The gravitational potential and its derivatives of a right rectangular prism with depth-dependent density following an n-th degree polynomial

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ABSTRACT

The direct gravity problem and its solution belong to the basis of the gravimetry. The solutions of this problem are well known for wide class of the source bodies with the constant density contrast. The non-uniform density approximation leads to the relatively complicated mathematical formalism. The analytical solutions for this type of sources are rare and currently these bodies are very useful in the gravimetrical modeling. The solution for the vertical component of the gravitational attraction vector for the 3D right rectangular prism is known in the geophysical literature for the density variations described by the 3-rd degree polynomial. We generalize this solution for an n-th degree, not only for the vertical component, but for the horizontal components, the second-order derivatives and the potential as well. The 2D modifications of all given formulae are presented, too. The presented general solutions, which involve a hypergeometric functions, can be used as they are, or as an auxiliary tool to derive desired solution for the given degree of the density polynomial as a sum of the elementary functions. The pros-and-cons of these approaches (the complexity of the programming codes, runtimes) are discussed, too.

Keywords: gravimetry, direct problem, variable density

1. INTRODUCTION

The right rectangular prism played and still plays an important role in the direct and inverse gravity problems (e.g., *Snopek and Casten, 2006*; *Shin et al., 2006*; *Dubey and Tiwary, 2016*; *Zhang and Jiang, 2017*). In the article by *Banerjee and Gupta (1977)*, a full derivation is presented for computing of the gravitational attraction caused by the rectangular prism with the constant density. The similar solution was developed earlier by *Sorokin (1951)*, but without explanation of how it was obtained in a low-accessible textbook. The formula of *Sorokin (1951)* is mentioned in the paper of *Nagy (1966)*, what makes this paper, paradoxically, the most cited one, where the solution of the gravitational attraction of the rectangular prism occurs, even if the solution of author himself is not properly derived and it is not used by a community. The one of the earliest solutions can be found in *Everest (1830, p. 97)* or *Bessel (1813)*, the solutions for the potential, the first, the second and the third-order derivatives as well, can be found together in *Mader (1951)*.

The constant density prism has been a useful tool in geophysical interpretation, but in many geological settings this simple density assumption does not hold. For example, the density of a sedimentary basins increases often exponentially with the depth. Moreover, the geological structure of a sedimentary basin can be complicated by other geological processes. Because of them, the simple constant-density cannot capture the complexities in the sedimentary basins. This problem can be managed in two basic ways: to approximate the non-constant density, which is varying from prism to prism or, to derive the solution for the prism with the depth-dependent density.

Only a few authors have treated the forward gravity problem for the case of non-constant density variations in the vertical direction. The solution for the case of the quadratic dependent density has been derived by *Bhaskara Rao (1990)*. The solution, which combines analytical and numerical methods of the integration, was provided by *García-Abdeslem (1992)*. Next, the solution for the case of the cubic dependent density has been derived by *García-Abdeslem (2005)*. Finally, *Zhang and Jiang (2017)* gave the general solution for the arbitrary polynomial degree, where all three components of the gravitational attraction vector occur. According to our opinion, their solution is quite complicated (in used symbolism, the appearance of the final solutions etc.), does not contain the solution for the potential and the second-order derivatives. More, what is quite important, these authors published, in fact very similar solution (in different symbolism) for the vertical component only, in another journal in the same time (!) (both papers were published online in May, 2017), see *Jiang et al.* (2017). According to our knowledge, there is no solution for the gravitational potential or the second-order derivatives for the prism with depth-dependent density.

In the present work, we derive the analytical solution to compute the gravitational potential and its first- and second-order derivatives caused by a rectangular prism with density, which varies as function of the depth following an *n*-th degree polynomial law. As example, the 3-rd degree polynomial in the case of the potential and the 4-th polynomial degree in the case of the first and the second-order derivatives are shown to demonstrate derived general solutions. The solutions for the 2D case (the prism with one dimension, mostly in the *y* direction, is enlarge to the plus/minus infinity) are presented, too. Despite the theoretical nature of the presented paper, we think that the practical application could be found in the future, e.g., in the modelling of the sedimentary basins.

2. FORMULATION OF THE PROBLEM

The gravitational potential observed at point $P(x, y, z)$ outside a 3D rectangular prism, in Cartesian coordinate system, is given by (e.g., *LaFehr and Nabighian, 2012, Eq. (5), p. 8*):

$$V(x, y, z) = G \iiint_{\xi\,\eta\,\zeta} \sigma(\xi, \eta, \zeta) \frac{1}{\left[(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2\right]^{1/2}} \, d\xi \, d\eta \, d\zeta \,, \quad (1)$$

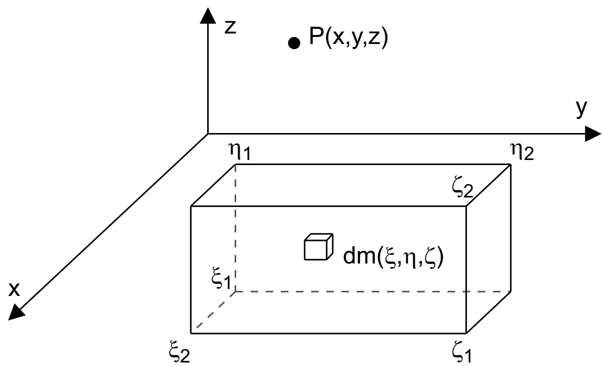

Fig. 1. The scheme of the mass element *dm* and 3D rectangular prism placed in Cartesian coordinate system with position of the calculation point *P*.

where the Greek letters are connected with the mass element, the Latin letters stands for coordinates of calculation point, $\sigma(\xi,\eta,\zeta)$ is the density function and G is the Newton's gravitational constant. The boundaries of the prism are ξ_i, η_i and ζ_i, $i = 1, 2$ (Fig. 1). The positive part of the vertical axis of the coordinate system is oriented upwards, while one can find the papers (e.g., *Bhaskara Rao, 1990*; *Shin et al., 2006*; *Dubey and Tiwary, 2016*), where it is oriented in "geological way" with positive part pointed downwards.

In our particular case we assume the density varies in vertical (ζ) direction, i.e. $\sigma(\xi,\eta,\zeta)=\sigma(\zeta)$. According to the possible practical applications, we will name it as a depth-dependent density. Most of the continuous function can be approximated by *n*-th degree of polynomial so, the discussed density function is then:

$$\sigma(\zeta)=\sum_{k=0}^{n} a_k \zeta^k \,. \tag{2}$$

Note, that the special case, where the density is not a continuous function, could be solved by dividing the prism into the parts, with the continuous density distribution.

By using the substitution $X=\xi-x$, $Y=\eta-y$, $Z=\zeta-z$, $d\xi=dX$, $d\eta=dY$, $d\zeta=dZ$, Eq. (1) can be written as:

$$V(x,y,\mathrm{z})=G\int_{X_1}^{X_2}\int_{Y_1}^{Y_2}\int_{Z_1}^{Z_2}\left(\sum_{k=0}^{n} a_k\left(Z+z\right)^k\right)\frac{1}{R}\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z\,, \tag{3}$$

where $R=\sqrt{X^2+Y^2+Z^2}$ and the integration limits are given by: $X_1=\xi_1-x$, $X_2=\xi_2-x$, $Y_1=\eta_1-y$, $Y_2=\eta_2-y$, $Z_1=\zeta_1-z$, $Z_2=\zeta_2-z$. The goal is to solve the last integral for any non-negative integer k. Finally, the gravitational attraction vector is then given by:

$$\boldsymbol{g} = \nabla V = \frac{\partial V}{\partial x}\boldsymbol{i} + \frac{\partial V}{\partial y}\boldsymbol{j} + \frac{\partial V}{\partial z}\boldsymbol{k} , \tag{4}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors in the directions of the coordinate axes. Practically, it is simpler to carry out the derivatives of integrated function in Eq. (3) firstly, and then finish the integration itself to obtain all the components of the gravitational attraction vector. We did our calculations in this manner, too.

To complete our task, the solutions for the second order derivatives of the gravitational potential are required, too. The five linearly independent components are:

$$V_{xx} = \frac{\partial^2 V}{\partial x^2} , \quad V_{yy} = \frac{\partial^2 V}{\partial x^2} , \quad V_{xy} = \frac{\partial^2 V}{\partial x \, \partial y} , \quad V_{xz} = \frac{\partial^2 V}{\partial x \, \partial z} , \quad V_{yz} = \frac{\partial^2 V}{\partial y \, \partial z} .$$

Note, that interchanging of the integration and differentiation is valid for the second and higher-order derivatives only, if the calculation point lies outside of the prism.

3. GRAVITATIONAL POTENTIAL

The integrals with respect to the variables X and Y are carried out firstly in Eq. (3) and we will obtain:

$$V = G \int_Z \left[\sum_{p=0}^{n} C_p Z^p \right] \left[Y \ln\left(X + R\right) + X \ln\left(Y + R\right) - Z \arctan \frac{XY}{ZR} \right]_{X,Y} dZ , \tag{5}$$

where we introduce the parameters C_p as:

$$C_p = \sum_{k=p}^{n} a_k \binom{k}{p} z^{k-p} , \tag{6}$$

i.e. for k-th degree of the density polynomial we will obtain a specified form of this constant, e.g., for $n = 5$, we will have:

$$\begin{aligned} C_0 &= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 , \\ C_1 &= a_1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 , \\ C_2 &= a_2 + 3a_3 z + 6a_4 z^2 + 10a_5 z^3 , \\ C_3 &= a_3 + 4a_4 z + 10a_5 z^2 , \\ C_4 &= a_4 + 5a_5 z , \\ C_5 &= a_5 . \end{aligned} \tag{7}$$

We can see, that integer coefficients in Eq. (7) are in a form of the Pascal triangle in the direction highlighted by an arrow. According to Eq. (5) we have to solve the following integrals:

$$I_1 = Y \int Z^p \ln\left(X + R\right) dZ , \quad I_2 = X \int Z^p \ln\left(Y + R\right) dZ , \quad I_3 = -\int Z^{p+1} \arctan \frac{XY}{ZR} dZ . \tag{8}$$

The per partes method is used in all three integrals and we have:

$$I_1 = \frac{Y}{p+1}\left[Z^{p+1}\ln\left(X+R\right)+X\int\frac{Z^{p+2}}{\left(Y^2+Z^2\right)R}\,dZ\right],$$

$$I_2 = \frac{X}{p+1}\left[Z^{p+1}\ln\left(Y+R\right)+Y\int\frac{Z^{p+2}}{\left(Y^2+Z^2\right)R}\,dZ\right], \tag{9}$$

$$I_3 = -\frac{1}{p+2}\left[Z^{p+2}\arctan\frac{XY}{ZR}+XY\int\frac{Z^{p+2}}{\left(Y^2+Z^2\right)R}\,dZ+XY\int\frac{Z^{p+2}}{\left(X^2+Z^2\right)R}\,dZ\right].$$

Note that terms which do not contain all three variables are already omitted, while they will be neglected after the boundaries substitution. We can see, that the task is reduced now to the solution of the typical integral:

$$\int\frac{Z^{p+2}}{\left(a^2+Z^2\right)R}\,dZ, \tag{10}$$

where $a = X^2 \vee a = Y^2$.

Next, we need the decomposition of $Z^{p+2}\Big/\left(a^2+Z^2\right)$ into the elementary fractions what will give us:

$$\frac{Z^{p+2}}{a^2+Z^2} = \begin{cases} \sum\limits_{k=0}^{p/2}\left(-1\right)^k Z^{p-2k}a^{2k} + \mathrm{i}^{p+2}\dfrac{a^{p+2}}{a^2+Z^2} & \text{if } p = 0, 2, 4, \ldots, \\ \sum\limits_{k=0}^{(p-1)/2}\left(-1\right)^k Z^{p-2k}a^{2k} + \mathrm{i}^{p+1}\dfrac{a^{p+1}Z}{a^2+Z^2} & \text{if } p = 1, 3, 5, \ldots. \end{cases} \tag{11}$$

We can see, that the situation depends on whether p is even or odd number. This will lead us to the if-else statements in the program realization. To unify the solution, we can join these two decompositions with help of new functions $f_1\left(p\right)$ and $f_2\left(p\right)$ which will contain the odd/even property:

$$\begin{aligned}\frac{Z^{p+2}}{a^2+Z^2} &= f_1\left(p\right)\left[\sum_{k=0}^{p/2}\left(-1\right)^k Z^{p-2k}a^{2k} + \frac{\mathrm{i}^{p+2}a^{p+2}}{a^2+Z^2}\right] \\ &+ f_2\left(p\right)\left[\sum_{k=0}^{(p-1)/2}\left(-1\right)^k Z^{p-2k}a^{2k} + \frac{\mathrm{i}^{p+1}a^{p+1}Z}{a^2+Z^2}\right],\end{aligned} \tag{12}$$

where

$$f_1\left(p\right) = \frac{\left(-1\right)^p+1}{2} = \begin{cases} 1 & \text{if } p = 0, 2, 4, \ldots, \\ 0 & \text{if } p = 1, 3, 5, \ldots, \end{cases}$$

and
$$f_2(p) = \frac{(-1)^{p+1}+1}{2} = \begin{cases} 0 & \text{if } p = 0, 2, 4, \ldots, \\ 1 & \text{if } p = 1, 3, 5, \ldots. \end{cases}$$

After substituting the last formula into the integrals I_1, I_2 and I_3 (Eq. (9)) and some rearranging, we will obtain the following integrals to be solved:

$$\int \frac{dZ}{\left(a^2+Z^2\right)R}, \quad \int \frac{Z\,dZ}{\left(a^2+Z^2\right)R}, \quad \int \frac{Z^{p-2k}}{R}\,dZ\,, \tag{13}$$

where $a = X^2 \vee a = Y^2$. Their solutions are in the Appendix A.

Now, after combining all the partial results, we can write the final solution for the gravitational potential of the rectangular prism with the depth-dependent density described by an n-th degree of the polynomial:

$$\begin{aligned} V = G \sum_{p=0}^{n} \frac{C_p}{(p+1)(p+2)} & \Bigg\{ (p+2) Z^{p+1} \left[X \ln(Y+R) + Y \ln(X+R) \right] - (p+1) Z^{p+2} \arctan\frac{XY}{ZR} \\ & + f_1(p) \Bigg[\sum_{k=0}^{p/2} \Bigg(\frac{(-1)^k \left(X^{2k}+Y^{2k}\right) XY}{p-2k+1} \frac{Z^{p-2k+1}}{\sqrt{X^2+Y^2}} \\ & \times {}_2F_1\left(\frac{1}{2}, \frac{p-2k+1}{2}; \frac{p-2k+3}{2}; -\frac{Z^2}{X^2+Y^2} \right) \Bigg) \\ & + \mathrm{i}^{p+2} \cdot \left(X^{p+2} \arctan\frac{YZ}{XR} + Y^{p+2} \arctan\frac{XZ}{YR} \right) \Bigg] \\ & + f_2(p) \Bigg[\sum_{k=0}^{(p-1)/2} \Bigg(\frac{(-1)^k \left(X^{2k}+Y^{2k}\right) XY}{p-2k+1} \frac{Z^{p-2k+1}}{\sqrt{X^2+Y^2}} \\ & \cdot {}_2F_1\left(\frac{1}{2}, \frac{p-2k+1}{2}; \frac{p-2k+3}{2}; -\frac{Z^2}{X^2+Y^2} \right) \Bigg) \\ & + \mathrm{i}^{p-1} \cdot \left(X^{p+2} \ln(Y+R) + Y^{p+2} \ln(X+R) \right) \Bigg] \Bigg\}_{X_1, Y_1, Z_1}^{X_2, Y_2, Z_2}, \end{aligned} \tag{14}$$

where $f(t) = {}_2F_1(a,b,c,t)$ is the Gauss's hypergeometric function. This function has closed analytical expression for each p and k in our case. As example, the solution for the 3-rd degree of the density polynomial $\left(\sigma(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3\right)$ is presented in the Appendix B. The modification for the 2D case is presented in the Appendix E.

4. GRAVITATIONAL ATTRACTION

Substituting the result for the potential (Eq. (14)) into Eq. (4), we will obtain the formula for the gravitational attraction. Practically, it is more convenient to make the derivative of Eq. (1) firstly, and then carry out the integration (although both approaches lead to the same result). The partial results from the previous solution will be helpful. The components of the gravitational attraction vector are:

$$V_x = \frac{\partial V}{\partial x} = G \int_\xi \int_\eta \int_\zeta \sigma(\xi,\eta,\zeta) \frac{(\xi - x)}{\left[(\xi-x)^2 + (\eta-y)^2 + (\zeta-z)^2\right]^{3/2}} d\xi \, d\eta \, d\zeta \,, \quad (15)$$

$$V_y = \frac{\partial V}{\partial y} = G \int_\xi \int_\eta \int_\zeta \sigma(\xi,\eta,\zeta) \frac{(\eta - y)}{\left[(\xi-x)^2 + (\eta-y)^2 + (\zeta-z)^2\right]^{3/2}} d\xi \, d\eta \, d\zeta \,, \quad (16)$$

$$V_z = \frac{\partial V}{\partial z} = G \int_\xi \int_\eta \int_\zeta \sigma(\xi,\eta,\zeta) \frac{(\zeta - z)}{\left[(\xi-x)^2 + (\eta-y)^2 + (\zeta-z)^2\right]^{3/2}} d\xi \, d\eta \, d\zeta \,. \quad (17)$$

One can find the statements of some authors, that gravitational effect of the body and component V_z are synonymous, or that gravity-meters are measuring V_Z. This is not true, while it is the size of gravitational attraction vector $|\boldsymbol{g}|$ what is measured quantity with use of the gravity-meters. Even if the horizontal components can be small in the practice, in the direct problem branch we should not to be restricted to the just one component of the gravitational attraction vector. The integrals (15)–(17) have some properties in common (one can find kind of the symmetry in them, V_y is, in fact, just the variation of V_x) what will make the solution search less complicated. We assume the same density distribution and the substitution of the variables as in the case of the potential. After the integration with respect to the variables X and Y we have:

$$V_x = -G \sum_{p=0}^{n} \frac{C_p}{p+1} \left[Z^{p+1} \ln(Y+R) + Y \int_Z \frac{Z^{p+2}}{\left(X^2+Z^2\right)R} dZ \right]_{X,Y} ,$$

$$V_y = -G \sum_{p=0}^{n} \frac{C_p}{p+1} \left[Z^{p+1} \ln(X+R) + X \int_Z \frac{Z^{p+2}}{\left(Y^2+Z^2\right)R} dZ \right]_{X,Y} ,$$

$$V_z = G \sum_{p=0}^{n} \frac{C_p}{p+1} \left[Z^{p+1} \arctan \frac{XY}{ZR} + XY \int_Z \frac{Z^{p+1}}{\left(Y^2+Z^2\right)R} dZ + XY \int_Z \frac{Z^{p+1}}{\left(X^2+Z^2\right)R} dZ \right]_{X,Y} . \quad (18)$$

We can see that these integrals are the same (or close to) as were solved in the case of the potential. With help of Eq. (11) we can write for V_x and V_y:

$$\begin{aligned} V_x = -G \sum_{p=0}^{n} \frac{C_p}{p+1} \Bigg\{ & Z^{p+1} \ln\left(Y+R\right) + f_1\left(p\right) \left[\sum_{k=0}^{p/2} \left(\frac{(-1)^k Z^{p-2k+1}}{p-2k+1} \frac{X^{2k} Y}{\sqrt{X^2+Y^2}} \right.\right. \\ & \left.\left. \times {}_2F_1\left(\frac{1}{2}, \frac{p-2k+1}{2}; \frac{p-2k+3}{2}; -\frac{Z^2}{X^2+Y^2} \right) \right) - \mathrm{i}^p X^{p+1} \arctan\frac{YZ}{XR} \right] \\ & + f_2\left(p\right) \left[\sum_{k=0}^{(p-1)/2} \left(\frac{(-1)^k Z^{p-2k+1}}{p-2k+1} \frac{X^{2k} Y}{\sqrt{X^2+Y^2}} \right.\right. \\ & \left.\left. \times {}_2F_1\left(\frac{1}{2}, \frac{p-2k+1}{2}; \frac{p-2k+3}{2}; -\frac{Z^2}{X^2+Y^2} \right) \right) + \mathrm{i}^{p-1} X^{p+1} \ln\left(Y+R\right) \right] \Bigg\}_{X_1,Y_1,Z_1}^{X_2,Y_2,Z_2} , \end{aligned} \tag{19}$$

$$\begin{aligned} V_y = -G \sum_{p=0}^{n} \frac{C_p}{p+1} \Bigg\{ & Z^{p+1} \ln\left(X+R\right) + f_1\left(p\right) \left[\sum_{k=0}^{p/2} \left(\frac{(-1)^k Z^{p-2k+1}}{p-2k+1} \frac{X Y^{2k}}{\sqrt{X^2+Y^2}} \right.\right. \\ & \left.\left. \times {}_2F_1\left(\frac{1}{2}, \frac{p-2k+1}{2}; \frac{p-2k+3}{2}; -\frac{Z^2}{X^2+Y^2} \right) \right) - \mathrm{i}^p Y^{p+1} \arctan\frac{YZ}{XR} \right] \\ & + f_2\left(p\right) \left[\sum_{k=0}^{(p-1)/2} \left(\frac{(-1)^k Z^{p-2k+1}}{p-2k+1} \frac{X Y^{2k}}{\sqrt{X^2+Y^2}} \right.\right. \\ & \left.\left. \times {}_2F_1\left(\frac{1}{2}, \frac{p-2k+1}{2}; \frac{p-2k+3}{2}; -\frac{Z^2}{X^2+Y^2} \right) \right) + \mathrm{i}^{p-1} Y^{p+1} \ln\left(X+R\right) \right] \Bigg\}_{X_1,Y_1,Z_1}^{X_2,Y_2,Z_2} , \end{aligned} \tag{20}$$

The vertical component contains the integrals we did not solve yet. It is clear we need the decomposition of the fraction $Z^{p+1} \big/ \left(a^2 + Z^2\right)$, where $a = X^2 \vee a = Y^2$, what is:

$$\frac{Z^{p+1}}{a^2+Z^2} = \begin{cases} \displaystyle\sum_{k=0}^{p/2} (-1)^k \operatorname{sgn}\left(p-2k\right) Z^{p-2k-1} a^{2k} + \mathrm{i}^p \frac{a^p Z}{a^2+Z^2} & \text{if } p = 0, 2, 4 \ldots, \\ \displaystyle\sum_{k=0}^{(p-1)/2} (-1)^k Z^{p-2k-1} a^{2k} + \mathrm{i}^{p+1} \frac{a^{p+1}}{a^2+Z^2} & \text{if } p = 1, 3, 5 \ldots. \end{cases} \tag{21}$$

Now, we can write the solution for V_z:

$$
\begin{aligned}
V_z = G \sum_{p=0}^{n} \frac{C_p}{p+1} \Bigg\{ & Z^{p+1} \arctan \frac{XY}{ZR} \\
& + f_1(p) \left[\sum_{k=0}^{p/2} \left(\frac{(-1)^k \operatorname{sgn}(p-2k) XY}{p-2k+1-\operatorname{sgn}(p-2k)} \frac{\left(X^{2k}+Y^{2k}\right) Z^{p-2k}}{\sqrt{X^2+Y^2}} \right.\right. \\
& \left.\left. \times {}_2F_1\left(\frac{1}{2}, \frac{p-2k}{2}; \frac{p-2k+2}{2}; \frac{-Z^2}{X^2+Y^2} \right) \right) - \mathrm{i}^p \left(X^{p+1} \ln(Y+R) + Y^{p+1} \ln(X+R) \right) \right] \\
& + f_2(p) \left[\sum_{k=0}^{(p-1)/2} \left(\frac{(-1)^k XY}{p-2k+1-\operatorname{sgn}(p-2k)} \frac{\left(X^{2k}+Y^{2k}\right) Z^{p-2k}}{\sqrt{X^2+Y^2}} \right.\right. \\
& \left. \times {}_2F_1\left(\frac{1}{2}, \frac{p-2k}{2}; \frac{p-2k+2}{2}; \frac{-Z^2}{X^2+Y^2} \right) \right) \\
& \left.\left. + \mathrm{i}^{p-1} \left(X^{p+1} \arctan \frac{YZ}{XR} + Y^{p+1} \arctan \frac{XZ}{YR} \right) \right] \Bigg\} \Bigg|_{X_1, Y_1, Z_1}^{X_2, Y_2, Z_2} ,
\end{aligned}
\tag{22}
$$

The examples of given components of the gravitational attraction vector for the 4-th degree of the density polynomial are in the Appendix C and solution for the 2D case is placed in the Appendix E. The reader can check, that the solution for the third degree of the density polynomial is the same as one presented in *García-Abdeslem (2005)*.

5. SECOND-ORDER DERIVATIVES

This task will give us, after all the initial processes as in the previous cases, i.e. the substitution and the integration with the respect to the variables X and Y, the following integrals:

$$
\begin{aligned}
& V_{xx} = -G \sum_{p=0}^{n} C_p \left[XY \int_Z \frac{Z^p}{\left(X^2+Z^2\right) R} dZ \right]_{X,Y} , \\
& V_{yy} = -G \sum_{p=0}^{n} C_p \left[XY \int_Z \frac{Z^p}{\left(Y^2+Z^2\right) R} dZ \right]_{X,Y} , \\
& V_{xy} = G \sum_{p=0}^{n} C_p \left[\int_Z \frac{Z^p}{R} dZ \right]_{X,Y} , \quad V_{xz} = -G \sum_{p=0}^{n} C_p \left[Y \int_Z \frac{Z^{p+1}}{\left(X^2+Z^2\right) R} dZ \right]_{X,Y} , \\
& V_{yz} = -G \sum_{p=0}^{n} C_p \left[X \int_Z \frac{Z^{p+1}}{\left(Y^2+Z^2\right) R} dZ \right]_{X,Y} , \quad V_{zz} = -\left(V_{xx}+V_{yy}\right) .
\end{aligned}
\tag{23}
$$

There is only one expression we did not use yet - the fraction $Z^p \big/ \left(a^2 + Z^2\right)$, where $a = X^2 \vee a = Y^2$ and the required expression is the decomposition of this fraction into the elementary parts, which is:

$$\frac{Z^p}{a^2 + Z^2} = \begin{cases} \sum\limits_{k=0}^{p/2} \left[(-1)^k \operatorname{sgn}(p-2k) Z^{p-2k-2} a^{2k} + \mathrm{i}^p \dfrac{a^p}{a^2+Z^2} \right], \; p = 0,2,\ldots, \\ \sum\limits_{k=0}^{(p-1)/2} \left[(-1)^k \operatorname{sgn}(p-2k-1) Z^{p-2k-1} a^{2k} + \mathrm{i}^{p-1} \dfrac{Z a^{p-1}}{a^2+Z^2} \right], \; p = 1,3,\ldots. \end{cases} \tag{24}$$

According to this property and to the previous results we can write:

$$\begin{aligned} V_{xx} = G \sum_{p=0}^{n} C_p \Bigg\{ & f_1(p) \Bigg[\sum_{k=0}^{p/2} \Bigg(\frac{(-1)^k \operatorname{sgn}(p-2k)(-XY)}{p-2k-\operatorname{sgn}(p-2k-1)} \frac{X^{2k} Z^{p-2k-1}}{\sqrt{X^2+Y^2}} \\ & \times {}_2F_1\left(\frac{1}{2}, \frac{p-2k-1}{2}; \frac{p-2k+1}{2}; \frac{-Z^2}{X^2+Y^2}\right)\Bigg) - \mathrm{i}^p X^p \arctan \frac{YZ}{XR} \Bigg] \\ & + f_2(p) \Bigg[\sum_{k=0}^{(p-1)/2} \Bigg(\frac{(-1)^k \operatorname{sgn}(p-2k-1)(-XY)}{p-2k-\operatorname{sgn}(p-2k-1)} \frac{X^{2k} Z^{p-2k-1}}{\sqrt{X^2+Y^2}} \\ & \times {}_2F_1\left(\frac{1}{2}, \frac{p-2k-1}{2}; \frac{p-2k+1}{2}; \frac{-Z^2}{X^2+Y^2}\right)\Bigg) + \mathrm{i}^{p-1} X^p \ln(Y+R) \Bigg] \Bigg\} \Bigg|_{X_1, Y_1, Z_1}^{X_2, Y_2, Z_2}, \end{aligned} \tag{25}$$

$$\begin{aligned} V_{yy} = G \sum_{p=0}^{n} C_p \Bigg\{ & f_1(p) \Bigg[\sum_{k=0}^{p/2} \Bigg(\frac{(-1)^k \operatorname{sgn}(p-2k)(-XY)}{p-2k-\operatorname{sgn}(p-2k-1)} \frac{Y^{2k} Z^{p-2k-1}}{\sqrt{X^2+Y^2}} \\ & \times {}_2F_1\left(\frac{1}{2}, \frac{p-2k-1}{2}; \frac{p-2k+1}{2}; \frac{-Z^2}{X^2+Y^2}\right)\Bigg) - \mathrm{i}^p Y^p \arctan \frac{XZ}{YR} \Bigg] \\ & + f_2(p) \Bigg[\sum_{k=0}^{(p-1)/2} \Bigg(\frac{(-1)^k \operatorname{sgn}(p-2k-1)(-XY)}{p-2k-\operatorname{sgn}(p-2k-1)} \frac{Y^{2k} Z^{p-2k-1}}{\sqrt{X^2+Y^2}} \\ & \times {}_2F_1\left(\frac{1}{2}, \frac{p-2k-1}{2}; \frac{p-2k+1}{2}; \frac{-Z^2}{X^2+Y^2}\right)\Bigg) + \mathrm{i}^{p-1} Y^p \ln(X+R) \Bigg] \Bigg\} \Bigg|_{X_1, Y_1, Z_1}^{X_2, Y_2, Z_2}, \end{aligned} \tag{26}$$

$$\begin{aligned} V_{xz} = G \sum_{p=0}^{n} C_p \Bigg\{ f_1(p) \Bigg[\sum_{k=0}^{p/2} \Bigg(\frac{(-1)^k \operatorname{sgn}(p-2k)(-Y)}{p-2k+1-\operatorname{sgn}(p-2k)} \frac{X^{2k} Z^{p-2k}}{\sqrt{X^2+Y^2}} \\ \times {}_2F_1\left(\frac{1}{2}, \frac{p-2k}{2}; \frac{p-2k+2}{2}; \frac{-Z^2}{X^2+Y^2}\right)\Bigg) + \mathrm{i}^p X^p \ln(Y+R) \Bigg] \\ + f_2(p) \Bigg[\sum_{k=0}^{(p-1)/2} \Bigg(\frac{(-1)^k (-Y)}{p-2k+1-\operatorname{sgn}(p-2k)} \frac{X^{2k} Z^{p-2k}}{\sqrt{X^2+Y^2}} \\ \times {}_2F_1\left(\frac{1}{2}, \frac{p-2k}{2}; \frac{p-2k+2}{2}; \frac{-Z^2}{X^2+Y^2}\right)\Bigg) - \mathrm{i}^{p+1} X^p \arctan\frac{YZ}{XR} \Bigg] \Bigg\}_{X_1, Y_1, Z_1}^{X_2, Y_2, Z_2} , \end{aligned} \tag{27}$$

$$\begin{aligned} V_{yz} = G \sum_{p=0}^{n} C_p \Bigg\{ f_1(p) \Bigg[\sum_{k=0}^{p/2} \Bigg(\frac{(-1)^k \operatorname{sgn}(p-2k)(-X)}{p-2k+1-\operatorname{sgn}(p-2k)} \frac{Y^{2k} Z^{p-2k}}{\sqrt{X^2+Y^2}} \\ \times {}_2F_1\left(\frac{1}{2}, \frac{p-2k}{2}; \frac{p-2k+2}{2}; \frac{-Z^2}{X^2+Y^2}\right)\Bigg) + \mathrm{i}^p Y^p \ln(X+R) \Bigg] \\ + f_2(p) \Bigg[\sum_{k=0}^{(p-1)/2} \Bigg(\frac{(-1)^k \operatorname{sgn}(p-2k-1)(-X)}{p-2k+1-\operatorname{sgn}(p-2k)} \frac{Y^{2k} Z^{p-2k}}{\sqrt{X^2+Y^2}} \\ \times {}_2F_1\left(\frac{1}{2}, \frac{p-2k}{2}; \frac{p-2k+2}{2}; \frac{-Z^2}{X^2+Y^2}\right)\Bigg) - \mathrm{i}^{p+1} Y^p \arctan\frac{XZ}{YR} \Bigg] \Bigg\}_{X_1, Y_1, Z_1}^{X_2, Y_2, Z_2} , \end{aligned} \tag{28}$$

$$V_{xy} = G \sum_{p=0}^{n} C_p \left[\frac{Z^{p+1}}{(p+1)\sqrt{X^2+Y^2}} \cdot {}_2F_1\left(\frac{1}{2}, \frac{p+1}{2}; \frac{p+3}{2}; -\frac{Z^2}{X^2+Y^2}\right) \right]_{X_1, Y_1, Z_1}^{X_2, Y_2, Z_2} . \tag{29}$$

The solutions for the 4-th degree of the density polynomial are presented in the Appendix D. While the 2D prism was/is widely used in gravimetry, mainly in the modelling of the various geological situations, the formulae for this special case are placed in the Appendix E.

Note that the presented formulae for the potential and its first derivatives are fully valid even for the calculation points located inside of the prism. The formulae for the second derivatives are fully valid outside of the prism, while the changing of the integration and the differentiation order is possible only there.

To avoid the problems with the apparent singularities on the surface of the prism and in other (source-free) points, indicated in *Nagy et al. (2000),* we add a small number, named ε, to the coordinates of such calculation points to make a little shift of them outside of the crucial positions. The numerical tests of the presented formulae show us that it is enough to set $\varepsilon \approx 10^{-5}$ m. Another possibility how to avoid such problems is, e.g., the proper choosing of the calculation step a.o.

6. PROGRAM REALIZATION AND VERIFICATION OF THE FORMULAE

Some important issues has to be mentioned here. Some of the presented formulae cannot be instantly programmed as they are. The reason for this are the upper limits of the summations in them, e.g., if the p is even number, one of these limits are not integers. This is problem for some programming environments as, e.g., MatLab, while in some, e.g., in Mathematica system, this is not a complication. Anyway, these problems can be easily avoided by proper programming, if the parts for even and odd number p are programmed separately, as we did. More, if realized in this manner, the functions $f_1(p)$ and $f_2(p)$ are not necessary any more. While we did use the MatLab environment to the program realization of the presented formulae, one can face the problem with calculation of the Gauss's hypergeometric function with the help of a built-in hypergeom function. This function is available only in the MatLab Symbolic Toolbox and is not the part of a basic installation (it must be purchased separately). More, this function works really slowly. Fortunately, the MatLab community is wide and one can find the realizations of the hypergeometric function which are for free and work far faster than a built-in function. Of course, the validity and the accuracy of these codes must be checked and tested precisely (but this is beyond the scope of the paper).

Another way out of the problems with program realizations of the presented solutions could be as follows: just use the general solutions (Eqs (14), (19), (20), (22) and (25)–(29)) to produce the required expressions with the elementary functions, as presented in the Appendices B, C and D. This means to rewrite the hypergeometric function into the summation of the elementary functions for the input integers p and k. The programming codes for such summation work far faster, but, on the other hand, are more complicated.

To validate the analytic solution we used the density values of the sedimentary rocks obtained from 4 boreholes situated in the Neogene Danube Basin (SW Slovakia, Central Europe region) which, if displayed in the density vs. depth graph, creates a density cross-plot (Fig. 2). We fit this data (used the LSQ method) with the 5-th degree of polynomial to obtain some reasonable coefficients a_i for the modelling purposes. The coefficients of the fitting polynomial (density function) are: $a_0 = 1743.957$, $a_1 = -0.768$, $a_2 = -0.722 \times 10^{-3}$, $a_3 = -5.097 \times 10^{-7}$, $a_4 = -1.685 \times 10^{-10}$, $a_5 = -2.017 \times 10^{-14}$.

As we can see from the Fig. 2, the density values starts at $z = -600$ m. While the task is just to verify the derived formulae, we did use the obtained density function even for the rest of a depth values from $z = 0$ m, while the density values for this depth interval are reasonable, too. Note, that used density field could be approximated by a lower degree of the polynomial, too, but the aim is to verify the derived formulae, so we decided to use the fifth degree.

The potential and its derivatives were computed along the profile identical with the x axis on the surface ($z = 0$ m). These results were compared with the layered model, i.e. the body was divided into the system of the layers with the constant density, which are

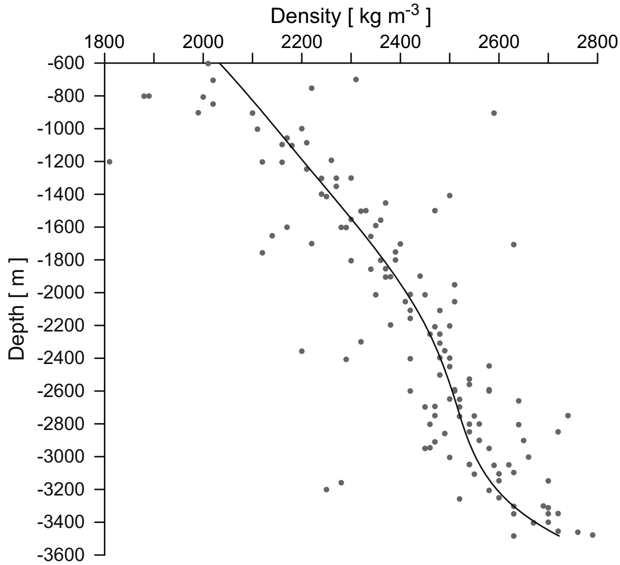

Fig. 2. The density field (dots, measured density values from wells SP-1, SP-3, SP-4 and SP-5, Danube Basin, Slovakia) with fitting polynomial (5-th degree, solid curve).

changed in the steps to follow the given density distribution. The source is placed symmetrically below the origin of the coordinate system - see Table 1 for the parameters. In the described modeling process this source was divided into the 100, 150 and 200 layers. The size of each one in the vertical direction was then 40, 26.66 or 20 m, respectively. The (constant) density of each layer was calculated for the center of it, with help of the previously obtained density polynomial. This testing were made for the all derived formulae. Next, the differences between layered model and the compact model were calculated. As example, the difference between V_z curves (3D case) is displayed (Fig. 3). We can see, that, in this particular example, it is necessary to have about 100 layers to obtain the difference below 0.005 mGal (what is an accuracy of the latest gravity-meters).

To demonstrate the speed problem connected with the MatLab built-in hypergeom function we did calculations of the V_z component (and for all other presented formulae as well) for the presented model according to the Eq. (22) and next, we did calculations with help of the elementary functions as demonstrated in the Appendix B (i.e. the hypergeometric function was rewritten as the summation of the elementary functions).

Table 1. Parameters of the 3D prism model.

Upper boundary	$\xi_2 = 500$ m	$\eta_2 = 500$ m	$\zeta_2 = 0$ m
Lower boundary	$\xi_1 = -500$ m	$\eta_1 = -500$ m	$\zeta_1 = -4000$ m

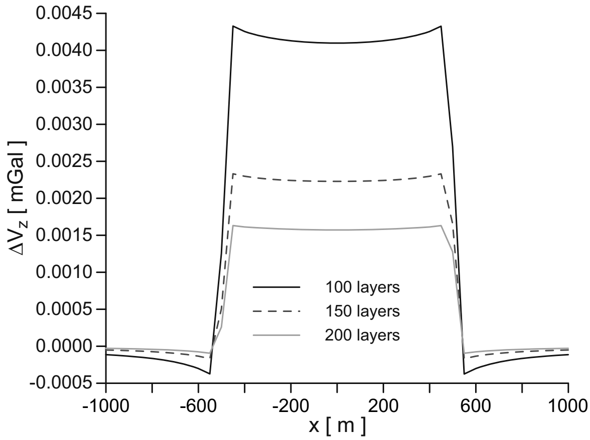

Fig. 3. Differences in vertical component of the gravitational effect (V_z) for the layered model with the constant density in each layer and the analytical solution with the density distribution given by polynomial of the 5-th degree.

Next, the calculation times were compared. Despite the fact, that used computer was “warhorse” (Intel Core Duo 2.8 GHz, 8 GB RAM) the difference in the calculation times is enormous: the formula with the built-in hypergeom function requires almost 4 s (!) to produce the final value for the one calculation point, in the comparison with 0.3 s required for the 100 calculation points, if the solution with the elementary functions is used - what is about 1300 times faster (for the one point). While the relative runtimes (the ratio between runtimes for the both approaches) are highly dependent of a computer used, they can be lower if a more powerful technology is used. Another possibility is to find the faster program realization of the hypergeometric function in the future to make the derived solutions more useful. The possible approach could be the usage of the integral representation or the continuous fractions representation of this function. While the ${}_2F_1(a, b; c; x)$ is the one, among the family of the hypergeometric functions, which arises most frequently in the physical problems, it occurs as a built-in function in the other languages, too. So, we can assume, that the speed problem could not be common throughout the programming environments.

Note that solutions for the 2D case does not suffer with such problems, while they consist only of the elementary functions.

7. CONCLUSION

The derived formulae for the gravitational attraction of a right rectangular prism with the density, approximated by an n-th polynomial (2D case is solved, too) is a new contribution to the direct problem formalism in the gravimetry. It generalizes the linear, quadratic and the cubic approximations of the density variations with the depth, which have been published before (e.g., *Bhaskara Rao, 1990*; *García-Abdeslem, 1992, 2005*). In the special situations it can be used to describe the change of the selected sequences of the sedimentary basin filling and it can be utilized during the stripping procedures in the applied gravimetry, e.g., *Bielik et al. (2013)*. The derived formulae can be used in two ways - to be programmed as described above, or to be used to write down the solutions for the given density polynomial degree by rewriting the hypergeometric component with the help of the elementary functions. The applicability of the given general expressions depends of the complexity of the density distribution within the model body and a desired accuracy. The polynomial function can approximated large set of the given continuous density distribution functions which are used to describe the density in e.g. sedimentary filling (e.g., cubic, hyperbolic, exponential etc.). As demonstrated in the comparison between a layered model and a general solution, the accuracy in the case of complicated density distributions, can be improved, too. The main complication connected with the first approach can be the calculation time. We used the MatLab environment and a built-in codes for the hypergeometric functions are slow. Despite the low calculation speed, the programming and the final code of obtained general formulae, is simpler. The second approach will result into faster calculations, but the formulae are getting complicated as the degree of the density polynomial goes up. The selection between these two approaches depends on the necessity of the solutions with the high degree of the density polynomial.

The presented approach, we used to find the solutions of this particular direct problem, can be used even in other cases, e.g., polynomial density distribution in the horizontal directions or in the vertical and horizontal direction combined, the third-order derivations, etc.

APPENDIX A

The task is to solve the following integrals:

$$\int \frac{dZ}{\left(a^2+Z^2\right)R}, \quad \int \frac{Z\,dZ}{\left(a^2+Z^2\right)R}, \quad \int \frac{Z^{p-2k}}{R}\,dZ$$

The first one can be solved through the substitution, e.g., *Gradshteyn and Ryzhik (1962)*:

$$v = \frac{t}{\sqrt{t^2+b}} \quad \rightarrow \quad t^2 = \frac{v^2 b}{1-v^2} \quad \rightarrow \quad dt = \frac{bv}{t\left(1-v^2\right)^2}dv = \frac{\sqrt{v}}{\left(1-v^2\right)^{3/2}}dv\,.$$

Finally, we have:

$$\int \frac{dZ}{\left(a^2+Z^2\right)R} = \begin{cases} \dfrac{1}{XY}\arctan\dfrac{YZ}{XR} & \text{if } a = X^2\,, \\ \dfrac{1}{XY}\arctan\dfrac{XZ}{YR} & \text{if } a = Y^2\,. \end{cases}.$$

The second integration is solvable by a simple substitution and yields:

$$\int \frac{Z\,dZ}{\left(a^2+Z^2\right)R} = \begin{cases} -\dfrac{1}{Y}\operatorname{arctanh}\dfrac{R}{Y} & \text{if } a = X^2, \\ -\dfrac{1}{X}\operatorname{arctanh}\dfrac{R}{X} & \text{if } a = Y^2. \end{cases}$$

When the integration boundaries are substituted into these two results, they can be simplified to:

$$-\frac{1}{Y}\operatorname{arctanh}\frac{R}{Y} = -\frac{1}{Y}\ln\left(Y+R\right), \quad -\frac{1}{X}\operatorname{arctanh}\frac{R}{X} = -\frac{1}{Y}\ln\left(X+R\right).$$

The last integration is more complicated. Firstly, we use the identity (*Wolfram Research, 2017a*):

$$\frac{1}{\sqrt{1+Z^2}} = {}_2F_1\left[\frac{1}{2}, b; b; -Z^2\right],$$

where ${}_2F_1\left[a, b; c; z\right]$ is the Gauss's hypergeometric function (for the definition of used hypergeometric functions, see below). This relation and the per partes method lead us to the integral (*Wolfram Research, 2017b*):

$$\int z^{\alpha-1}\, {}_2F_1\left[a, b; c; z\right] dz = \frac{z^{\alpha}}{\alpha}\, {}_3F_2\left[a, b, \alpha; c, \alpha+1; z\right],$$

where ${}_3F_2\left[a, b, c; d, e; z\right]$ is the generalized hypergeometric function. In our particular case we obtained ${}_3F_2\left[1/2, 1, c; 1, e; z\right]$ what can be, directly from the definition of the generalized hypergeometric function, simplified to:

$${}_3F_2\left[\frac{1}{2}, 1, c; 1, e; z\right] = {}_2F_1\left[\frac{1}{2}, c; e; z\right].$$

According to these relations, we can write the desired solution of our integration as:

$$\int \frac{Z^{p-2k}}{R}\,dZ = \frac{Z^{p-2k+1}}{p-2k+1}\cdot\frac{1}{\sqrt{X^2+Y^2}}\cdot {}_2F_1\left[\frac{1}{2}, \frac{p-2k+1}{2}; \frac{p-2k+3}{2}; -\frac{Z^2}{X^2+Y^2}\right].$$

Note: The definitions of the generalized hypergeometric functions are (e.g., *Gradshteyn and Ryzhik (1962)*):

$${}_3F_2\left[a_1, a_2, a_3; b_1, b_2; x\right] = \sum_{k=0}^{\infty} \frac{\left(a_1\right)_k \left(a_2\right)_k \left(a_3\right)_k x^k}{\left(b_1\right)_k \left(b_2\right)_k k!},$$

$$ {}_2F_1\left[a_1, a_2; b_1; x\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k x^k}{(b_1)_k k!} , $$

where $(a)_k$ is the Pochhammer symbol.

APPENDIX B

The gravitational potential for the density polynomial of the 3-rd degree $\left(\sigma(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3\right)$ is:

$$ \begin{aligned} V(x, y, z) = G\Bigg\{ & C_0\left[XZ\ln(Y+R) + YZ\ln(X+R) + XY\ln(Z+R) - \frac{X^2}{2}\arctan\frac{YZ}{XR} \right. \\ & \left. - \frac{Y^2}{2}\arctan\frac{XZ}{YR} - \frac{Z^2}{2}\arctan\frac{XY}{ZR} \right] \\ & + C_1\left[\frac{Y\left(3Z^2+Y^2\right)}{6}\ln(X+R) + \frac{X\left(3Z^2+X^2\right)}{6}\ln(Y+R) - \frac{Z^3}{3}\mathit{arctg}\frac{XY}{ZR} + \frac{XYR}{3} \right] \\ & + C_2\left[\frac{YZ^3}{3}\ln(X+R) + \frac{XZ^3}{3}\ln(Y+R) - \frac{XY}{6}\left(X^2+Y^2\right)\ln(Z+R) \right. \\ & \left. - \frac{Z^4}{4}\arctan\frac{XY}{ZR} + \frac{Y^4}{12}\arctan\frac{XZ}{YR} + \frac{X^4}{12}\arctan\frac{YZ}{XR} + \frac{XYZR}{12} \right] \\ & + C_3\left[\frac{Y}{20}\left(5Z^4 - Y^4\right)\ln(X+R) + \frac{X}{20}\left(5Z^4 - X^4\right)\ln(Y+R) \right. \\ & \left. \left. - \frac{Z^5}{5}\arctan\frac{YX}{ZR} + \frac{XYR}{60}\left[2Z^2 - 7\left(X^2+Y^2\right)\right] \right] \Bigg\} \Bigg|_{X_1, Y_1, Z_1}^{X_2, Y_2, Z_2} . \end{aligned} $$

APPENDIX C

The components of vector $\boldsymbol{g}$ for the 4-th degree of the density polynomial are:

$$V_x = -G\left\{ C_0 \left[Z \ln(Y+R) + Y \ln(Z+R) - X \arctan\frac{YZ}{XR} \right] \right.$$
$$+ C_1 \left[\frac{X^2+Z^2}{2} \ln(Y+R) + \frac{YR}{2} \right]$$
$$+ C_2 \left[\frac{Z^3}{3} \ln(Y+R) - \frac{Y}{6}\left(3X^2+Y^2\right) \ln(Z+R) + \frac{X^3}{3} \arctan\frac{YZ}{XR} + \frac{YZR}{6} \right]$$
$$+ C_3 \left[\frac{Z^4 - X^4}{4} \ln(Y+R) - \frac{YR\left(5X^2+2Y^2-Z^2\right)}{12} \right] ,$$
$$+ C_4 \left[\frac{Z^5}{5} \ln(Y+R) - \frac{X^5}{5} \arctan\frac{YZ}{XR} + \frac{YZR\left(2Z^2-7X^2-3Y^2\right)}{40} \right.$$
$$\left.\left. + \frac{Y}{40}\left(15X^4+10X^2Y^2+3Y^4\right) \ln(Z+R) \right] \right\}_{X_1,Y_1,Z_1}^{X_2,Y_2,Z_2} ,$$

$$V_y = -G\left\{ C_0 \left[Z \ln(X+R) + X \ln(Z+R) - Y \arctan\frac{XZ}{YR} \right] \right.$$
$$+ C_1 \left[\frac{Y^2+Z^2}{2} \ln(X+R) + \frac{XR}{2} \right]$$
$$+ C_2 \left[\frac{Z^3}{3} \ln(X+R) - \frac{X}{6}\left(3Y^2+X^2\right) \ln(Z+R) + \frac{Y^3}{3} \arctan\frac{XZ}{YR} + \frac{XZR}{6} \right]$$
$$+ C_3 \left[\frac{Z^4 - Y^4}{4} \ln(X+R) - \frac{XR\left(5Y^2+2X^2-Z^2\right)}{12} \right] ,$$
$$+ C_4 \left[\frac{Z^5}{5} \ln(X+R) - \frac{Y^5}{5} \arctan\frac{XZ}{YR} + \frac{XZR\left(2Z^2-7Y^2-3X^2\right)}{40} \right.$$
$$\left.\left. + \frac{X}{40}\left(15Y^4+10X^2Y^2+3X^4\right) \ln(Z+R) \right] \right\}_{X_1,Y_1,Z_1}^{X_2,Y_2,Z_2} ,$$

$$
\begin{aligned}
V_z = G\Bigg\{ & C_0\left[Z\arctan\frac{XY}{ZR} - X\ln\left(Y+R\right) - Y\ln\left(X+R\right)\right] \\
& + C_1\left[\frac{Z^2}{2}\arctan\frac{XY}{ZR} - \frac{X^2}{2}\arctan\frac{YZ}{XR} - \frac{Y^2}{2}\arctan\frac{XZ}{YR} + XY\ln\left(Z+R\right)\right] \\
& + C_2\left[\frac{Z^3}{3}\arctan\frac{XY}{ZR} + \frac{X^3}{3}\ln\left(Y+R\right) + \frac{Y^3}{3}\ln\left(X+R\right) + \frac{2}{3}XYR\right] \\
& + C_3\left[\frac{Z^4}{4}\arctan\frac{XY}{ZR} + \frac{X^4}{4}\arctan\frac{YZ}{XR} + \frac{Y^4}{4}\arctan\frac{XZ}{YR}\right. \\
& \left. - \frac{XY\left(X^2+Y^2\right)}{2}\ln\left(Z+R\right) + \frac{XYZR}{4}\right] \\
& + C_4\left[\frac{Z^5}{5}\arctan\frac{XY}{ZR} - \frac{X^5}{5}\ln\left(Y+R\right) - \frac{Y^5}{5}\ln\left(X+R\right)\right. \\
& \left.\left. + \frac{XYR\left(-7X^2-7Y^2+2Z^2\right)}{15}\right]\Bigg\}\Bigg|_{X_1,Y_1,Z_1}^{X_2,Y_2,Z_2} .
\end{aligned}
$$

APPENDIX D

The second-order derivatives of the gravitational potential V for the 4-th degree of the density polynomial are:

$$
\begin{aligned}
V_{xx} = G\Bigg\{ & C_0\left[-\arctan\frac{YZ}{XR}\right] \\
& + C_1\left[X\ln\left(Y+R\right)\right] \\
& + C_2\left[X^2\arctan\frac{YZ}{XR} - \mathrm{XY}\ln\left(Z+R\right)\right] \\
& + C_3\left[-XYR - X^3\ln\left(Y+R\right)\right] \\
& + C_4\left[\frac{XY\left(3X^2+Y^2\right)}{2}\ln\left(Z+R\right) - \frac{XYZR}{2} - X^4\arctan\frac{YZ}{XR}\right]\Bigg\}\Bigg|_{X_1,Y_1,Z_1}^{X_2,Y_2,Z_2} ,
\end{aligned}
$$

$$V_{yy}=G\left\{C_0\left[-\arctan\frac{XZ}{YR}\right]\right.$$
$$+C_1\left[Y\ln\left(X+R\right)\right]$$
$$+C_2\left[Y^2\arctan\frac{XZ}{YR}-\mathrm{XY}\ln\left(Z+R\right)\right]$$
$$+C_3\left[-XYR-Y^3\ln\left(X+R\right)\right]$$
$$\left.+C_4\left[\frac{XY\left(3Y^2+X^2\right)}{2}\ln\left(Z+R\right)-\frac{XYZR}{2}-Y^4\arctan\frac{XZ}{YR}\right]\right\}_{X_1,Y_1,Z_1}^{X_2,Y_2,Z_2},$$

$$V_{xz}=G\left\{C_0\left[\ln\left(Y+R\right)\right]\right.$$
$$+C_1\left[X\arctan\frac{YZ}{XR}-Y\ln\left(Z+R\right)\right]$$
$$+C_2\left[-YR-X^2\ln\left(Y+R\right)\right]$$
$$+C_3\left[\frac{Y\left(3X^2+Y^2\right)}{2}\ln\left(Z+R\right)-X^3\arctan\frac{YZ}{XR}-\frac{YZR}{2}\right]$$
$$\left.+C_4\left[X^4\ln\left(Y+R\right)+\frac{YR\left(5X^2+2Y^2-Z^2\right)}{3}\right]\right\}_{X_1,Y_1,Z_1}^{X_2,Y_2,Z_2},$$

$$V_{yz}=G\left\{C_0\left[\ln\left(X+R\right)\right]\right.$$
$$+C_1\left[Y\arctan\frac{XZ}{YR}-X\ln\left(Z+R\right)\right]$$
$$+C_2\left[-XR-Y^2\ln\left(X+R\right)\right]$$
$$+C_3\left[\frac{X\left(3Y^2+X^2\right)}{2}\ln\left(Z+R\right)-Y^3\arctan\frac{XZ}{YR}-\frac{XZR}{2}\right]$$
$$\left.+C_4\left[Y^4\ln\left(X+R\right)+\frac{XR\left(5Y^2+2X^2-Z^2\right)}{3}\right]\right\}_{X_1,Y_1,Z_1}^{X_2,Y_2,Z_2},$$

$$V_{xy} = G\Big\{C_0\left[\ln\left(Z+R\right)\right]$$
$$+C_1\left[R\right]$$
$$+C_2\left[\frac{ZR}{2}-\frac{\left(X^2+Y^2\right)}{2}\ln\left(Z+R\right)\right]$$
$$+C_3\left[-\frac{R\left(2X^2+2Y^2-Z^2\right)}{3}\right]$$
$$+C_4\left[\frac{3\left(X^2+Y^2\right)^2\ln\left(Z+R\right)-ZR\left(3X^2+3Y^2-2Z^2\right)}{8}\right]\Big\}_{X_1,Y_1,Z_1}^{X_2,Y_2,Z_2} .$$

APPENDIX E

The gravitational potential and its derivatives for the 2D rectangular prism:

$$V = 2G\sum_{p=0}^{n}\frac{C_p}{\left(p+1\right)\left(p+2\right)}$$
$$\times\Big\{\left(p+2\right)X\frac{Z^{p+1}}{2}\left[\ln\left(X^2+Z^2\right)-2\right]+\left(p+1\right)Z^{p+2}\arctan\frac{X}{Z}$$
$$-f_1\left(p\right)\left[\sum_{k=0}^{p/2}\left(\frac{\left(-1\right)^k X^{2k+1}Z^{p-2k+1}}{p-2k+1}\right)+\mathrm{i}^{p+2}X^{p+2}\arctan\frac{Z}{X}\right]$$
$$-f_2\left(p\right)\left[\sum_{k=0}^{(p-1)/2}\left(\frac{\left(-1\right)^k X^{2k+1}Z^{p-2k+1}}{p-2k+1}\right)+\mathrm{i}^{p+1}\frac{X^{p+2}}{2}\ln\left(X^2+Z^2\right)\right]\Big\}_{X_1,Z_1}^{X_2,Z_2} ,$$

$$V_x = -2G\sum_{p=0}^{n}\frac{C_p}{p+1}\Big\{\frac{Z^{p+1}}{2}\ln\left(X^2+Z^2\right)$$
$$-f_1\left(p\right)\left[\sum_{k=0}^{p/2}\left(\frac{\left(-1\right)^k X^{2k+1}Z^{p-2k+1}}{p-2k+1}\right)-\mathrm{i}^{p}X^{p+1}\arctan\frac{Z}{X}\right]$$
$$-f_2\left(p\right)\left[\sum_{k=0}^{(p-1)/2}\left(\frac{\left(-1\right)^k X^{2k}Z^{p-2k+1}}{p-2k+1}\right)+\mathrm{i}^{p+1}\frac{X^{p+1}}{2}\ln\left(X^2+Z^2\right)\right]\Big\}_{X_1,Z_1}^{X_2,Z_2} ,$$

$$V_z = -2G \sum_{p=0}^{n} \frac{C_p}{p+1} \left\{ Z^{p+1} \arctan \frac{X}{Z} \right.$$

$$+ f_1(p) \left[\sum_{k=0}^{p/2} \left(\frac{\operatorname{sgn}(p-2k)(-1)^k X^{2k+1} Z^{p-2k}}{p-2k+1-\operatorname{sgn}(p-2k)} \right) + \mathrm{i}^p \frac{X^{p+1}}{2} \ln\left(X^2+Z^2\right) \right]$$

$$\left. + f_2(p) \left[\sum_{k=0}^{(p-1)/2} \left(\frac{(-1)^k X^{2k+1} Z^{p-2k}}{p-2k} \right) + \mathrm{i}^{p+1} X^{p+1} \arctan \frac{Z}{X} \right] \right\}_{X_1, Z_1}^{X_2, Z_2} ,$$

$$V_{xx} = 2G \sum_{p=0}^{n} C_p$$

$$\times \left\{ f_1(p) \left[\sum_{k=0}^{p/2} \left(\frac{\operatorname{sgn}(p-2k)(-1)^k X^{2k+1} Z^{p-2k-1}}{p-2k-\operatorname{sgn}(p-2k-1)} \right) + \mathrm{i}^p X^p \arctan \frac{Z}{X} \right] \right.$$

$$+ f_2(p) \left[\sum_{k=0}^{(p-1)/2} \left(\frac{\operatorname{sgn}(p-2k-1)(-1)^k X^{2k+1} Z^{p-2k-1}}{p-2k-\operatorname{sgn}(p-2k-1)} \right) \right.$$

$$\left. \left. + \mathrm{i}^{p-1} \frac{X^p}{2} \ln\left(X^2+Z^2\right) \right] \right\}_{X_1, Z_1}^{X_2, Z_2} ,$$

$$V_{xz} = 2G \sum_{p=0}^{n} C_p \left\{ f_1(p) \left[\sum_{k=0}^{p/2} \left(\frac{\operatorname{sgn}(p-2k)(-1)^k X^{2k} Z^{p-2k}}{\operatorname{sgn}(p-2k)+p-2k+1} \right) \right. \right.$$

$$\left. + \mathrm{i}^p \frac{X^p}{2} \ln\left(X^2+Z^2\right) \right]$$

$$\left. + f_2(p) \left[\sum_{k=0}^{(p-1)/2} \left(\frac{(-1)^k X^{2k} Z^{p-2k}}{p-2k} \right) + \mathrm{i}^{p+1} X^p \arctan \frac{Z}{X} \right] \right\}_{X_1, Z_1}^{X_2, Z_2} ,$$

$$V_{zz} = -V_{xx} .$$

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