orthorhombic (OR) or transverse isotropy (TI) symmetries. Resulting A-parameters in the global coordinate system are left without indices. In the case of orthorhombic symmetry, we consider the symmetry planes to be parallel with coordinate planes of the crystal coordinate system. In the case of transverse isotropy, we assume that the axis of symmetry is parallel to the third axis of the crystal coordinate system.

In the end of the section, we present transformation rules for A-parameters for the case when the reference velocities $\alpha$ and $\beta$ are changed.

### 4.1. Triclinic symmetry

Twenty one A-parameters of a triclinic symmetry specified in the crystal coordinate system transform into twenty one A-parameters in the global coordinate system through the equations given below:

$$\begin{aligned}\varepsilon_x = {} & \varepsilon_x^{TR} R_{11}^2 + \varepsilon_y^{TR} R_{12}^2 + \varepsilon_z^{TR} R_{13}^2 + 2\chi_x^{TR} R_{12} R_{13} + 2\chi_y^{TR} R_{11} R_{13} + 2\chi_z^{TR} R_{11} R_{12} \\ & + \eta_x^{TR} R_{12}^2 R_{13}^2 + \eta_y^{TR} R_{11}^2 R_{13}^2 + \eta_z^{TR} R_{11}^2 R_{12}^2 \\ & - 2\xi_{15}^{TR} R_{11}^3 R_{13} - 2\xi_{16}^{TR} R_{11}^3 R_{12} - 2\xi_{24}^{TR} R_{12}^3 R_{13} \\ & - 2\xi_{26}^{TR} R_{12}^3 R_{11} - 2\xi_{34}^{TR} R_{13}^3 R_{12} - 2\xi_{35}^{TR} R_{13}^3 R_{11} \,,\end{aligned} \tag{10a}$$

$$\begin{aligned}\varepsilon_y = {} & \varepsilon_x^{TR} R_{21}^2 + \varepsilon_y^{TR} R_{22}^2 + \varepsilon_z^{TR} R_{23}^2 + 2\chi_x^{TR} R_{22} R_{23} + 2\chi_y^{TR} R_{21} R_{23} + 2\chi_z^{TR} R_{21} R_{22} \\ & + \eta_x^{TR} R_{22}^2 R_{23}^2 + \eta_y^{TR} R_{21}^2 R_{23}^2 + \eta_z^{TR} R_{21}^2 R_{22}^2 \\ & - 2\xi_{15}^{TR} R_{21}^3 R_{23} - 2\xi_{16}^{TR} R_{21}^3 R_{22} - 2\xi_{24}^{TR} R_{22}^3 R_{23} \\ & - 2\xi_{26}^{TR} R_{22}^3 R_{21} - 2\xi_{34}^{TR} R_{23}^3 R_{22} - 2\xi_{35}^{TR} R_{23}^3 R_{21} \,,\end{aligned} \tag{10b}$$

$$\begin{aligned}\varepsilon_z = {} & \varepsilon_x^{TR} R_{31}^2 + \varepsilon_y^{TR} R_{32}^2 + \varepsilon_z^{TR} R_{33}^2 + 2\chi_x^{TR} R_{32} R_{33} + 2\chi_y^{TR} R_{31} R_{33} + 2\chi_z^{TR} R_{31} R_{32} \\ & + \eta_x^{TR} R_{32}^2 R_{33}^2 + \eta_y^{TR} R_{31}^2 R_{33}^2 + \eta_z^{TR} R_{31}^2 R_{32}^2 \\ & - 2\xi_{15}^{TR} R_{31}^3 R_{33} - 2\xi_{16}^{TR} R_{31}^3 R_{32} - 2\xi_{24}^{TR} R_{32}^3 R_{33} \\ & - 2\xi_{26}^{TR} R_{32}^3 R_{31} - 2\xi_{34}^{TR} R_{33}^3 R_{32} - 2\xi_{35}^{TR} R_{33}^3 R_{31} \,,\end{aligned} \tag{10c}$$

$$\begin{aligned}\chi_x = {} & \varepsilon_x^{TR} R_{21} R_{31} + \varepsilon_y^{TR} R_{22} R_{32} + \varepsilon_z^{TR} R_{23} R_{33} + \chi_x^{TR} D_{11} + \chi_y^{TR} D_{12} + \chi_z^{TR} D_{13} \\ & + \eta_x^{TR} \left( D_{21} D_{31} + R_{12} R_{13} D_{11} \right) + \eta_y^{TR} \left( D_{32} D_{22} + R_{11} R_{13} D_{12} \right) \\ & + \eta_z^{TR} \left( D_{33} D_{23} + R_{11} R_{12} D_{13} \right) - 3\xi_{15}^{TR} R_{11} \left( R_{31} D_{32} + R_{21} D_{22} \right) \\ & - 3\xi_{16}^{TR} R_{11} \left( R_{31} D_{33} + R_{21} D_{23} \right) - 3\xi_{24}^{TR} R_{12} \left( R_{32} D_{31} + R_{22} D_{21} \right) \\ & - 3\xi_{26}^{TR} R_{12} \left( R_{32} D_{33} + R_{22} D_{23} \right) - 3\xi_{34}^{TR} R_{13} \left( R_{33} D_{31} + R_{23} D_{21} \right) \\ & - 3\xi_{35}^{TR} R_{13} \left( R_{33} D_{32} + R_{23} D_{22} \right) ,\end{aligned} \tag{10d}$$

$$\begin{aligned}\chi_y = {} & \varepsilon_x^{TR} R_{11} R_{31} + \varepsilon_y^{TR} R_{12} R_{32} + \varepsilon_z^{TR} R_{13} R_{33} + \chi_x^{TR} D_{21} + \chi_y^{TR} D_{22} + \chi_z^{TR} D_{23} \\ & + \eta_x^{TR} \left( D_{11} D_{31} + R_{22} R_{23} D_{21} \right) + \eta_y^{TR} \left( D_{12} D_{32} + R_{21} R_{23} D_{22} \right) \\ & + \eta_z^{TR} \left( D_{13} D_{33} + R_{21} R_{22} D_{23} \right) - 3\xi_{15}^{TR} R_{21} \left( R_{11} D_{12} + R_{31} D_{32} \right) \\ & - 3\xi_{16}^{TR} R_{21} \left( R_{11} D_{13} + R_{31} D_{33} \right) - 3\xi_{24}^{TR} R_{22} \left( R_{12} D_{11} + R_{32} D_{31} \right) \\ & - 3\xi_{26}^{TR} R_{22} \left( R_{12} D_{13} + R_{32} D_{33} \right) - 3\xi_{34}^{TR} R_{23} \left( R_{13} D_{11} + R_{33} D_{31} \right) \\ & - 3\xi_{35}^{TR} R_{23} \left( R_{13} D_{12} + R_{33} D_{32} \right),\end{aligned} \tag{10e}$$

$$\begin{aligned}\chi_z = {} & \varepsilon_x^{TR} R_{11} R_{21} + \varepsilon_y^{TR} R_{12} R_{22} + \varepsilon_z^{TR} R_{13} R_{23} + \chi_x^{TR} D_{31} + \chi_y^{TR} D_{32} + \chi_z^{TR} D_{33} \\ & + \eta_x^{TR} \left( D_{11} D_{21} + R_{32} R_{33} D_{31} \right) + \eta_y^{TR} \left( D_{12} D_{22} + R_{31} R_{33} D_{32} \right) \\ & + \eta_z^{TR} \left( D_{13} D_{23} + R_{31} R_{32} D_{33} \right) - 3\xi_{15}^{TR} R_{31} \left( R_{11} D_{12} + R_{21} D_{22} \right) \\ & - 3\xi_{16}^{TR} R_{31} \left( R_{11} D_{13} + R_{21} D_{23} \right) - 3\xi_{24}^{TR} R_{32} \left( R_{12} D_{11} + R_{22} D_{21} \right) \\ & - 3\xi_{26}^{TR} R_{32} \left( R_{12} D_{13} + R_{22} D_{23} \right) - 3\xi_{34}^{TR} R_{33} \left( R_{13} D_{11} + R_{23} D_{21} \right) \\ & - 3\xi_{35}^{TR} R_{33} \left( R_{13} D_{12} + R_{23} D_{22} \right),\end{aligned} \tag{10f}$$

$$\begin{aligned}\eta_x = {} & \eta_x^{TR} \left[ \left( 1 - R_{12}^2 \right) \left( 1 - R_{13}^2 \right) - 2 \left( R_{32} R_{33} - R_{22} R_{23} \right)^2 \right] \\ & + \eta_y^{TR} \left[ \left( 1 - R_{13}^2 \right) \left( 1 - R_{11}^2 \right) - 2 \left( R_{33} R_{31} - R_{23} R_{21} \right)^2 \right] \\ & + \eta_z^{TR} \left[ \left( 1 - R_{11}^2 \right) \left( 1 - R_{12}^2 \right) - 2 \left( R_{31} R_{32} - R_{21} R_{22} \right)^2 \right] \\ & + 2\xi_{15}^{TR} \left( R_{21}^3 R_{23} + R_{31}^3 R_{33} - 3 R_{21} R_{31} D_{12} \right) + 2\xi_{16}^{TR} \left( R_{21}^3 R_{22} + R_{31}^3 R_{32} - 3 R_{21} R_{31} D_{13} \right) \\ & + 2\xi_{24}^{TR} \left( R_{22}^3 R_{23} + R_{32}^3 R_{33} - 3 R_{22} R_{32} D_{11} \right) + 2\xi_{26}^{TR} \left( R_{22}^3 R_{21} + R_{32}^3 R_{31} - 3 R_{22} R_{32} D_{13} \right) \\ & + 2\xi_{34}^{TR} \left( R_{23}^3 R_{22} + R_{33}^3 R_{32} - 3 R_{23} R_{33} D_{11} \right) + 2\xi_{35}^{TR} \left( R_{23}^3 R_{21} + R_{33}^3 R_{31} - 3 R_{23} R_{33} D_{12} \right),\end{aligned} \tag{10g}$$

$$\begin{aligned}\eta_y = {} & \eta_x^{TR} \left[ \left( 1 - R_{22}^2 \right) \left( 1 - R_{23}^2 \right) - 2 \left( R_{32} R_{33} - R_{12} R_{13} \right)^2 \right] \\ & + \eta_y^{TR} \left[ \left( 1 - R_{23}^2 \right) \left( 1 - R_{21}^2 \right) - 2 \left( R_{33} R_{31} - R_{13} R_{11} \right)^2 \right] \\ & + \eta_z^{TR} \left[ \left( 1 - R_{21}^2 \right) \left( 1 - R_{22}^2 \right) - 2 \left( R_{31} R_{32} - R_{11} R_{12} \right)^2 \right] \\ & + 2\xi_{15}^{TR} \left( R_{11}^3 R_{13} + R_{31}^3 R_{33} - 3 R_{11} R_{31} D_{22} \right) + 2\xi_{16}^{TR} \left( R_{11}^3 R_{12} + R_{31}^3 R_{32} - 3 R_{11} R_{31} D_{23} \right) \\ & + 2\xi_{24}^{TR} \left( R_{12}^3 R_{13} + R_{32}^3 R_{33} - 3 R_{12} R_{32} D_{21} \right) + 2\xi_{26}^{TR} \left( R_{12}^3 R_{11} + R_{32}^3 R_{31} - 3 R_{12} R_{32} D_{23} \right) \\ & + 2\xi_{34}^{TR} \left( R_{13}^3 R_{12} + R_{33}^3 R_{32} - 3 R_{13} R_{33} D_{21} \right) + 2\xi_{35}^{TR} \left( R_{13}^3 R_{11} + R_{33}^3 R_{31} - 3 R_{13} R_{33} D_{22} \right),\end{aligned} \tag{10h}$$

$$\begin{aligned}
\eta_z &= \eta_x^{TR}\left[\left(1-R_{32}^2\right)\left(1-R_{33}^2\right)-2\left(R_{22}R_{23}-R_{12}R_{13}\right)^2\right] \\
&+\eta_y^{TR}\left[\left(1-R_{33}^2\right)\left(1-R_{31}^2\right)-2\left(R_{23}R_{21}-R_{13}R_{11}\right)^2\right] \\
&+\eta_z^{TR}\left[\left(1-R_{31}^2\right)\left(1-R_{32}^2\right)-2\left(R_{21}R_{22}-R_{11}R_{12}\right)^2\right] \\
&+2\xi_{15}^{TR}\left(R_{11}^3R_{13}+R_{21}^3R_{23}-3R_{11}R_{21}D_{32}\right)+2\xi_{16}^{TR}\left(R_{11}^3R_{12}+R_{21}^3R_{22}-3R_{11}R_{21}D_{33}\right) \\
&+2\xi_{24}^{TR}\left(R_{12}^3R_{13}+R_{22}^3R_{23}-3R_{12}R_{22}D_{31}\right)+2\xi_{26}^{TR}\left(R_{12}^3R_{11}+R_{22}^3R_{21}-3R_{12}R_{22}D_{33}\right) \\
&+2\xi_{34}^{TR}\left(R_{13}^3R_{12}+R_{23}^3R_{22}-3R_{13}R_{23}D_{31}\right)+2\xi_{35}^{TR}\left(R_{13}^3R_{11}+R_{23}^3R_{21}-3R_{13}R_{23}D_{32}\right),
\end{aligned} \tag{10i}$$

$$\begin{aligned}
\xi_{15} &= \eta_x^{TR}\left[D_{11}D_{31}+D_{21}\left(R_{22}R_{23}-R_{12}R_{13}\right)\right] \\
&+\eta_y^{TR}\left[D_{12}D_{32}+D_{22}\left(R_{23}R_{21}-R_{13}R_{11}\right)\right]+\eta_z^{TR}\left[D_{13}D_{33}+D_{23}\left(R_{21}R_{22}-R_{11}R_{12}\right)\right] \\
&+\xi_{15}^{TR}\left[\left(R_{11}^2-3R_{21}^2\right)D_{22}+2R_{11}R_{31}\left(R_{13}R_{11}-3R_{23}R_{21}\right)\right] \\
&+\xi_{16}^{TR}\left[\left(R_{11}^2-3R_{21}^2\right)D_{23}+2R_{11}R_{31}\left(R_{11}R_{12}-3R_{21}R_{22}\right)\right] \\
&+\xi_{24}^{TR}\left[\left(R_{12}^2-3R_{22}^2\right)D_{21}+2R_{12}R_{32}\left(R_{12}R_{13}-3R_{22}R_{23}\right)\right] \\
&+\xi_{26}^{TR}\left[\left(R_{12}^2-3R_{22}^2\right)D_{23}+2R_{12}R_{32}\left(R_{11}R_{12}-3R_{21}R_{22}\right)\right] \\
&+\xi_{34}^{TR}\left[\left(R_{13}^2-3R_{23}^2\right)D_{21}+2R_{13}R_{33}\left(R_{12}R_{13}-3R_{22}R_{23}\right)\right] \\
&+\xi_{35}^{TR}\left[\left(R_{13}^2-3R_{23}^2\right)D_{22}+2R_{13}R_{33}\left(R_{13}R_{11}-3R_{23}R_{21}\right)\right],
\end{aligned} \tag{10j}$$

$$\begin{aligned}
\xi_{16} &= \eta_x^{TR}\left[D_{11}D_{21}+D_{31}\left(R_{32}R_{33}-R_{12}R_{13}\right)\right] \\
&+\eta_y^{TR}\left[D_{12}D_{22}+D_{32}\left(R_{33}R_{31}-R_{13}R_{11}\right)\right]+\eta_z^{TR}\left[D_{13}D_{23}+D_{33}\left(R_{31}R_{32}-R_{11}R_{12}\right)\right] \\
&+\xi_{15}^{TR}\left[\left(R_{11}^2-3R_{31}^2\right)D_{32}+2R_{11}R_{21}\left(R_{13}R_{11}-3R_{33}R_{31}\right)\right] \\
&+\xi_{16}^{TR}\left[\left(R_{11}^2-3R_{31}^2\right)D_{33}+2R_{11}R_{21}\left(R_{11}R_{12}-3R_{31}R_{32}\right)\right] \\
&+\xi_{24}^{TR}\left[\left(R_{12}^2-3R_{32}^2\right)D_{31}+2R_{12}R_{22}\left(R_{12}R_{13}-3R_{32}R_{33}\right)\right] \\
&+\xi_{26}^{TR}\left[\left(R_{12}^2-3R_{32}^2\right)D_{33}+2R_{12}R_{22}\left(R_{11}R_{12}-3R_{31}R_{32}\right)\right] \\
&+\xi_{34}^{TR}\left[\left(R_{13}^2-3R_{33}^2\right)D_{31}+2R_{13}R_{23}\left(R_{12}R_{13}-3R_{32}R_{33}\right)\right] \\
&+\xi_{35}^{TR}\left[\left(R_{13}^2-3R_{33}^2\right)D_{32}+2R_{13}R_{23}\left(R_{13}R_{11}-3R_{33}R_{31}\right)\right],
\end{aligned} \tag{10k}$$

$$
\begin{aligned}
\xi_{24} &= \eta_x^{TR}\left[D_{21}D_{31}+D_{11}\left(R_{12}R_{13}-R_{22}R_{23}\right)\right]\\
&+\eta_y^{TR}\left[D_{22}D_{32}+D_{12}\left(R_{13}R_{11}-R_{23}R_{21}\right)\right]+\eta_z^{TR}\left[D_{23}D_{33}+D_{13}\left(R_{11}R_{12}-R_{21}R_{22}\right)\right]\\
&+\xi_{15}^{TR}\left[\left(R_{21}^2-3R_{11}^2\right)D_{12}+2R_{21}R_{31}\left(R_{23}R_{21}-3R_{13}R_{11}\right)\right]\\
&+\xi_{16}^{TR}\left[\left(R_{21}^2-3R_{11}^2\right)D_{13}+2R_{21}R_{31}\left(R_{21}R_{22}-3R_{11}R_{12}\right)\right]\\
&+\xi_{24}^{TR}\left[\left(R_{22}^2-3R_{12}^2\right)D_{11}+2R_{22}R_{32}\left(R_{22}R_{23}-3R_{12}R_{13}\right)\right]\\
&+\xi_{26}^{TR}\left[\left(R_{22}^2-3R_{12}^2\right)D_{13}+2R_{22}R_{32}\left(R_{21}R_{22}-3R_{11}R_{12}\right)\right]\\
&+\xi_{34}^{TR}\left[\left(R_{23}^2-3R_{13}^2\right)D_{11}+2R_{23}R_{33}\left(R_{22}R_{23}-3R_{12}R_{13}\right)\right]\\
&+\xi_{35}^{TR}\left[\left(R_{23}^2-3R_{13}^2\right)D_{12}+2R_{23}R_{33}\left(R_{23}R_{21}-3R_{11}R_{13}\right)\right],
\end{aligned}
\tag{10l}
$$

$$
\begin{aligned}
\xi_{26} &= \eta_x^{TR}\left[D_{11}D_{21}+D_{31}\left(R_{32}R_{33}-R_{22}R_{23}\right)\right]\\
&+\eta_y^{TR}\left[D_{12}D_{22}+D_{32}\left(R_{31}R_{33}-R_{21}R_{23}\right)\right]+\eta_z^{TR}\left[D_{13}D_{23}+D_{33}\left(R_{31}R_{32}-R_{21}R_{22}\right)\right]\\
&+\xi_{15}^{TR}\left[\left(R_{21}^2-3R_{31}^2\right)D_{32}+2R_{11}R_{21}\left(R_{23}R_{21}-3R_{31}R_{33}\right)\right]\\
&+\xi_{16}^{TR}\left[\left(R_{21}^2-3R_{31}^2\right)D_{33}+2R_{11}R_{21}\left(R_{21}R_{22}-3R_{31}R_{32}\right)\right]\\
&+\xi_{24}^{TR}\left[\left(R_{22}^2-3R_{32}^2\right)D_{31}+2R_{12}R_{22}\left(R_{22}R_{23}-3R_{32}R_{33}\right)\right]\\
&+\xi_{26}^{TR}\left[\left(R_{22}^2-3R_{32}^2\right)D_{33}+2R_{12}R_{22}\left(R_{21}R_{22}-3R_{31}R_{32}\right)\right]\\
&+\xi_{34}^{TR}\left[\left(R_{23}^2-3R_{33}^2\right)D_{31}+2R_{13}R_{23}\left(R_{22}R_{23}-3R_{32}R_{33}\right)\right]\\
&+\xi_{35}^{TR}\left[\left(R_{23}^2-3R_{33}^2\right)D_{32}+2R_{13}R_{23}\left(R_{21}R_{23}-3R_{31}R_{33}\right)\right],
\end{aligned}
\tag{10m}
$$

$$
\begin{aligned}
\xi_{34} = \eta_x^{TR} \left[ D_{21} D_{31} + D_{11} \left( R_{12} R_{13} - R_{32} R_{33} \right) \right] \\
+\eta_y^{TR} \left[ D_{22} D_{32} + D_{12} \left( R_{11} R_{13} - R_{31} R_{33} \right) \right] + \eta_z^{TR} \left[ D_{23} D_{33} + D_{13} \left( R_{11} R_{12} - R_{31} R_{32} \right) \right] \\
+\xi_{15}^{TR} \left[ \left( R_{31}^2 - 3R_{11}^2 \right) D_{12} + 2R_{21} R_{31} \left( R_{33} R_{31} - 3R_{11} R_{13} \right) \right] \\
+\xi_{16}^{TR} \left[ \left( R_{31}^2 - 3R_{11}^2 \right) D_{13} + 2R_{21} R_{31} \left( R_{31} R_{32} - 3R_{11} R_{12} \right) \right] \\
+\xi_{24}^{TR} \left[ \left( R_{32}^2 - 3R_{12}^2 \right) D_{11} + 2R_{22} R_{32} \left( R_{32} R_{33} - 3R_{12} R_{13} \right) \right] \\
+\xi_{26}^{TR} \left[ \left( R_{32}^2 - 3R_{12}^2 \right) D_{13} + 2R_{22} R_{32} \left( R_{31} R_{32} - 3R_{11} R_{12} \right) \right] \\
+\xi_{34}^{TR} \left[ \left( R_{33}^2 - 3R_{13}^2 \right) D_{11} + 2R_{23} R_{33} \left( R_{32} R_{33} - 3R_{12} R_{13} \right) \right] \\
+\xi_{35}^{TR} \left[ \left( R_{33}^2 - 3R_{13}^2 \right) D_{12} + 2R_{23} R_{33} \left( R_{31} R_{33} - 3R_{11} R_{13} \right) \right],
\end{aligned}
\tag{10n}
$$

$$
\begin{aligned}
\xi_{35} = \eta_x^{TR} \left[ D_{31} D_{11} + D_{21} \left( R_{22} R_{23} - R_{32} R_{33} \right) \right] \\
+\eta_y^{TR} \left[ D_{32} D_{12} + D_{22} \left( R_{21} R_{23} - R_{31} R_{33} \right) \right] + \eta_z^{TR} \left[ D_{33} D_{13} + D_{23} \left( R_{21} R_{22} - R_{31} R_{32} \right) \right] \\
+\xi_{15}^{TR} \left[ \left( R_{31}^2 - 3R_{21}^2 \right) D_{22} + 2R_{11} R_{31} \left( R_{33} R_{31} - 3R_{21} R_{23} \right) \right] \\
+\xi_{16}^{TR} \left[ \left( R_{31}^2 - 3R_{21}^2 \right) D_{23} + 2R_{11} R_{31} \left( R_{31} R_{32} - 3R_{21} R_{22} \right) \right] \\
+\xi_{24}^{TR} \left[ \left( R_{32}^2 - 3R_{22}^2 \right) D_{21} + 2R_{12} R_{32} \left( R_{32} R_{33} - 3R_{22} R_{23} \right) \right] \\
+\xi_{26}^{TR} \left[ \left( R_{32}^2 - 3R_{22}^2 \right) D_{23} + 2R_{32} R_{12} \left( R_{31} R_{32} - 3R_{21} R_{22} \right) \right] \\
+\xi_{34}^{TR} \left[ \left( R_{33}^2 - 3R_{23}^2 \right) D_{21} + 2R_{33} R_{13} \left( R_{32} R_{33} - 3R_{22} R_{23} \right) \right] \\
+\xi_{35}^{TR} \left[ \left( R_{33}^2 - 3R_{23}^2 \right) D_{22} + 2R_{33} R_{13} \left( R_{31} R_{33} - 3R_{21} R_{23} \right) \right],
\end{aligned}
\tag{10o}
$$

$$
\begin{aligned}
\gamma_x = \gamma_x^{TR} R_{11}^2 + \gamma_y^{TR} R_{12}^2 + \gamma_z^{TR} R_{13}^2 - \varepsilon_{45}^{TR} R_{12} R_{11} - \varepsilon_{46}^{TR} R_{11} R_{13} - \varepsilon_{56}^{TR} R_{13} R_{12} \\
+r^2 \Big( \eta_x^{TR} R_{22} R_{32} R_{23} R_{33} + \eta_y^{TR} R_{21} R_{31} R_{23} R_{33} + \eta_z^{TR} R_{21} R_{31} R_{22} R_{32} \\
-\xi_{15}^{TR} R_{21} R_{31} D_{12} - \xi_{16}^{TR} R_{21} R_{31} D_{13} - \xi_{24}^{TR} R_{22} R_{32} D_{11} \\
-\xi_{26}^{TR} R_{22} R_{32} D_{13} - \xi_{34}^{TR} R_{23} R_{33} D_{11} - \xi_{35}^{TR} R_{23} R_{33} D_{12} \Big),
\end{aligned}
\tag{10p}
$$

$$\begin{aligned}\gamma_y = {} & \gamma_x^{TR} R_{21}^2 + \gamma_y^{TR} R_{22}^2 + \gamma_z^{TR} R_{23}^2 - \varepsilon_{45}^{TR} R_{22} R_{21} - \varepsilon_{46}^{TR} R_{21} R_{23} - \varepsilon_{56}^{TR} R_{23} R_{22} \\ & + r^2 \Big( \eta_x^{TR} R_{12} R_{32} R_{13} R_{33} + \eta_y^{TR} R_{11} R_{13} R_{31} R_{33} + \eta_z^{TR} R_{11} R_{31} R_{12} R_{32} \\ & - \xi_{15}^{TR} R_{11} R_{31} D_{22} - \xi_{16}^{TR} R_{11} R_{31} D_{23} - \xi_{24}^{TR} R_{12} R_{32} D_{21} \\ & - \xi_{26}^{TR} R_{12} R_{32} D_{23} - \xi_{34}^{TR} R_{13} R_{33} D_{21} - \xi_{35}^{TR} R_{13} R_{33} D_{22} \Big) ,\end{aligned} \tag{10q}$$

$$\begin{aligned}\gamma_z = {} & \gamma_x^{TR} R_{31}^2 + \gamma_y^{TR} R_{32}^2 + \gamma_z^{TR} R_{33}^2 - \varepsilon_{45}^{TR} R_{32} R_{31} - \varepsilon_{46}^{TR} R_{31} R_{33} - \varepsilon_{56}^{TR} R_{33} R_{32} \\ & + r^2 \Big( \eta_x^{TR} R_{12} R_{22} R_{13} R_{23} + \eta_y^{TR} R_{11} R_{21} R_{13} R_{23} + \eta_z^{TR} R_{11} R_{21} R_{12} R_{22} \\ & - \xi_{15}^{TR} R_{11} R_{21} D_{32} - \xi_{16}^{TR} R_{11} R_{21} D_{33} - \xi_{24}^{TR} R_{12} R_{22} D_{31} \\ & - \xi_{26}^{TR} R_{12} R_{22} D_{33} - \xi_{34}^{TR} R_{13} R_{23} D_{31} - \xi_{35}^{TR} R_{13} R_{23} D_{32} \Big) ,\end{aligned} \tag{10r}$$

$$\begin{aligned}\varepsilon_{45} = {} & -2\gamma_x^{TR} R_{11} R_{21} - 2\gamma_y^{TR} R_{12} R_{22} - 2\gamma_z^{TR} R_{13} R_{23} + \varepsilon_{45}^{TR} D_{33} + \varepsilon_{46}^{TR} D_{32} + \varepsilon_{56}^{TR} D_{31} \\ & + r^2 \Big[ \eta_x^{TR} R_{32} R_{33} D_{31} + \eta_y^{TR} R_{31} R_{33} D_{32} + \eta_z^{TR} R_{31} R_{32} D_{33} \\ & - \xi_{15}^{TR} R_{31} \left( 2R_{11} R_{21} R_{33} + R_{31} D_{32} \right) - \xi_{16}^{TR} R_{31} \left( 2R_{11} R_{21} R_{32} + R_{31} D_{33} \right) \\ & - \xi_{24}^{TR} R_{32} \left( 2R_{12} R_{22} R_{33} + R_{32} D_{31} \right) - \xi_{26}^{TR} R_{32} \left( 2R_{12} R_{22} R_{31} + R_{32} D_{33} \right) \\ & - \xi_{34}^{TR} R_{33} \left( 2R_{13} R_{23} R_{32} + R_{33} D_{31} \right) - \xi_{35}^{TR} R_{33} \left( 2R_{13} R_{23} R_{31} + R_{33} D_{32} \right) \Big] ,\end{aligned} \tag{10s}$$

$$\begin{aligned}\varepsilon_{46} = {} & -2\gamma_x^{TR} R_{11} R_{31} - 2\gamma_y^{TR} R_{12} R_{32} - 2\gamma_z^{TR} R_{13} R_{33} + \varepsilon_{45}^{TR} D_{23} + \varepsilon_{46}^{TR} D_{22} + \varepsilon_{56}^{TR} D_{21} \\ & + r^2 \Big[ \eta_x^{TR} R_{22} R_{23} D_{21} + \eta_y^{TR} R_{21} R_{23} D_{22} + \eta_z^{TR} R_{21} R_{22} D_{23} \\ & - \xi_{15}^{TR} R_{21} \left( 2R_{11} R_{23} R_{31} + R_{21} D_{22} \right) - \xi_{16}^{TR} R_{21} \left( 2R_{11} R_{22} R_{31} + R_{21} D_{23} \right) \\ & - \xi_{24}^{TR} R_{22} \left( 2R_{12} R_{23} R_{32} + R_{22} D_{21} \right) - \xi_{26}^{TR} R_{22} \left( 2R_{12} R_{21} R_{32} + R_{22} D_{23} \right) \\ & - \xi_{34}^{TR} R_{23} \left( 2R_{13} R_{22} R_{33} + R_{23} D_{21} \right) - \xi_{35}^{TR} R_{23} \left( 2R_{13} R_{21} R_{33} + R_{23} D_{22} \right) \Big] ,\end{aligned} \tag{10t}$$

$$\begin{aligned}\varepsilon_{56} = {} & -2\gamma_x^{TR} R_{21} R_{31} - 2\gamma_y^{TR} R_{22} R_{32} - 2\gamma_z^{TR} R_{23} R_{33} + \varepsilon_{45}^{TR} D_{13} + \varepsilon_{46}^{TR} D_{12} + \varepsilon_{56}^{TR} D_{11} \\ & + r^2 \Big[ \eta_x^{TR} R_{12} R_{13} D_{11} + \eta_y^{TR} R_{11} R_{13} D_{12} + \eta_z^{TR} R_{11} R_{12} D_{13} \\ & - \xi_{15}^{TR} R_{11} \left( 2R_{13} R_{21} R_{31} + R_{11} D_{12} \right) - \xi_{16}^{TR} R_{11} \left( 2R_{12} R_{21} R_{31} + R_{11} D_{13} \right) \\ & - \xi_{24}^{TR} R_{12} \left( 2R_{13} R_{22} R_{32} + R_{12} D_{11} \right) - \xi_{26}^{TR} R_{12} \left( 2R_{11} R_{22} R_{32} + R_{12} D_{13} \right) \\ & - \xi_{34}^{TR} R_{13} \left( 2R_{12} R_{23} R_{33} + R_{13} D_{11} \right) - \xi_{35}^{TR} R_{13} \left( 2R_{11} R_{23} R_{33} + R_{13} D_{12} \right) \Big] .\end{aligned} \tag{10u}$$

The symbols $R_{ij}$ represent elements of the rotation matrix R, see Appendix A. The symbol $r$ is the ratio of reference P- and S-wave velocities, $r = \alpha/\beta$, see Eq. (9). The symbols $D_{ij}$ used in Eqs (10a–u) have the following meaning:

$$\begin{aligned} &D_{11} = R_{22}R_{33} + R_{23}R_{32}\,, \quad D_{12} = R_{21}R_{33} + R_{23}R_{31}\,, \quad D_{13} = R_{22}R_{31} + R_{21}R_{32}\,, \\ &D_{21} = R_{12}R_{33} + R_{13}R_{32}\,, \quad D_{22} = R_{11}R_{33} + R_{13}R_{31}\,, \quad D_{23} = R_{12}R_{31} + R_{11}R_{32}\,, \\ &D_{31} = R_{12}R_{23} + R_{13}R_{22}\,, \quad D_{32} = R_{13}R_{21} + R_{11}R_{23}\,, \quad D_{33} = R_{11}R_{22} + R_{12}R_{21}\,. \end{aligned} \qquad (11)$$

### 4.2. Orthorhombic symmetry

Here we specify the above transformation equations for nine A-parameters, $\varepsilon_x^{OR}$, $\varepsilon_y^{OR}$, $\varepsilon_z^{OR}$, $\eta_x^{OR}$, $\eta_y^{OR}$, $\eta_z^{OR}$, $\gamma_x^{OR}$, $\gamma_y^{OR}$ and $\gamma_z^{OR}$, describing orthorhombic symmetry in the crystal coordinate system, whose coordinate planes coincide with the symmetry planes. The above nine orthorhombic A-parameters transform into 21 A-parameters in the global coordinate system through the equations:

$$\varepsilon_x = \varepsilon_x^{OR}R_{11}^2 + \varepsilon_y^{OR}R_{12}^2 + \varepsilon_z^{OR}R_{13}^2 + \eta_x^{OR}R_{12}^2R_{13}^2 + \eta_y^{OR}R_{11}^2R_{13}^2 + \eta_z^{OR}R_{11}^2R_{12}^2\,, \qquad (12a)$$

$$\varepsilon_y = \varepsilon_x^{OR}R_{21}^2 + \varepsilon_y^{OR}R_{22}^2 + \varepsilon_z^{OR}R_{23}^2 + \eta_x^{OR}R_{22}^2R_{23}^2 + \eta_y^{OR}R_{21}^2R_{23}^2 + \eta_z^{OR}R_{21}^2R_{22}^2\,, \qquad (12b)$$

$$\varepsilon_z = \varepsilon_x^{OR}R_{31}^2 + \varepsilon_y^{OR}R_{32}^2 + \varepsilon_z^{OR}R_{33}^2 + \eta_x^{OR}R_{32}^2R_{33}^2 + \eta_y^{OR}R_{31}^2R_{33}^2 + \eta_z^{OR}R_{31}^2R_{32}^2\,, \qquad (12c)$$

$$\begin{aligned} \chi_x = {}& \varepsilon_x^{OR}R_{21}R_{31} + \varepsilon_y^{OR}R_{22}R_{32} + \varepsilon_z^{OR}R_{23}R_{33} + \eta_x^{OR}\left(D_{21}D_{31} + R_{12}R_{13}D_{11}\right) \\ & + \eta_y^{OR}\left(D_{32}D_{22} + R_{11}R_{13}D_{12}\right) + \eta_z^{OR}\left(D_{33}D_{23} + R_{11}R_{12}D_{13}\right), \end{aligned} \qquad (12d)$$

$$\begin{aligned} \chi_y = {}& \varepsilon_x^{OR}R_{11}R_{31} + \varepsilon_y^{OR}R_{12}R_{32} + \varepsilon_z^{OR}R_{13}R_{33} + \eta_x^{OR}\left(D_{11}D_{31} + R_{22}R_{23}D_{21}\right) \\ & + \eta_y^{OR}\left(D_{12}D_{32} + R_{21}R_{23}D_{22}\right) + \eta_z^{OR}\left(D_{13}D_{33} + R_{21}R_{22}D_{23}\right), \end{aligned} \qquad (12e)$$

$$\begin{aligned} \chi_z = {}& \varepsilon_x^{OR}R_{11}R_{21} + \varepsilon_y^{OR}R_{12}R_{22} + \varepsilon_z^{OR}R_{13}R_{23} + \eta_x^{OR}\left(D_{11}D_{21} + R_{32}R_{33}D_{31}\right) \\ & + \eta_y^{OR}\left(D_{12}D_{22} + R_{31}R_{33}D_{32}\right) + \eta_z^{OR}\left(D_{13}D_{23} + R_{31}R_{32}D_{33}\right), \end{aligned} \qquad (12f)$$

$$\begin{aligned} \eta_x = {}& \eta_x^{OR}\left[\left(1-R_{12}^2\right)\left(1-R_{13}^2\right) - 2\left(R_{32}R_{33} - R_{22}R_{23}\right)^2\right] \\ & + \eta_y^{OR}\left[\left(1-R_{13}^2\right)\left(1-R_{11}^2\right) - 2\left(R_{33}R_{31} - R_{23}R_{21}\right)^2\right] \\ & + \eta_z^{OR}\left[\left(1-R_{11}^2\right)\left(1-R_{12}^2\right) - 2\left(R_{31}R_{32} - R_{21}R_{22}\right)^2\right], \end{aligned} \qquad (12g)$$

$$\eta_y = \eta_x^{OR}\left[\left(1-R_{22}^2\right)\left(1-R_{23}^2\right)-2\left(R_{32}R_{33}-R_{12}R_{13}\right)^2\right] + \eta_y^{OR}\left[\left(1-R_{23}^2\right)\left(1-R_{21}^2\right)-2\left(R_{33}R_{31}-R_{13}R_{11}\right)^2\right] + \eta_z^{OR}\left[\left(1-R_{21}^2\right)\left(1-R_{22}^2\right)-2\left(R_{31}R_{32}-R_{11}R_{12}\right)^2\right], \tag{12h}$$

$$\eta_z = \eta_x^{OR}\left[\left(1-R_{32}^2\right)\left(1-R_{33}^2\right)-2\left(R_{22}R_{23}-R_{12}R_{13}\right)^2\right] + \eta_y^{OR}\left[\left(1-R_{33}^2\right)\left(1-R_{31}^2\right)-2\left(R_{23}R_{21}-R_{13}R_{11}\right)^2\right] + \eta_z^{OR}\left[\left(1-R_{31}^2\right)\left(1-R_{32}^2\right)-2\left(R_{21}R_{22}-R_{11}R_{12}\right)^2\right], \tag{12i}$$

$$\xi_{15} = \eta_x^{OR}\left[D_{11}D_{31}+D_{21}\left(R_{22}R_{23}-R_{12}R_{13}\right)\right] + \eta_y^{OR}\left[D_{12}D_{32}+D_{22}\left(R_{23}R_{21}-R_{13}R_{11}\right)\right] + \eta_z^{OR}\left[D_{13}D_{33}+D_{23}\left(R_{21}R_{22}-R_{11}R_{12}\right)\right], \tag{12j}$$

$$\xi_{16} = \eta_x^{OR}\left[D_{11}D_{21}+D_{31}\left(R_{32}R_{33}-R_{12}R_{13}\right)\right] + \eta_y^{OR}\left[D_{12}D_{22}+D_{32}\left(R_{33}R_{31}-R_{13}R_{11}\right)\right] + \eta_z^{OR}\left[D_{13}D_{23}+D_{33}\left(R_{31}R_{32}-R_{11}R_{12}\right)\right], \tag{12k}$$

$$\xi_{24} = \eta_x^{OR}\left[D_{21}D_{31}+D_{11}\left(R_{12}R_{13}-R_{22}R_{23}\right)\right] + \eta_y^{OR}\left[D_{22}D_{32}+D_{12}\left(R_{13}R_{11}-R_{23}R_{21}\right)\right] + \eta_z^{OR}\left[D_{23}D_{33}+D_{13}\left(R_{11}R_{12}-R_{21}R_{22}\right)\right], \tag{12l}$$

$$\xi_{26} = \eta_x^{OR}\left[D_{11}D_{21}+D_{31}\left(R_{32}R_{33}-R_{22}R_{23}\right)\right] + \eta_y^{OR}\left[D_{12}D_{22}+D_{32}\left(R_{31}R_{33}-R_{21}R_{23}\right)\right] + \eta_z^{OR}\left[D_{13}D_{23}+D_{33}\left(R_{31}R_{32}-R_{21}R_{22}\right)\right], \tag{12m}$$

$$\xi_{34} = \eta_x^{OR}\left[D_{21}D_{31}+D_{11}\left(R_{12}R_{13}-R_{32}R_{33}\right)\right] + \eta_y^{OR}\left[D_{22}D_{32}+D_{12}\left(R_{11}R_{13}-R_{31}R_{33}\right)\right] + \eta_z^{OR}\left[D_{23}D_{33}+D_{13}\left(R_{11}R_{12}-R_{31}R_{32}\right)\right], \tag{12n}$$

$$\xi_{35} = \eta_x^{OR}\left[D_{31}D_{11} + D_{21}\left(R_{22}R_{23} - R_{32}R_{33}\right)\right]$$
$$+\eta_y^{OR}\left[D_{32}D_{12} + D_{22}\left(R_{21}R_{23} - R_{31}R_{33}\right)\right] \tag{12o}$$
$$+\eta_z^{OR}\left[D_{33}D_{13} + D_{23}\left(R_{21}R_{22} - R_{31}R_{32}\right)\right],$$

$$\gamma_x = \gamma_x^{OR}R_{11}^2 + \gamma_y^{OR}R_{12}^2 + \gamma_z^{OR}R_{13}^2$$
$$+r^2\left(\eta_x^{OR}R_{22}R_{32}R_{23}R_{33} + \eta_y^{OR}R_{21}R_{31}R_{23}R_{33} + \eta_z^{OR}R_{21}R_{31}R_{22}R_{32}\right), \tag{12p}$$

$$\gamma_y = \gamma_x^{OR}R_{21}^2 + \gamma_y^{OR}R_{22}^2 + \gamma_z^{OR}R_{23}^2$$
$$+r^2\left(\eta_x^{OR}R_{12}R_{32}R_{13}R_{33} + \eta_y^{OR}R_{11}R_{13}R_{31}R_{33} + \eta_z^{OR}R_{11}R_{31}R_{12}R_{32}\right), \tag{12q}$$

$$\gamma_z = \gamma_x^{OR}R_{31}^2 + \gamma_y^{OR}R_{32}^2 + \gamma_z^{OR}R_{33}^2$$
$$+r^2\left(\eta_x^{OR}R_{12}R_{22}R_{13}R_{23} + \eta_y^{OR}R_{11}R_{21}R_{13}R_{23} + \eta_z^{OR}R_{11}R_{21}R_{12}R_{22}\right), \tag{12r}$$

$$\varepsilon_{45} = -2\gamma_x^{OR}R_{11}R_{21} - 2\gamma_y^{OR}R_{12}R_{22} - 2\gamma_z^{OR}R_{13}R_{23}$$
$$+r^2\left(\eta_x^{OR}R_{32}R_{33}D_{31} + \eta_y^{OR}R_{31}R_{33}D_{32} + \eta_z^{OR}R_{31}R_{32}D_{33}\right), \tag{12s}$$

$$\varepsilon_{46} = -2\gamma_x^{OR}R_{11}R_{31} - 2\gamma_y^{OR}R_{12}R_{32} - 2\gamma_z^{OR}R_{13}R_{33}$$
$$+r^2\left(\eta_x^{OR}R_{22}R_{23}D_{21} + \eta_y^{OR}R_{21}R_{23}D_{22} + \eta_z^{OR}R_{21}R_{22}D_{23}\right), \tag{12t}$$

$$\varepsilon_{56} = -2\gamma_x^{OR}R_{21}R_{31} - 2\gamma_y^{OR}R_{22}R_{32} - 2\gamma_z^{OR}R_{23}R_{33}$$
$$+r^2\left(\eta_x^{OR}R_{12}R_{13}D_{11} + \eta_y^{OR}R_{11}R_{13}D_{12} + \eta_z^{OR}R_{11}R_{12}D_{13}\right). \tag{12u}$$

Symbol $r$ is given in Eq. (9), symbols $D_{ij}$ in Eq. (11).

### 4.3. Transverse isotropy

Five A-parameters, $\varepsilon_x^{TI}$, $\varepsilon_z^{TI}$, $\eta_x^{TI}$, $\gamma_x^{TI}$ and $\gamma_z^{TI}$, specifying transverse isotropy in the crystal coordinate system, in which the axis of symmetry coincides with the third axis of the crystal coordinate system, transform into twenty one A-parameters in the global coordinate system through the equations:

$$\varepsilon_x = \varepsilon_x^{TI}\left(1 - R_{13}^2\right) + \varepsilon_z^{TI}R_{13}^2 + \eta_x^{TI}R_{13}^2\left(1 - R_{13}^2\right), \tag{13a}$$

$$\varepsilon_y = \varepsilon_x^{TI}\left(1 - R_{23}^2\right) + \varepsilon_z^{TI}R_{23}^2 + \eta_x^{TI}R_{23}^2\left(1 - R_{23}^2\right), \tag{13b}$$

$$\varepsilon_z = \varepsilon_x^{TI}\left(1 - R_{33}^2\right) + \varepsilon_z^{TI}R_{33}^2 + \eta_x^{TI}R_{33}^2\left(1 - R_{33}^2\right), \tag{13c}$$

$$\chi_x = -\varepsilon_x^{TI} R_{23} R_{33} + \varepsilon_z^{TI} R_{23} R_{33} + \eta_x^{TI} R_{23} R_{33} \left(1 - 6R_{13}^2\right), \tag{13d}$$

$$\chi_y = -\varepsilon_x^{TI} R_{33} R_{13} + \varepsilon_z^{TI} R_{33} R_{13} + \eta_x^{TI} R_{33} R_{13} \left(1 - 6R_{23}^2\right), \tag{13e}$$

$$\chi_z = -\varepsilon_x^{TI} R_{13} R_{23} + \varepsilon_z^{TI} R_{13} R_{23} + \eta_x^{TI} R_{13} R_{23} \left(1 - 6R_{33}^2\right), \tag{13f}$$

$$\eta_x = \eta_x^{TI} \left[ 2\left(R_{23}^2 - R_{33}^2\right)^2 - \left(1 - R_{13}^2\right)^2 \right], \tag{13g}$$

$$\eta_y = \eta_x^{TI} \left[ 2\left(R_{33}^2 - R_{13}^2\right)^2 - \left(1 - R_{23}^2\right)^2 \right], \tag{13h}$$

$$\eta_z = \eta_x^{TI} \left[ 2\left(R_{13}^2 - R_{23}^2\right)^2 - \left(1 - R_{33}^2\right)^2 \right], \tag{13i}$$

$$\xi_{15} = 2\eta_x^{TI} R_{13} R_{33} \left(R_{13}^2 - 3R_{23}^2\right), \tag{13j}$$

$$\xi_{16} = 2\eta_x^{TI} R_{13} R_{23} \left(R_{13}^2 - 3R_{33}^2\right), \tag{13k}$$

$$\xi_{24} = 2\eta_x^{TI} R_{23} R_{33} \left(R_{23}^2 - 3R_{13}^2\right), \tag{13l}$$

$$\xi_{26} = 2\eta_x^{TI} R_{13} R_{23} \left(R_{23}^2 - 3R_{33}^2\right), \tag{13m}$$

$$\xi_{34} = 2\eta_x^{TI} R_{23} R_{33} \left(R_{33}^2 - 3R_{13}^2\right), \tag{13n}$$

$$\xi_{35} = 2\eta_x^{TI} R_{13} R_{33} \left(R_{33}^2 - 3R_{23}^2\right), \tag{13o}$$

$$\gamma_x = \gamma_x^{TI} \left(1 - R_{13}^2\right) + \gamma_z^{TI} R_{13}^2 - r^2 \eta_x^{TI} R_{23}^2 R_{33}^2, \tag{13p}$$

$$\gamma_y = \gamma_x^{TI} \left(1 - R_{23}^2\right) + \gamma_z^{TI} R_{23}^2 - r^2 \eta_x^{TI} R_{33}^2 R_{13}^2, \tag{13q}$$

$$\gamma_z = \gamma_x^{TI} \left(1 - R_{33}^2\right) + \gamma_z^{TI} R_{33}^2 - r^2 \eta_x^{TI} R_{13}^2 R_{23}^2, \tag{13r}$$

$$\varepsilon_{45} = 2\gamma_x^{TI} R_{13} R_{23} - 2\gamma_z^{TI} R_{13} R_{23} - 2r^2 \eta_x^{TI} R_{13} R_{23} R_{33}^2, \tag{13s}$$

$$\varepsilon_{46} = 2\gamma_x^{TI} R_{13} R_{33} - 2\gamma_z^{TI} R_{13} R_{33} - 2r^2 \eta_x^{TI} R_{13} R_{23}^2 R_{33}, \tag{13t}$$

$$\varepsilon_{56} = 2\gamma_x^{TI} R_{23} R_{33} - 2\gamma_z^{TI} R_{23} R_{33} - 2r^2 \eta_x^{TI} R_{13}^2 R_{23} R_{33}. \tag{13u}$$

Symbol $r$ is again given in Eq. (9).

Note that the transformation Eqs (13a–u) contain only the third column of the rotation matrix **R** specifying the orientation of the axis of symmetry in the global coordinate system.

### 4.4. Transformation of A-parameters due to change of reference velocities

Let us introduce new reference velocities $\alpha'$ and $\beta'$ instead of original reference velocities $\alpha$ and $\beta$. Let us mark A-parameters related to primed reference velocities by primes. Using factors $k_\alpha$ and $k_\beta$ defined as ratios of original and new reference velocities, $k_\alpha = \alpha/\alpha'$, $k_\beta = \beta/\beta'$, one can express primed A-parameters (related to $\alpha'$ and $\beta'$) in terms of non-primed A-parameters (related to $\alpha$ and $\beta$) in the following way:

$$
\begin{gathered}
\varepsilon'_x = \varepsilon_x k_\alpha^2 + \frac{k_\alpha^2 - 1}{2}, \quad \varepsilon'_y = \varepsilon_y k_\alpha^2 + \frac{k_\alpha^2 - 1}{2}, \quad \varepsilon'_z = \varepsilon_z k_\alpha^2 + \frac{k_\alpha^2 - 1}{2}, \\
\chi'_x = \chi_x k_\alpha^2, \quad \chi'_y = \chi_y k_\alpha^2, \quad \chi'_z = \chi_z k_\alpha^2, \\
\eta'_x = \eta_x k_\alpha^2, \quad \eta'_y = \eta_y k_\alpha^2, \quad \eta'_z = \eta_z k_\alpha^2, \\
\xi'_{15} = \xi_{15} k_\alpha^2, \ \xi'_{16} = \xi_{16} k_\alpha^2, \ \xi'_{24} = \xi_{24} k_\alpha^2, \ \xi'_{26} = \xi_{26} k_\alpha^2, \ \xi'_{34} = \xi_{34} k_\alpha^2, \ \xi'_{35} = \xi_{35} k_\alpha^2, \\
\gamma'_x = \gamma_x k_\beta^2 + \frac{k_\beta^2 - 1}{2}, \quad \gamma'_y = \gamma_y k_\beta^2 + \frac{k_\beta^2 - 1}{2}, \quad \gamma'_z = \gamma_z k_\beta^2 + \frac{k_\beta^2 - 1}{2}, \\
\varepsilon'_{45} = \varepsilon_{45} k_\beta^2, \quad \varepsilon'_{46} = \varepsilon_{46} k_\beta^2, \quad \varepsilon'_{56} = \varepsilon_{56} k_\beta^2 .
\end{gathered}
\tag{14}
$$

## 5. DISCUSSION AND CONCLUSIONS

We present a set of twenty one A-parameters, which represent a useful alternative of twenty one independent elements of the density-normalized stiffness tensor or twenty one independent density-normalized elastic parameters in the Voigt notation. A-parameters are non-dimensional; for a proper choice of reference velocities, they can all have a similar size. When used with the weak-anisotropy approximation, specifically in the first-order approximation, one can clearly distinguish a set of six A-parameters, which control only P-wave velocities, and another set of six A-parameters, which control only S-wave velocities. Remaining nine common A-parameters control velocities of both P and S waves. The described separation of A-parameters can be also observed in transformation equations of A-parameters from one coordinate system to another. The presented transformation equations are applicable to anisotropy of any symmetry. Specifications to orthorhombic symmetry and transverse isotropy are also presented. P-wave A-parameters in the new coordinate system depend only on P-wave and common A-parameters from the original coordinate system. S-wave A-parameters in the new coordinate system depend only on S-wave and common A-parameters in the original coordinate system. Common A-parameters in the new coordinate system depend on all nine common A-parameters in the original coordinate system. They depend on neither P- nor S-wave A-parameters in the original coordinate system.

Behaviour of polarization vectors in the first-order weak-anisotropy approximation is slightly different from behaviour of velocities. P-wave polarization vector and common

S-wave polarization plane depend on only P-wave and common A-parameters, it is on fifteen A-parameters. S-wave polarization vectors of separate S waves depend on all twenty one A-parameters.

A-parameters were successfully used in several applications. *Pšenčík et al. (2018)* used the complete set of A-parameters for the inversion of anisotropy of a laboratory rock sample. *Pšenčík et al. (2020)* used only P-wave A-parameters for the inversion of P-wave traveltimes from synthetic VSP data sets. *Farra and Pšenčík (2023)* used the complete set of A-parameters in the weak-anisotropy approximation of reflection moveout of converted PS waves and specified it for the TTI case.

## APPENDIX A
## ROTATION MATRIX R IN TERMS OF EULER ANGLES

The transformation of A-parameters from the crystal to the global coordinate system is performed with the use of the rotation matrix **R**. We present here such a matrix, expressed in terms of three Euler angles $\varphi$, $\theta$ and $\nu$. The Euler angles $\varphi$ and $\theta$ represent the azimuth and polar angles of the third coordinate axis of the crystal coordinate system in the global coordinate system. The angle $\nu$ represents a rotation around this axis. The rotation matrix **R** has then the form:

$$\mathbf{R} = \begin{pmatrix} \cos\varphi\cos\theta\cos\nu - \sin\varphi\sin\nu & -\cos\varphi\cos\theta\sin\nu - \sin\varphi\cos\nu & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta\cos\nu + \cos\varphi\sin\nu & -\sin\varphi\cos\theta\sin\nu + \cos\varphi\cos\nu & \sin\varphi\sin\theta \\ -\sin\theta\cos\nu & \sin\theta\sin\nu & \cos\theta \end{pmatrix}. \tag{A.1}$$

If the TI symmetry is considered, and its symmetry axis is oriented along the third coordinate axis of the crystal coordinate system, the angle $\nu$ representing the rotation around the axis of symmetry looses its meaning and can be set to 0. Equation (A.1) then reduces to:

$$\mathbf{R} = \begin{pmatrix} \cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta \\ \sin\varphi\cos\theta & \cos\varphi & \sin\varphi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{A.2}$$

*Acknowledgement:* We are grateful to the Research Project 20-06887S of the Czech Science Foundation for support.



*Funding Note:* Open access publishing supported by the National Technical Library in Prague.

*References*

Bakker P., 2002. Coupled anisotropic shear-wave ray tracing in situations where associated slowness sheets are almost tangent. *Pure Appl. Geophys.*, **159**, 1403–1417

Bond W., 1943. The mathematics of the physical properties of crystals. *Bell System Technical Journal*, **22**, https://doi.org/10.1002/j.1538-7305.1943.tb01304.x

Carcione J.M., 2014. *Wave Fields in Real Media: Wave Propagation in Anisotropic, Anelastic, Porous and Electromagnetic Media*. 3rd Edition. Elsevier Science, Oxford, U.K.

Červený V., 2001. *Seismic Ray Theory*. Cambridge University Press, Cambridge, U.K.

Chapman C.H., 2004. *Fundamentals of Seismic Wave Propagation*. Cambridge University Press, Cambridge, U.K.

Farra V. and Pšenčík I., 2003. Properties of the zero-, first- and higher-order approximations of attributes of elastic waves in weakly anisotropic media. *J. Acoust. Soc. Am.*, **114**, 1366–1378

Farra V. and Pšenčík I., 2008. First-order ray computations of coupled S waves in inhomogeneous weakly anisotropic media. *Geophys. J. Int.*, **173**, 979–989

Farra V. and Pšenčík I., 2016. Weak-anisotropy approximations of P-wave phase and ray velocities for anisotropy of arbitrary symmetry. *Stud. Geophys. Geod.*, **60**, 403–418

Farra V. and Pšenčík I., 2023. PS reflection moveout in a homogeneous anisotropic layer of arbitrary symmetry and tilt. *J. Appl. Geophys.*, **215**, Art.No. 105062, https://doi.org/10.1016/j.jappgeo.2023.105062

Farra V., Pšenčík I. and Jílek P., 2016. Weak-anisotropy moveout approximations for P waves in homogeneous layers of monoclinic or higher anisotropy symmetries. *Geophysics*, **81**, C39–C59

Fedorov F.I., 1968. *Theory of Elastic Waves in Crystals*. Plenum Publ., New York

Gomes E., Zheng X., Pšenčík I., Horne S. and Leaney S., 2004. Local determination of weak anisotropy parameters from a walkaway VSP qP-wave data in the Java Sea region. *Stud. Geophys. Geod.*, **48**, 215–230

Klimeš L., 2006. Common-ray tracing and dynamic ray tracing for S waves in a smooth elastic anisotropic medium. *Stud. Geophys. Geod.*, **50**, 449–461

Mensch T. and Rasolofosaon P., 1997. Elastic-wave velocities in anisotropic media of arbitrary symmetry - generalization of Thomsens parameters ε, δ and γ. *Geophys. J. Int.*, **128**, 43–64

Pšenčík I. and Gajewski D., 1998. Polarization, phase velocity and NMO velocity of qP waves in arbitrary weakly anisotropic media. *Geophysics*, **63**, 1754–1766

Pšenčík I., Růžek B. and Jílek 2020. Practical concept of arbitrary anisotropy applied in traveltime inversion of simulated P-wave VSP data. *Geophysics*, **85**, C107–C123

Pšenčík I., Růžek B., Lokajíček T. and Svitek T., 2018. Determination of rock-sample anisotropy from P- and S-wave traveltime inversion. *Geophys. J. Int.*, **214**, 1088–1104

Sayers C.M., 1994. P-wave propagation in weakly anisotropic media. *Geophys. J. Int.*, **116**, 799–805

Thomsen L., 1986. Weak elastic anisotropy. *Geophysics*, **51**, 1954–1966

Tsvankin I., 1997. Anisotropic parameters and P-wave velocities for orthorhombic media. *Geophysics*, **62**, 1292–1309

Tsvankin I. and Grechka V., 2011. Seismology of azimuthally anisotropic media and seismic fracture characterization. Society of Exploration Geophysicists, Tulsa, OK

Zheng X. and Pšenčík I., 2002. Local determination of weak anisotropy parameters from the qP-wave slowness and particle motion measurements. *Pure Appl. Geophys.*, **159**, 1881–1905