Does Poisson's downward continuation give physically meaningful results?

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ABSTRACT

The downward continuation (DWC) of the gravity anomalies from the Earth's surface to the geoid is still probably the most problematic step in the precise geoid determination. It is this step that motivates the quasi-geoid users to opt for Molodenskij's rather than Stokes's theory. The reason for this is that the DWC is perceived as suffering from two major flaws: first, a physically meaningful DWC technique requires the knowledge of the irregular topographical density; second, the Poisson DWC, which is the only physically meaningful technique we know, presents itself mathematically in the form of Fredholm integral equation of the 1st kind. As Fredholm integral equations are often numerically ill-conditioned, this makes some people believe that the DWC problem is physically ill-posed. According to a revered French mathematician Hadamard, the DWC problem is physically well-posed and as such gives always a finite and unique solution. The necessity of knowing the topographical density is, of course, a real problem but one that is being solved with an ever increasing accuracy; so sooner or later it will allow us to determine the geoid with the centimetre accuracy.

Keywords: inverse problem; convergence; Jacobi iterations; Poisson integral equation; convexity of equipotential surfaces; physical constraints

1. INTRODUCTION

The motivation for writing this paper came from the ongoing discussions between the geodesists who believe that the original Stokes theory (*Stokes, 1849*) is still the best technique for (gravimetric) geoid determination and those, who would opt for Molodenskij's approach (*Molodenskij et al., 1960*). The main reason why Molodenskij came up with his idea was that he did not believe it would be possible to ever know the topographical density well enough to allow us to compute the geoid to a "good enough" accuracy. Presumably, Molodenskij believed that the contribution of topographical density in-homogeneities would be larger than they actually turned out to be. Since 1960 it has been shown that details of topographical density matter much less than originally assumed they would and that, with collaboration of geologists and geophysicists, it is possible to get at least the lateral distribution of topographical density realistic enough to enable us to compute the geoid to an accuracy of a couple of centimetres (*Martinec et al., 1995*; *Huang et al., 2001*; *Hwang and Hsiao 2003*; *Kingdon et al., 2009*; *Vaníček et al., 2012*). This leaves us, who believe in Stokes's theory, with the task of proving that the downward continuation (DWC) can be carried out to a sufficient accuracy as well.

We began the investigation of Poisson's DWC process more than 20 years ago (*Vaníček et al., 1996*) by discovering that the DWC can be done relatively easily over, say, a $1° \times 1°$ area with surface data given on a $5'$ by $5'$ grid. Problems with its numerical instability begin to appear when the step of the surface grid becomes smaller than $2'$, as predicted by *Martinec (1996)* and it worsens with growing topographical heights. Seeking the exact solution for a smaller step size of the grid and larger topographical heights requires the necessity to iterate the solution many thousands of times resulting in a solution that is clearly marred by some high frequency noise. This noise manifested itself by spike-like features - see, e.g., Fig. 1 - which, as we realized, reflects errors in observations or irregular/uneven distribution of the observed gravity values. Please note that we are not attempting here to test a computational procedure, in which case the use of a synthetic field in a closed loop might make sense. Rather, we intend to test whether the actual gravity field's physical behaviour is reflected in the mathematical operations we are applying to it.

This realization led us to seek a way of smoothing the solution in a physically meaningful way, as compared to an arbitrary mathematical way. To do so had obviously required to show that the iterated results had a tendency to be outside of the realm of physically acceptable values. This was not a simple task as the extreme values of Helmert's anomalies on the geoid are not easy to obtain. However, it seems that the values tend to remain physically acceptable, which would support the fact that the DWC is a physically well-posed problem. We realize, of course, that the discovery that the iterated results merely keep within physically reasonable bounds does not constitute a proof that the DWC is a physically meaningful process. If our results fell outside of those bounds we would conclude that it is not physically meaningful, but not the other way around.

We had decided to use a set of real gravity and topographical data from the Auvergne region in central France. These real data, noisy as they are, have been used by scores of researchers and should be thus well known, at least from the literature. From the geological point of view, the area is interesting because of its basalt intrusions and thus the presence of high-density material in topography. During these computations with real gravity and topographical data from Auvergne, France, we realized that it makes no sense to seek the exact solution even when we know that it exists. From a certain number of iterations onwards, further iterations only model the noise in the input data; thus the results must be sought by means of estimating the most probable values on the geoid. But that aspect is considered outside the interest of this paper and will be tackled later on.

The paper is divided in 6 sections of which this introduction is the first. The second section explains the fundamentals and the mathematics of Poisson's downward continuation process; it also shows how a particular exact solution that uses real data ends up being contaminated by a spurious high-frequency noise. The third section reviews the mathematical relations among different gravity field functionals needed in the next, fourth, section which is devoted to a discussion of what might the physically acceptable values of these functionals be in the space of artificially modelled density distribution (space with density defined by Helmert's 2nd condensation method in our case, that we call "Helmert space" for short). The fifth section presents numerical results obtained from real data taken from 3 zones of very different characteristics located in the area of Auvergne in central France. The last, sixth, section spells out the conclusions we were able to arrive at, based on the numerical results from the Auvergne data. The main discoveries are that 1) there is no need to enforce physically meaningful regularization in the iterative process and the question we set out to answer becomes moot, and 2) the solution will have to be sought in the realm of statistics, i.e., least-squares (also known as minimization of L2 norm) and the iterative process will no longer be necessary.

2. POISSON's DOWNWARD CONTINUATION

Poisson in his investigation of the behavior of potentials (*MacMillan, 1930*) formulated the theory for upward and downward continuation of harmonic functions. What is a harmonic function and who needs it? A harmonic function is one that satisfies the Laplace partial differential equation in a specific region. The following equation is valid for any function F harmonic in a region $Q \subset \Re_3$:

$$\forall \vec{r} \in Q: \; \nabla^2 F(\vec{r}) = 0 \;, \tag{1}$$

using any coordinate system. We note that harmonicity of F guarantees that F is defined in Q as a function of a 3D argument (position). Neglecting the atmosphere, the disturbing potential of the Earth T as well as the product of a "solid gravity anomaly" (*Vaníček et al., 2004*) and geocentric distance, $r\Delta g$, are both harmonic functions above the Earth surface. By the same token, also the Helmert disturbing potential T^H and the product of the Helmert gravity anomaly, which is solid (*Vaníček et al., 2004*), and the geocentric distance, $r\Delta g^H$, are harmonic in Helmert space above the Helmert co-geoid (*Ellmann and Vaníček, 2006*). Other mass-model spaces exist, where the subspace between the geoid and the surface of the Earth, occupied by topography in the real space, is void. In all these spaces, the appropriate disturbing potential as well as the associated solid gravity anomaly multiplied by geocentric distance are both harmonic functions (see, e.g., *Martinec, 1998*).

In geoid computations we need to continue gravity anomalies mostly downward (when the values of topographical heights reckoned in the orthometric height system are positive) but sometimes also upward (when the values are negative) from the Earth surface to the geoid and vice versa. This cannot be done in the real space because there the space Q contains topographical masses and the functions T and $r\Delta g$ are not harmonic. We do not know how to physically correctly downward/upward continue other functions than harmonic and therefore we cannot downward/upward continue gravity anomalies to the geoid in the real space. However, it is possible to upward/downward continue the Helmert gravity anomalies, in Helmert space, i.e., to do the following transformation:

$$\Delta g_g^H \to \Delta g_t^H \ ,$$

where, for simplicity, the subscript *g* represents the values on the geoid and *t* the values on the Earth surface. The upward continuation is written as a surface integral, known as the Poisson integral, usually in a spherical approximation which is numerically accurate enough for all our computations, as

$$\Delta g^H \left(R + H(\Omega), \Omega \right) = \frac{R}{4\pi \left(R + H(\Omega) \right)} \iint_{\Omega'} K\left(H(\Omega), \Omega, \Omega' \right) \, \Delta g^H \left(R, \Omega' \right) \mathrm{d}\Omega' \ , \quad (2)$$

where $K\left(H(\Omega), \Omega, \Omega' \right)$ is the Poisson integral kernel, Ω represents the horizontal position of the point of interest, located on the Earth surface, and Ω' stands for the horizontal position of the integration element on the geoid. In spherical approximation, the point on topography has coordinates $\left(R + H(\Omega), \Omega \right)$, the point on the geoid has coordinates $\left(R, \Omega' \right)$. The Poisson kernel can then be expressed as (*MacMillan, 1930*)

$$K\left(H(\Omega), \Omega, \Omega' \right) = \sum_{j=2}^{\infty} (2j+1) \left(\frac{R}{R + H(\Omega)} \right)^{j+1} P_j\left(\cos\psi \right) \ , \quad (3)$$

or in a closed form as

$$K\left(H(\Omega), \Omega, \Omega' \right) = R \left[\frac{\left(2RH + H^2 \right)}{\ell^3 \left(R + H, \Omega; R, \Omega' \right)} - \frac{1}{r} - \frac{3R}{r^2} \cos\psi \right] , \quad (4)$$

where ℓ is the Euclidian and ψ the spherical distance between the point of interest $\left(R + H(\Omega), \Omega \right)$ and the integration point $\left(R, \Omega' \right)$. Poisson's kernel is regular everywhere, except points where simultaneously $H(\Omega) = 0$ and $\Omega = \Omega'$. At these points the kernel degenerates to Dirac distribution δ and Poisson's integral just reproduces Δg_t^H as it should (*Sun and Vaníček, 1996*).

The inverse transformation, $\Delta g_t^H \to \Delta g_g^H$, also known as the downward continuation, is given by the Poisson integral in Eq. (2), where the unknown, Δg_g^H is embedded within the integral. It is the Fredholm integral equation of 1st kind and these equations are known to be notoriously numerically unstable. When the solution to the inverse Poisson problem is sought, the Poisson integral equation is discretized converting it to a system of linear equations, which is the standard way of solving integral equations. Even though the problem is physically well-posed (*Hadamard, 1923*), this system of linear equations is often numerically ill-conditioned.

Tests with real Helmert's gravity anomalies have shown that the final solution often looks as if it is plagued by a high frequency noise, with large positive and negative spikes. As an example we show in Fig. 1 the downward continued "observed" Helmert gravity anomalies from the Earth surface on to the geoid. The spikes seen in the south-east corner of this orthographic plot are associated with the highest topographical features in the area, which is nothing else but Zone 1 as used later (cf., Fig. 2 later). These spikes are clearly too large, up to 600 mGal, and of a too high spatial frequency to be real. The largest real changes one can encounter in the DWC process are, perhaps, of the order of a few tens of mGal and there is no geological evidence in Auvergne to explain why gravity on the geoid should be changing by 1 Gal within a horizontal distance of 500 m. Hence we have no other explanation but that they are caused by errors in the observed anomalies and as such should not be part of the solution, i.e., the solution should look considerably "smoother". It is not the intention of this study to demonstrate how the noise in the observations is transmitted to the geoid and magnified by the DWC process - it will, for sure, become part of a subsequent study.

We had been using the Jacobi iterative technique (*Ralston, 1965*) for solving the system of linear equations that originates from the discretization of the Fredholm integral of 1st kind. In this approach it becomes quite natural to smooth/regularize the solution by stopping the iterative process after a certain number of iterations (*Martinec, 1996*; *Vaníček et al., 1996*). It was possible to formulate a criterion for stopping the iterations, see Eq. (25), after a reasonable number of iterations which became our natural way of regularizing the solution; it has worked quite satisfactorily (*Kingdon and Vaníček, 2011*). The only question remained: is the physics of downward continuation violated by the computational process? Or, in other words, would it be necessary to come up with a physically meaningful constraint on the computation to make sure that the solution makes a physical sense?

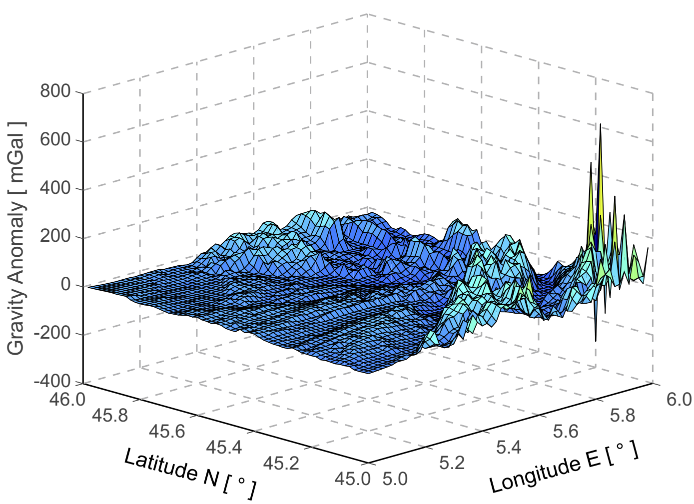

Fig. 1. Helmert's gravity anomalies on the geoid as downward continued from the Earth's surface, after 5000 iterations using the Jacobi iterative algorithm.

This paper is an attempt to answer this question. We have decided to work only within the Helmert space, because the computations in that space are less complicated than those in other spaces that would be candidates for this investigation. We may, however, investigate the behaviour of the downward continuation in these other mass-model spaces such as the "No Topography (NT)" space - this is the space inhabited by spherical complete Bouguer anomalies which are solid - later on.

3. SOME RELATIONS IN HELMERT SPACE

We know that for a harmonic function, in our case the Helmert disturbing potential T^H in a vacuum, the following relation holds

$$T_{zz}^{H} = -\left(T_{xx}^{H} + T_{yy}^{H}\right) , \tag{5}$$

where the double subscript denotes the second derivative in the implied directions and x, y are reckoned, for simplicity, in the local geodetic (LG) coordinate system (see, e.g., *Vaníček and Krakiwsky, 1986*). It is, of course, a consequence of Laplace equation validity. Here the Helmert disturbing potential is understood in terms of the Helmert second condensation method as used in the UNB interpretation (*Vaníček and Martinec, 1993*).

Neglecting the curvature of the normal plumbline, which is never large enough to play a significant role, we can write on the geoid to a sufficient accuracy using the standard notation (*Vaníček and Krakiwsky, 1986*):

$$-\xi_g^H = \frac{\partial N^H}{\partial x} \approx \frac{1}{g}\frac{\partial T^H}{\partial x} = \frac{1}{g} T_x^H \tag{6}$$

and, consequently,

$$T_x^H \approx -g\xi_g^H . \tag{7}$$

To the same degree of approximation as above we get,

$$T_y^H \approx -g\eta_g^H , \tag{8}$$

where ξ and η are the components of the deflection of the vertical. Further differentiation of Eq. (7) with respect to x yields

$$T_{xx}^H = \frac{\partial T_x^H}{\partial x} \approx -g\frac{\partial \xi_g^H}{\partial x} , \tag{9}$$

and similarly

$$T_{yy}^H = \frac{\partial T_y^H}{\partial y} \approx -g\frac{\partial \eta_g^H}{\partial y} \tag{10}$$

Finally, we get

$$T_{zz}^{H} \approx g \left(\frac{\partial \xi_g^H}{\partial x} + \frac{\partial \eta_g^H}{\partial y} \right) . \tag{11}$$

Considering a total deflection of the vertical θ in its azimuth α, we can obviously write

$$\xi_g^H = \theta_g^H \cos\alpha \ , \quad \eta_g^H = \theta_g^H \sin\alpha \ , \tag{12}$$

so that Eq. (11) becomes

$$T_{zz}^{H} \approx g \left(\frac{\partial \theta_g^H}{\partial x} \cos\alpha + \frac{\partial \theta_g^H}{\partial y} \sin\alpha \right) . \tag{13}$$

For the direction $\alpha = 0$, which we can always assume as the situation is rotation invariant, it is therefore enough to focus just on the magnitude of the deflection θ_g^H on the geoid and investigate its behaviour. The following is thus the basic equation we shall be dealing with

$$T_{zz}^{H} = g \frac{\partial \theta_g^H}{\partial s} \ , \tag{14}$$

where s is the horizontal distance in the direction specified by α.

4. EXTREME PHYSICALLY ADMISSIBLE VALUES

The question now is: what might be considered the extreme (largest or smallest), physically admissible value of T_{zz}^{H} or other related quantities? The absolutely extreme value for T_{zz}^{H} (in Helmert space) can be obtained from the knowledge that equipotential surfaces of the Earth gravity field are convex everywhere in the harmonic space. Convexity implies that the curvature of the equipotential surface must be positive everywhere and in every horizontal direction. According to *Heiskanen and Moritz (1967, Eq. (2-19))*, the mean curvature of the geoid, J, is given by

$$J = \frac{-\left(W_{xx} + W_{yy} \right)}{2g} \ . \tag{15}$$

In order for the geoid to be convex, it is necessary that J be positive; in other words (since g never goes to infinity or zero) the following inequality must hold

$$-\left(W_{xx} + W_{yy} \right) > 0 \ . \tag{16}$$

If W is a harmonic quantity, as it is in the Helmert space external to the geoid, then:

$$-\left(W_{xx}+W_{yy}\right)>W_{zz}\ . \tag{17}$$

How about the rotational part of the potential which is not harmonic anywhere? The rotational part of the potential is eliminated by the transition from the real potential W to the disturbing potential T (*Vaníček and Krakiwsky, 1986*). The inequality (16) is decomposed into:

$$W_{zz}=U_{zz}+T_{zz}>0\ , \tag{18}$$

and the geoid will be convex as long as:

$$T_{zz}>-U_{zz}\ . \tag{19}$$

This should always be the case! U_{zz} is approximately equal to $-2\gamma/R$, or about –308.6 μGal/m. Any positive T_{zz} will naturally meet the condition of Eq. (19) but a too small negative value of T_{zz} could violate the requirement of convexity if it were smaller than –308.6 μGal/m = –3086 E.

For convexity criterion, the value we are trying to use as an acceptable extreme value from the physical point of view, is really the vertical gradient of the gravity anomaly on the geoid, which, for all intents and purposes, can be defined as

$$T_{zz}^{H}\left(r_g\left(\Omega\right)\right)\approx\left.\frac{\partial\Delta g^{H}\left(r,\Omega\right)}{\partial H}\right|_{r_g\left(\Omega\right)}\approx\frac{\Delta g_t^{H}\left(\Omega\right)-\Delta g_g^{H}\left(\Omega\right)}{H\left(\Omega\right)}\ . \tag{20}$$

These values can readily be obtained from the numerical results of the downward continuation process.

There are various other possible ways of how to estimate this extreme quantity. Probably, the most obvious one is to use one of the global models, such as the EGM08 (*Pavlis, 2012*) or the EIGEN-6C4 (*Förste et al., 2014*) that are based on observed data. It should be noted that it would not be advisable to use any of the synthetic gravity fields in this context as none of these synthetic fields is formulated to have realistic vertical gravity anomaly gradients. The formal expression for the vertical gradient of gravity anomaly on the geoid is

$$\left.\frac{\partial\Delta g}{\partial H}\right|_{r_g}\approx\frac{GM}{R^3}\sum_j\left[\left(j-1\right)\left(j+2\right)\right]T_j\left(\Omega\right)\ . \tag{21}$$

Here the term $T_j\left(\Omega\right)$ is the azimuthal average of the disturbing potential component of order j at the location defined by the direction Ω.

There are several problems associated with this approach. First, the potential coefficients T_j describe the disturbing potential in the real space while we need them defined in the Helmert space. As already mentioned above, the disturbed gravity field in real space is not harmonic within topography and downward continuation in the physically meaningful sense cannot thus be even attempted (*Vaníček et al., 2004*). Second,

Table 1. Statistics of real space values of T_{zz} in E units (from EGM2008 and EIGEN-6C4 to the degree and order 2190). *RMS*: root mean square error.

Model	*Min*	*Max*	*Mean*	*RMS*
EGM2008	−801	634	0	12
EIGEN-6C4	−804	630	0	12

as such, it is not certain that these coefficients describe the real gravity field (gravity field in the real space) on the geoid. The real field has been downward continued through topography using a technique called “analytical downward continuation” which is not based on physics and thus unpredictable in its performance. But this is the best we can do.

Nevertheless, we have used Eq. (21) with EGM2008 and EIGEN-6C4 coefficients taken to the degree and order 2190, setting the altitude to 0 m. Table 1 summarizes the statistical behaviour of the T_{zz} values obtained from EGM2008 and EIGEN-6C4 values around the world. The minimal value is −804 E, about 26% of the minimum permissible value indicated by the conditions of convexity, see Table 1.

Another way of looking at the problem of physically meaningful extreme values is to use the relation of T_{zz} to topographical density. According to (*Vaníček and Krakiwsky, 1986, Eqs 21.32 and 21.37*) we write for the Helmert space

$$T_{zz}^{H}\left(r_g(\Omega)\right) \approx \left.\frac{\partial \Delta g^{H}(r,\Omega)}{\partial H}\right|_{r_g(\Omega)} = -4\pi G \rho^{H}\left(r_g(\Omega)\right) , \tag{22}$$

where $\rho^{H}\left(r_g(\Omega)\right)$ is the density at point $r_g(\Omega)$ on the co-geoid in Helmert space. We note that the density at a point infinitesimally above the co-geoid must equal to 0. The Helmert density for the smallest permissible value of T_{zz} based on requirement of convexity thus becomes 3.651 g cm^{-3}, slightly larger than the density of magma. For the minimum value of T_{zz} obtained from Table 1, Eq. (22) gives us the value of 0.957 g cm^{-3}, roughly the density of water. This sounds quite reasonable for the real space but it is unclear just how this can be interpreted in Helmert space, or, whether in fact this approach can be used in Helmert space at all.

So let us go back to the global gravitational models. The way to avoid some of the problems with global fields is to avoid areas of positive topography all together and work only in oceanic areas. We have done so in our investigation of the values of the horizontal gradients of the two components of the deflections of the vertical in the N-S and E-W directions (over the oceanic areas) as indicated by the EGM2008 model (the EIGEN-6C4 field was not considered in this context). The maximum values of the two horizontal gradients are 3.71″/km and 3.72″/km, respectively; the minimum values are −2.53″/km and −3.32″/km, respectively. Using the largest of these extreme values, 3.72″/km, in Eq. (11), we obtain the minimal value of T_{zz} to be −129 E. This represents only about 4% of the value based on the requirement of convexity.

One objection to the above described estimation may be that the EGM2008 model is known only to the degree and order 2190 and is therefore missing the shortest wavelength information content. This is probably a moot point as the very shortest wavelength features are caused mostly by topography and at sea, the values obtained from the real field are only slightly affected by topography. It seems to us that the extreme values of T_{zz} obtained from EGM2008 and EIGEN-6C4 using Eq. (21) look more realistic, while the absolute values from the oceanic areas, using Eq. (14), are just too small. The reason for this discrepancy is unclear to us but looking at the contribution of topography itself, in addition to the somewhat lower frequencies of the rest of the global fields, does not look like a realistic possibility as we are interested in the values encountered on the geoid rather than on the topography.

5. TEST ON REAL DATA IN THE AUVERGNE REGION

The data obtained from the French Institute Geographic Nationale in Paris (*Duquenne, 2007*) were cleaned first; three outliers and numerous duplicate observed points were eliminated. Then NT (also known as complete spherical Bouguer) gravity anomalies as well as corresponding Helmert's anomalies at the Earth surface were constructed on the 30″ × 30″ grid. These gridded data were then continued downward in both Helmert's and the NT spaces onto the geoid using the Jacobi algorithm and Eq. (24) for the optimum number of iterations. This was done for 3 topographically very different 1° × 1° zones, 1, 2 and 3, with a 30′ border strip. Zone 1 is the topographically most challenging, Zone 3 is the least challenging. In each zone, instead of seeking the maximum value of $T_{zz}^{H}\left(r(\Omega)\right)$, the maximum mean value of the vertical gradient within topography

$$T_{zz}^{H} \approx \text{mean}\left(\frac{\partial \Delta g}{\partial H}\right) = \frac{1}{H(\Omega)} \int_{0}^{H} \frac{\partial \Delta g^{H}(r, \Omega)}{\partial H} \, \mathrm{d}H(\Omega) \tag{23}$$

was sought. Then from the differences of corresponding values on the Earth surface and on the geoid we obtained values corresponding to Eq. (20).

The Helmert numerical results (results for Helmert's anomalies) in Zones 1 to 3 are shown below. The 2D plots, see Fig. 2b and later Figs 5b and 8b, show the "optimal" results, i.e., results obtained after the optimal number of iterations as determined from the formula (*Kingdon and Vaníček, 2011*)

$$\Delta g_{max_{i+1}} - \Delta g_{max_i} > \text{con}\left(\mathbf{B}\right)\sigma_{\Delta g} , \tag{24}$$

where $\text{con}(\mathbf{B})$ is the condition number of the matrix **B**. The iterations are stopped for such an i when the inequality that involves the difference between two maximum values Δg_{max} in two successive iterations, the condition number of **B** (where **B** is the matrix of linear equations created by the discretization of the Fredholm integral equation) and the standard deviation $\sigma_{\Delta g}$ of a typical gravity anomaly Δg at the Earth surface) ceases being satisfied. The orthographic plots, see Fig. 3 and later Figs 6 and 9, display the same "optimal" results as the 2D plots in Fig. 2b and later in Figs 5b and 8b; we have opted to include these plots as well as they show better the character of the present noise. The plots of the extreme values of T_{zz}^{H} , see Fig. 4 and later Figs 7 and 10, show the evolution of these values with the number of iterations going all the way to 1000.

Note that Fig. 1 shows the values of T_{zz}^{H} after $i = 5000$ iterations for Zone 1. Comparison of Fig. 1 with Fig. 3 tells the story of what happens when one tries to come up with the exact (and unique) solution. Clearly the higher order iterations are taken up by the process for trying to satisfy the random errors in the observations on the Earth surface.

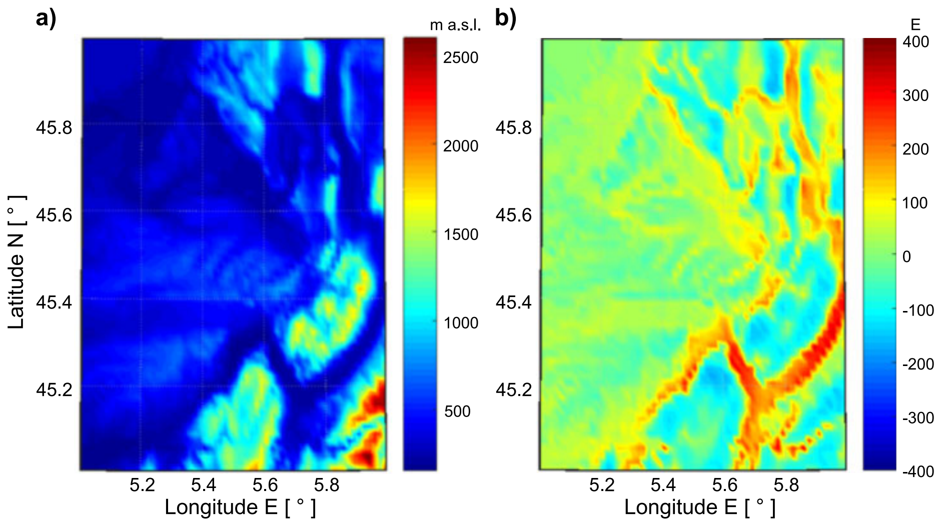

Fig. 2. Zone 1: **a)** topography, **b)** second vertical derivative of disturbing potential T_{zz} (*Max* = 346 E, *Min* = –169 E, *Mean* = 15 E and *St.Dev.* = 81 E).

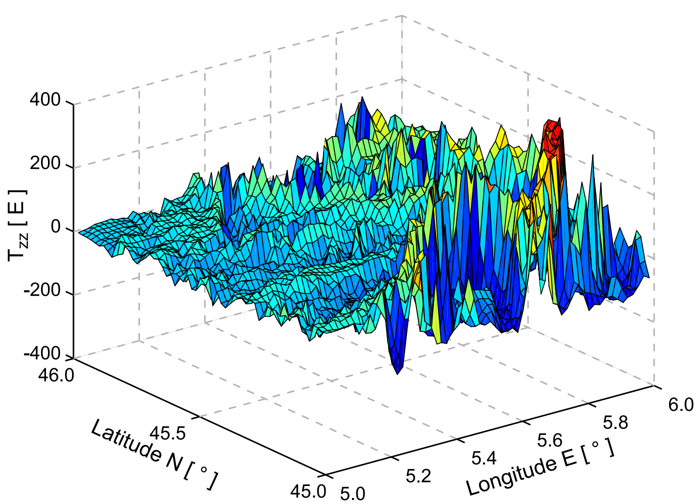

Fig. 3. Orthographic plot of the second vertical derivative of disturbing potential T_{zz} in Zone 1.

Table 2 contains the statistics of the results in the 3 zones: the maximum, minimum, mean and the standard deviation of the values of T_{zz}^{H}. The table shows the statistics for both the optimal number of iterations and after 1000 iterations. For a good measure, we have also included the results for T_{zz}^{NT} obtained from the DWC in the NT space which

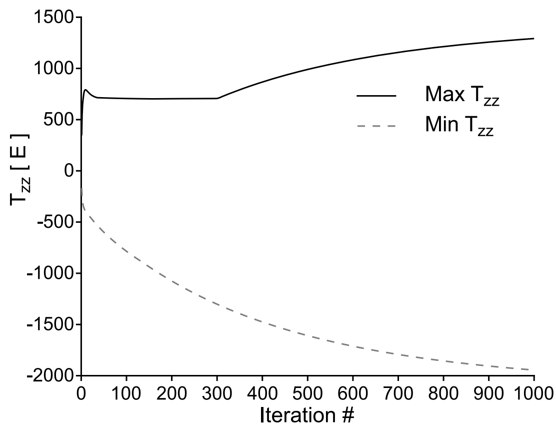

Fig. 4. Maximum and minimum values of the second vertical derivative of disturbing potential T_{zz} in Zone 1.

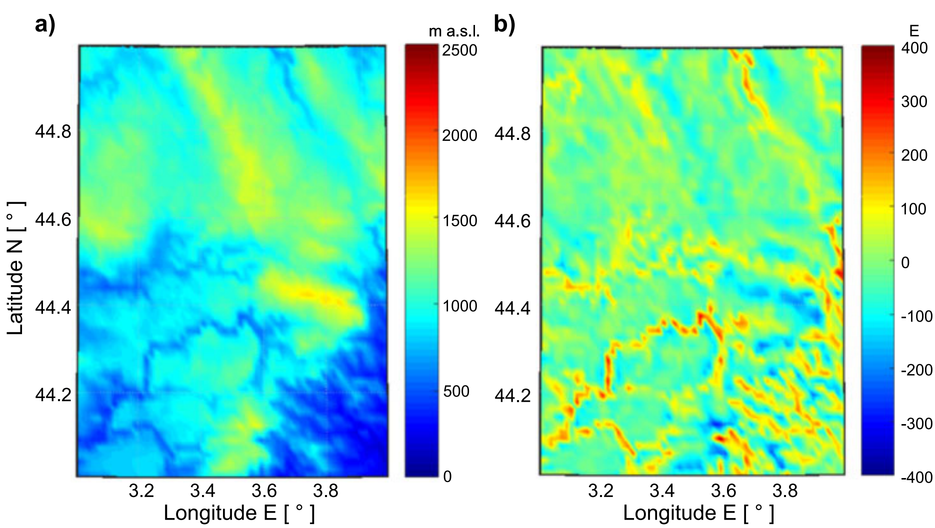

Fig. 5. Zone 2: **a)** topography, **b)** second vertical derivative of disturbing potential T_{zz} (*Max* = 385 E, *Min* = –300 E, *Mean* = –7 E and *St.Dev.* = 73 E).

only illustrate that the gravity field in the NT (also known as the spherical complete Bouguer anomaly) space is smoother than that in the Helmert space.

It seems that the results in all 3 zones and in both Helmert's and NT spaces are way larger than −3086 E, the limit needed to violate the convexity requirement. This means that the DWC process does not seem to violate the loosest physical condition we could come up with, which is the condition of convexity. As far as the value of −3086 E is

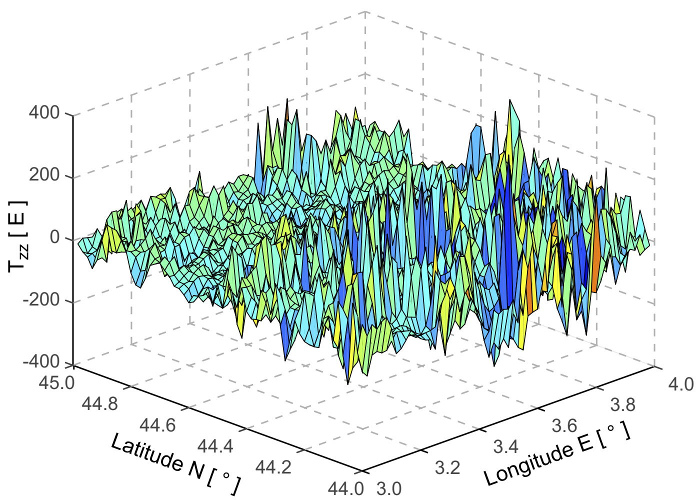

Fig. 6. Orthographic plot of the second vertical derivative of disturbing potential T_{zz} in Zone 2.

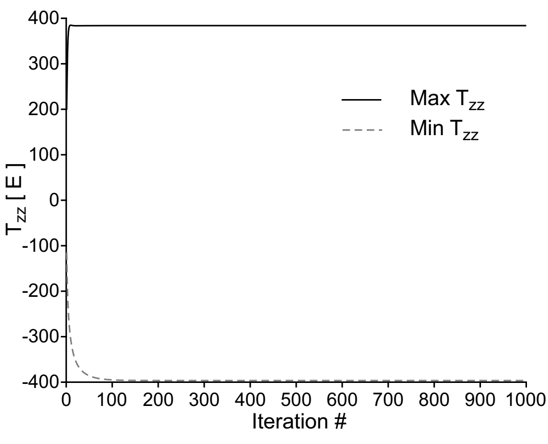

Fig. 7. Maximum and minimum values of the second vertical derivative of disturbing potential T_{zz} in Zone 2.

concerned, it represents good news and bad news. The good news is that none of our results based on real data approaches this value. So from this point of view, the Poisson DWC certainly cannot be classified as violating physics. The bad news is that the critical value seems to be too small to be really useful.

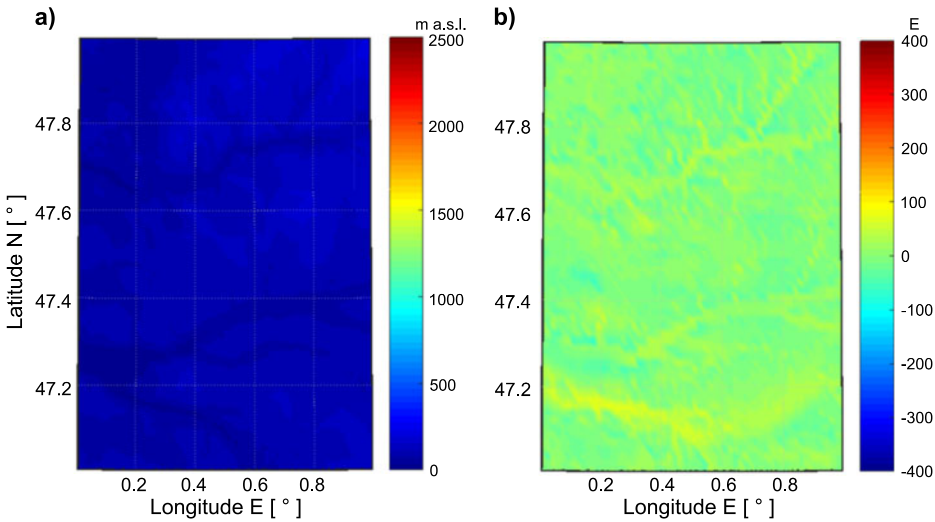

Fig. 8. Zone 3: **a)** topography, **b)** second vertical derivative of disturbing potential T_{zz} (*Max* = 79 E, *Min* = –74 E, *Mean* = 2 E and *St.Dev.* = 19 E).

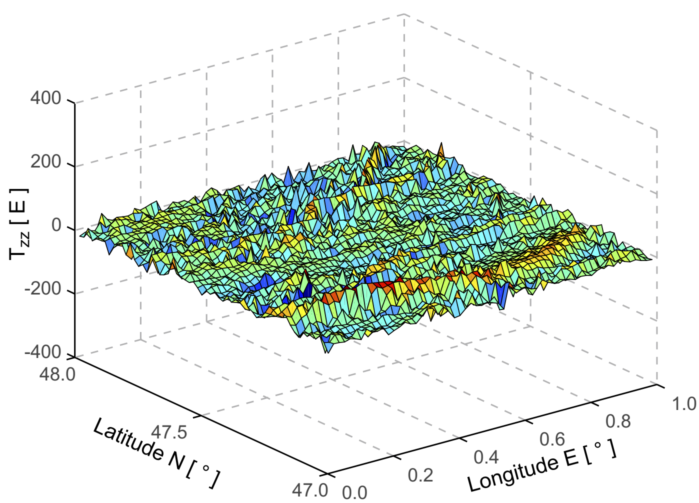

Fig. 9. Orthographic plot of the second vertical derivative of disturbing potential T_{zz} in Zone 3.

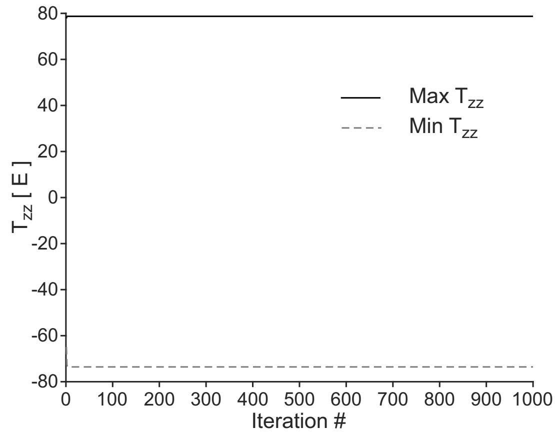

Fig. 10. Maximum and minimum values of the second vertical derivative of disturbing potential T_{zz} in Zone 3.

Some of the actual values are, however, way below the minima implied by the global gravitational models. The values of T_{zz} vs. the horizontal gradients of the deflection components seem to be somewhat inconsistent. The deflection data are from the oceanic regions only while the T_{zz} data are from around the globe. The higher frequencies (above $n = 2190$) in both the deflection and T_{zz} data are missing completely. Also, the question mark that hangs above the theory used for the global field construction and thus also the numerical results of EGM08 must be considered. Even then our results of –169 E, –300 E

Table 2. Recapitulation of numerical results for the actual second vertical derivative of disturbing potential T_{zz}. Values in E.

Zone	Statistics	Helmert		No Topography	
		Optimal Iterations	1000 Iterations	Optimal Iterations	1000 Iterations
1	*Max*	346	1291	90	618
	Min	–169	–1945	–119	–929
	Mean	15	19	–4	–4
	St.Dev.	81	151	13	31
2	*Max*	385	384	102	126
	Min	–300	–396	–82	–93
	Mean	–7	–7	–1	–1
	St.Dev.	73	76	17	19
3	*Max*	79	79	39	40
	Min	–74	–74	–71	–71
	Mean	2	2	–20	–20
	St.Dev.	19	19	12	12

and −74 E obtained after the optimal number of iterations in Helmert space (and similarly for the NT space) are all well above the value of −804 E for the anomalies from Table 1. Furthermore, half of them are smaller than the global minimum of −129 E indicated by the deflections of the vertical at the oceans and it is not clear what to make out this.

6. CONCLUSIONS

So what can we say about these results? Our main conclusion is that the Poisson DWC seems to work fine, as Hadamard had recognized, as long as the number of iterations is kept low. The number is regulated by Eq. (24) which gives the highest number of iterations of 11 for Zone 2, the numbers of iterations for Zones 1 (the roughest topography) and 3 (the smoothest topography) are much smaller. When the number of iterations is allowed to grow naturally until the correct solution is reached, some of the actual values of the second vertical derivative of disturbing potential (T_{zz}) may reach values that are not compatible with physics, but we have not tried this. In any case, it seems that we should not expect any improvement in the DWC results by employing a physically meaningful regularisation - which was the question we wanted answered through our investigations described here.

That brings us to the final point we wish to make. Although the exact solution can always be reached, it should not be considered to be the ultimate goal. We are dealing with an essentially statistical problem as the observed gravity values are burdened with uncertainties resulting from the measuring process and it makes absolutely no logical sense to seek an exact solution under these circumstances. We should concentrate on formulating a statistically best estimate of the DWC values and start dealing with the DWC as a statistical problem that follows the statistical laws as we are used to see in other geodetic problems. In the continuation of our research, which will concentrate on statistical aspects of DWC, there will be room to employ synthetic fields as well as various simulation techniques.

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