A structured and constrained Total Least-Squares solution with cross-covariances

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ABSTRACT

A new proof is presented of the desirable property of the weighted total least-squares (WTLS) approach in preserving the structure of the coefficient matrix in terms of the functional independent elements. The WTLS considers the full covariance matrix of observed quantities in the observation vector and in the coefficient matrix; possible correlation between entries in the observation vector and the coefficient matrix are also considered. The WTLS approach is then equipped with constraints in order to produce the constrained structured TLS (CSTLS) solution. The proposed approach considers the correlation between the observation vector and the coefficient matrix of an Error-In-Variables model, which is not considered in other, recently proposed approaches. A rigid transformation problem is done by preservation of the structure and satisfying the constraints simultaneously.

Keywords: Total Least-Squares, Errors-In-Variables model, functionally independent errors, structured coefficient matrix, linear and quadratic constraints, rigid transformations

1. INTRODUCTION

The classical Least-Squares (LS) approach within the Gauss-Markov model (GMM) has been widely applied in geodesy and related fields. The use of the LS approach to the linear GMM makes the assumption that all elements of the coefficient matrix are absolutely error-free. However, this premise might not always be the case with many applications. The GMM with an uncertain coefficient matrix is known in the literature as the Errors-In-Variables (EIV) models (e.g., *Fuller, 1987*). The Total Least-Squares (TLS) approach, which is actually a synonym for an LS estimation, has been used as a standard technique to adjust the EIV model, see *Golub and van Loan (1980)*. The TLS technique has attracted considerable attention since *Golub and van Loan (1980)* coined the term 'total least squares' and presented their solution based on the singular value decomposition. The study of this method was further refined and generalized in *van Huffel and Vandwalle (1991)*. The TLS has also been widely applied in geodesy. In terms of the admissibleness of weights, the differences in these TLS solutions vary from special to fairly general to general to very general as in *Schaffrin and Felus (2008)*, *Schaffrin and Wieser (2008)*, *Mahboub (2012)* and *Fang (2011, 2013)*, respectively. Later *Snow (2012)* extended the very general weighted TLS under the condition that singular cofactor matrix exists. *Amiri-Simkooei and Jazaeri (2012)* and *Xu et al. (2012)* formulated WTLS based on the well-known nonlinear least squares theory allowing one to directly use the existing body of statistical knowledge of the least squares theory. Recently, *Mahboub et al. (2013)* proposed an iterative robust TLS method to control the influence of outliers to the parameter estimator.

It is quite common that constraints are properly incorporated into a system of equations such as the EIV model. A number of algorithms for constrained TLS problems have been investigated in the past. *Van Huffel and Vandwalle (1991, p. 275)* and *Dowling et al. (1992)* proposed a closed form solution based on the singular value decomposition for a TLS problem with linear constraints. *Schaffrin and Felus (2005)* and *Schaffrin (2006)* presented the iterative linear fixed and stochastic constrained TLS solutions based on a fixed form solution, which is separately detailed in *Schaffrin and Wieser (2008)* specifically for their unconstrained TLS solution. For the quadratically constrained TLS, *Golub et al. (1999)*, *Sima et al. (2004)* and *Beck and Ben-Tal (2006)* presented equivalent regularized algorithms. Later, *Schaffrin and Felus (2009)* proposed a solution to treat the mixed constrained TLS problem iteratively where linear and quadratic constraints are considered simultaneously. Although the mixed constrained solution is able to solve the straight-line fitting application proposed in their paper, the generalization from the TLS to the WTLS and the structured TLS (STLS) needs to be investigated for other geodetic applications such as transformations. The STLS problem is the TLS problem with structured coefficient matrix, i.e. the augmented error matrix has the same structure as the augmented data matrix (*Markovsky and van Huffel, 2006*). Recently *Mahboub and Sharifi (2013a,b)* proposed an approach which produces the constrained STLS (CSTLS) solution for the EIV model according to the interesting property of the WTLS approach proven by *Mahboub (2012)* in automatically preserving the structure of the coefficient matrix when based on the prefect description of the dispersion matrix; however, it does not consider the correlation between the observation vector and random design matrix of the EIV model.

Apart from the above mentioned constraints that refer to the unknown parameters, the ‘constrained TLS’ presented by *Abatzoglou et al. (1991)* relates constraints virtually to functional relationships among the observations. The so called ‘constrained TLS’ approach has been proven to be equivalent to the structured TLS approach, see *Lemmerling et al. (1996)*, and is referred to as the structured TLS technique in this paper to avoid any confusion.

In geodesy, coordinate transformations have always been an issue (*Fang, 2011*; *Mahboub, 2012*). In these applications of the transformations, a structured coefficient matrix is often encountered. *Mahboub (2012)* presented a WTLS strategy to imprint the structure in the special case where the repeating elements appear in identical and negative form since in geodetic applications, one usually encounters these cases, see also *Schaffrin et al. (2012)*. However, the functional restriction within the coefficient matrix can be generalized by a universal functional relationship, instead of only a special functional relationship.

The STLS solution with constraints should be very interesting in geodesy although it has not been significantly discussed. A large number of varying methodologies to compute the transformation parameters have been surveyed on the basis of the TLS principle for affine transformations (e.g., *Schaffrin and Felus, 2008*) and as the STLS for similarity transformations (e.g., *Felus and Schaffrin, 2005*; *Mahboub, 2012*). However, the symmetrical adjustment of rigid transformations, which should be formulated as the STLS problem with constraints, has rarely been treated. Furthermore, this contribution provides a reasonable stochastic model and target function to treat a STLS problem so that the proposed constrained STLS solution can be regarded as a generalization of the method first given by *Schaffrin and Felus (2009)* or even *Mahboub and Sharifi (2013a,b)* from a theoretical standpoint.

The main objective of this paper is to present a STLS solution with linear and quadratic constraints which considers the correlation between the observation vector and random design matrix of the EIV model. For this purpose, a well-designed structured EIV model including functionally independent errors instead of all elements within the coefficient matrix is elaborated while a reasonable stochastic model with respect to the functionally independent errors is established. When incorporating the restrictions for the parameters, the STLS problem with the constraints is then accordingly solved by iteration. The new interesting form of our approach is an extension of the mixture of two the contributions by *Mahboub (2012)* and *Mahboub and Sharifi (2013a,b)* in considering cross-covariances. In addition, an important aspect treated in this paper is the application of a rigid transformation problem.

2. STLS PROBLEM WITH LINEAR AND QUADRATIC CONSTRAINTS

Let the functional part of the unconstrained univariate EIV model be defined by

$$\mathbf{y} = \left(\mathbf{A} - \mathbf{E}_A\right)\boldsymbol{\xi} + \mathbf{e}_y = \mathbf{A}\boldsymbol{\xi} - \mathbf{B}\mathbf{e} \quad \operatorname{rank} \mathbf{A} = u < n . \tag{1}$$

In the above equation, $\mathbf{y}$ and $\mathbf{e}_y$ are the $n \times 1$ observation vector and the corresponding random error vector, respectively. $\mathbf{A}$ and $\mathbf{E}_A$ are the full column rank $n \times u$ stochastic coefficient matrix and the corresponding random error matrix, respectively. $\boldsymbol{\xi}$ is the unknown parameter vector with a dimension of $u \times 1$. $\mathbf{e}$ is the extended error vector, which is expressed by the vectorized augmented matrix $\operatorname{vec}\left(\left[\mathbf{E}_A \quad \mathbf{e}_y\right]\right)$, where ‘vec‘ denotes the operator that stacks one column of a matrix underneath the previous one, see *Schaffrin and Felus (2008)*. $\mathbf{B}_{n\times n(u+1)} = \left[\boldsymbol{\xi}^{\mathrm{T}} \otimes \mathbf{I}_n, -\mathbf{I}_n\right]$ is a matrix depending on the parameter vector with full row rank; $\otimes$ denotes the Kronecker product. $\mathbf{I}_n$ is the identity matrix of size $n \times n$. The identity of $-\mathbf{E}_A\boldsymbol{\xi} + \mathbf{e}_y = -\mathbf{B}\mathbf{e}$ is utilizing the matrix property given in *Koch (1999, p. 41)*, which has been widely used in the TLS field.

In the coefficient matrix the observed elements are sometimes restricted within condition equations that define relationships between these elements, for an example of this in transformation problems, see *Felus and Schaffrin (2005). Mahboub (2012)* argued that having a perfect description of the stochastic model guarantees that the WTLS approach preserves the structure of an error matrix in the EIV model. His paper, however, only proved to hold when the elements appear repeatedly in identical and negative forms or are fixed since in geodetic applications one usually encounters these cases. In order to generalize these special cases, a universal functional restriction within the coefficient matrix should be taken into consideration. Now, the functionally independent error vector $\boldsymbol{\varepsilon}_A$ with a dimension of $t \times 1$ and a given $nu \times t$ deterministic matrix $\mathbf{C}_A$ are introduced to form the vectorized error matrix by $\operatorname{vec}(\mathbf{E}_A)=\mathbf{C}_A\boldsymbol{\varepsilon}_A$. The matrix $\mathbf{C}_A$ is also called the *characteristic matrix* in *Felus and Schaffrin (2005)*. The compression of the symbols is here used for all functionally independent errors $\boldsymbol{\varepsilon} = \begin{bmatrix} \boldsymbol{\varepsilon}_A^{\mathrm{T}} & \mathbf{e}_y^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ and the matrix $\mathbf{S} = \begin{bmatrix} \mathbf{C}_A & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix}$, which represents the relationship $\mathbf{e} = \mathbf{S}\boldsymbol{\varepsilon}$. Thus, the functional part of the structured EIV is introduced as

$$\mathbf{y} = \mathbf{A}\boldsymbol{\xi} - \mathbf{B}\mathbf{S}\boldsymbol{\varepsilon}. \tag{2}$$

Here, the stochastic part of the structured EIV model representing the properties of the random errors, is presented by the expectation and the dispersion of functionally independent observations:

$$\boldsymbol{\varepsilon} \sim \left(\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \ \sigma_0^2 \mathbf{Q} = \sigma_0^2 \mathbf{P}^{-1} \right), \tag{3}$$

where σ_0^2 denotes the unknown variance component. $\mathbf{Q}$ and $\mathbf{P}$ are the positive-definite cofactor matrix and the weight matrix of the random error vector $\boldsymbol{\varepsilon}$ of the functionally independent observations. Note that WTLS refers to the very general weighted TLS approach where the matrices $\mathbf{Q}$ and $\mathbf{P}$ are arbitrary positive definite matrices.

After introducing the functional and stochastic model, the following linear and quadratic constraints are taken into consideration together with structured EIV model (see *Schaffrin and Felus, 2009*):

$$\mathbf{K}\boldsymbol{\xi} - \mathbf{k}_0 = \mathbf{0}, \qquad \boldsymbol{\xi}^{\mathrm{T}}\mathbf{M}\boldsymbol{\xi} - \alpha = 0, \tag{4}$$

where $\mathbf{K}$ is a fixed $m \times u$ matrix with full row rank and $\mathbf{k}_0$ is a fixed $m \times 1$ vector. $\mathbf{M}$ is a fixed symmetric and positive-definite $u \times u$ matrix and α is a positive constant.

3. THE LAGRANGE APPROACH TO A WTLS APPROACH WITH CONSTRAINTS

A typical approach to estimate a parameter vector within the structured EIV model Eq. (2) subject to constraints given by Eq. (4) is using the Lagrange multipliers. According to the traditional Lagrange approach the total least-squares target function is set up as follows:

$$\Phi(\boldsymbol{\varepsilon}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\mu}_1, \mu_2) = \boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{P}\boldsymbol{\varepsilon} + 2\boldsymbol{\lambda}^{\mathrm{T}}(\mathbf{y} - \mathbf{A}\boldsymbol{\xi} + \mathbf{B}\mathbf{S}\boldsymbol{\varepsilon}) + 2\boldsymbol{\mu}_1^{\mathrm{T}}(\mathbf{K}\boldsymbol{\xi} - \mathbf{k}_0) + \mu_2\left(\boldsymbol{\xi}^{\mathrm{T}}\mathbf{M}\boldsymbol{\xi} - \alpha\right), \tag{5}$$

where $\boldsymbol{\lambda}$, $\boldsymbol{\mu}_1$ and μ_2 are the Lagrange multiplier vectors corresponding to the structured EIV model, the linear constraints and the quadratic constraint, respectively.

Setting the partial derivatives of the target function with respect to all the variables $\boldsymbol{\xi}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}_1$, μ_2 equal to $\mathbf{0}$, gives the necessary conditions as

$$\frac{1}{2}\frac{\partial\Phi}{\partial\boldsymbol{\xi}}\bigg|_{\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\varepsilon}},\hat{\boldsymbol{\lambda}},\hat{\boldsymbol{\mu}}_1,\hat{\mu}_2} = -\mathbf{A}^{\mathrm{T}}\hat{\boldsymbol{\lambda}} + \hat{\mathbf{E}}_A^{\mathrm{T}}\hat{\boldsymbol{\lambda}} + \mathbf{K}^{\mathrm{T}}\hat{\boldsymbol{\mu}}_1 + \hat{\mu}_2\mathbf{M}\hat{\boldsymbol{\xi}} = \mathbf{0}\,, \tag{6}$$

$$\frac{1}{2}\frac{\partial\Phi}{\partial\boldsymbol{\varepsilon}}\bigg|_{\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\varepsilon}},\hat{\boldsymbol{\lambda}},\hat{\boldsymbol{\mu}}_1,\hat{\mu}_2} = \mathbf{P}\hat{\boldsymbol{\varepsilon}} + \mathbf{S}^{\mathrm{T}}\hat{\mathbf{B}}^{\mathrm{T}}\hat{\boldsymbol{\lambda}} = \mathbf{0}\,, \tag{7}$$

$$\frac{1}{2}\frac{\partial\Phi}{\partial\boldsymbol{\lambda}}\bigg|_{\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\varepsilon}},\hat{\boldsymbol{\lambda}},\hat{\boldsymbol{\mu}}_1,\hat{\mu}_2} = \mathbf{y} - \mathbf{A}\hat{\boldsymbol{\xi}} + \hat{\mathbf{B}}\mathbf{S}\hat{\boldsymbol{\varepsilon}} = \mathbf{0}\,, \tag{8}$$

$$\frac{1}{2}\frac{\partial\Phi}{\partial\boldsymbol{\mu}_1}\bigg|_{\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\varepsilon}},\hat{\boldsymbol{\lambda}},\hat{\boldsymbol{\mu}}_1,\hat{\mu}_2} = \mathbf{K}\hat{\boldsymbol{\xi}} - \mathbf{k}_0 = \mathbf{0}\,, \tag{9}$$

$$\frac{1}{2}\frac{\partial\Phi}{\partial\mu_2}\bigg|_{\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\varepsilon}},\hat{\boldsymbol{\lambda}},\hat{\boldsymbol{\mu}}_1,\hat{\mu}_2} = \hat{\boldsymbol{\xi}}^{\mathrm{T}}\mathbf{M}\hat{\boldsymbol{\xi}} - \alpha = \mathbf{0}\,, \tag{10}$$

where $\hat{\mathbf{B}} = \left[\hat{\boldsymbol{\xi}}^{\mathrm{T}} \otimes \mathbf{I}_n, -\mathbf{I}_n\right]$.

From Eq. (7) the error vector is estimated as

$$\hat{\boldsymbol{\varepsilon}} = -\mathbf{Q}\mathbf{S}^{\mathrm{T}}\hat{\mathbf{B}}^{\mathrm{T}}\hat{\boldsymbol{\lambda}}\,. \tag{11}$$

Using Eq. (11) in Eq. (8) allows $\hat{\boldsymbol{\lambda}}$ to be expressed by

$$\hat{\boldsymbol{\lambda}} = \left(\hat{\mathbf{B}}\mathbf{S}\mathbf{Q}\mathbf{S}^{\mathrm{T}}\hat{\mathbf{B}}^{\mathrm{T}}\right)^{-1}\left(\mathbf{y} - \mathbf{A}\hat{\boldsymbol{\xi}}\right), \tag{12}$$

assuming the invertibility of this matrix $\hat{\mathbf{B}}\mathbf{S}\mathbf{Q}\mathbf{S}^{\mathrm{T}}\hat{\mathbf{B}}^{\mathrm{T}}$ of size $n \times n$.

Apart from its restriction in the stochastic model formulation, the cofactor matrix $\mathbf{Q}_{ll}$ of all observations $\mathbf{l} = \mathrm{vec}\left(\left[\mathbf{A}, \mathbf{y}\right]\right)$ can, however, here be introduced to build the STLS solution by the cofactor propagation theorem, i.e., $\mathbf{Q}_{ll} = \mathbf{Q}_{ee} = \mathbf{S}\mathbf{Q}\mathbf{S}^{\mathrm{T}}$, which will be singular. In other words, its non-existing regular inverse may not appear in the target function but is not used thoroughly in the later derivation for this proposed solution.

By reinserting Eq. (12) into Eq. (11) the functionally independent random error vector and the extended random error vector can be determined as

$$\hat{\boldsymbol{\varepsilon}} = -\mathbf{Q}\mathbf{S}^{\mathrm{T}}\hat{\mathbf{B}}^{\mathrm{T}}\left(\hat{\mathbf{B}}\mathbf{Q}_{ll}\hat{\mathbf{B}}^{\mathrm{T}}\right)^{-1}\left(\mathbf{y} - \mathbf{A}\hat{\boldsymbol{\xi}}\right), \tag{13}$$

and

$$\hat{\mathbf{e}} = \begin{bmatrix} \hat{\mathbf{e}}_A := \mathrm{vec}\left(\hat{\mathbf{E}}_A\right) \\ \hat{\mathbf{e}}_y \end{bmatrix} = \mathbf{S}\hat{\boldsymbol{\varepsilon}} = -\mathbf{Q}_{ll}\hat{\mathbf{B}}^{\mathrm{T}}\left(\hat{\mathbf{B}}\mathbf{Q}_{ll}\hat{\mathbf{B}}^{\mathrm{T}}\right)^{-1}\left(\mathbf{y} - \mathbf{A}\hat{\boldsymbol{\xi}}\right), \tag{14}$$

leading to the simplified formula of the cost function

$$\hat{\boldsymbol{\varepsilon}}^{\mathrm{T}}\mathbf{P}\hat{\boldsymbol{\varepsilon}} = \left(\mathbf{y} - \mathbf{A}\hat{\boldsymbol{\xi}}\right)^{\mathrm{T}}\left(\hat{\mathbf{B}}\mathbf{Q}_{ll}\hat{\mathbf{B}}^{\mathrm{T}}\right)^{-1}\left(\mathbf{y} - \mathbf{A}\hat{\boldsymbol{\xi}}\right). \tag{15}$$

The cofactor matrix $\mathbf{Q}_{ll}$ contains the four block cofactor matrices $\mathbf{Q}_{AA}$, $\mathbf{Q}_{Ay}$, $\mathbf{Q}_{yA}$ and $\mathbf{Q}_{yy}$ which have the dimensions $nu \times nu$, $nu \times n$, $n \times nu$ and $n \times n$, respectively.

Certainly, the error matrix $\hat{\mathbf{E}}_A$ can be retrieved by $\hat{\mathbf{e}}_A$ in Eq. (14). Using the estimated $\hat{\mathbf{E}}_A$ and $\hat{\boldsymbol{\lambda}}$, Eq. (6) can be simplified as

$$\left(\mathbf{A}-\hat{\mathbf{E}}_A\right)^{\mathrm{T}}\left(\hat{\mathbf{B}}\mathbf{Q}_{ll}\hat{\mathbf{B}}^{\mathrm{T}}\right)^{-1}\left(\mathbf{y}-\hat{\mathbf{E}}_A\hat{\boldsymbol{\xi}}+\hat{\mathbf{E}}_A\hat{\boldsymbol{\xi}}-\mathbf{A}\hat{\boldsymbol{\xi}}\right)-\mathbf{K}^{\mathrm{T}}\hat{\boldsymbol{\mu}}_1-\hat{\mu}_2\mathbf{M}\hat{\boldsymbol{\xi}}=\mathbf{0}\,, \tag{16}$$

leading to

$$\mathbf{n}-\mathbf{N}\hat{\boldsymbol{\xi}}-\mathbf{K}^{\mathrm{T}}\hat{\boldsymbol{\mu}}_1-\hat{\mu}_2\mathbf{M}\hat{\boldsymbol{\xi}}=\mathbf{0}\,, \tag{17}$$

where

$$\mathbf{N}=\left(\mathbf{A}-\hat{\mathbf{E}}_A\right)^{\mathrm{T}}\left(\hat{\mathbf{B}}\mathbf{Q}_{ll}\hat{\mathbf{B}}^{\mathrm{T}}\right)^{-1}\left(\mathbf{A}-\hat{\mathbf{E}}_A\right),\quad \mathbf{n}=\left(\mathbf{A}-\hat{\mathbf{E}}_A\right)^{\mathrm{T}}\left(\hat{\mathbf{B}}\mathbf{Q}_{ll}\hat{\mathbf{B}}^{\mathrm{T}}\right)^{-1}\left(\mathbf{y}-\hat{\mathbf{E}}_A\hat{\boldsymbol{\xi}}\right). \tag{18}$$

Note that, for brevity, the matrix **N** and the vector **n** keep their symbol to express their respective estimates. Here, one can readily recognize that $\mathbf{N}^{-1}\mathbf{n}$ is the well-established estimator for the unconstrained STLS parameters of *Fang (2011)*. Combining Eqs (9) and (17), the constrained STLS solution can be formulated by the following augmented equations system:

$$\begin{bmatrix}\mathbf{N} & \mathbf{K}^{\mathrm{T}}\\ \mathbf{K} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{\xi}}\\ \hat{\boldsymbol{\mu}}_1\end{bmatrix}=\begin{bmatrix}\mathbf{n}-\hat{\mu}_2\mathbf{M}\hat{\boldsymbol{\xi}}\\ \mathbf{k}_0\end{bmatrix}, \tag{19}$$

where the estimate $\hat{\mu}_2$ still appear on the right hand side.

Based on the above equation the parameter estimates can be obtained by elimination (*Koch, 1999, p. 33)* of the vector $\hat{\boldsymbol{\mu}}_1$ as

$$\begin{aligned}\hat{\boldsymbol{\xi}}&=\mathbf{N}^{-1}\left\{\mathbf{n}-\hat{\mu}_2\mathbf{M}\hat{\boldsymbol{\xi}}+\mathbf{K}^{\mathrm{T}}\left(\mathbf{K}\mathbf{N}^{-1}\mathbf{K}^{\mathrm{T}}\right)^{-1}\left[\mathbf{k}_0-\mathbf{K}\mathbf{N}^{-1}\left(\mathbf{n}-\hat{\mu}_2\mathbf{M}\boldsymbol{\xi}\right)\right]\right\}\\ &=\underbrace{\mathbf{N}^{-1}\left[\mathbf{n}+\mathbf{K}^{\mathrm{T}}\left(\mathbf{K}\mathbf{N}^{-1}\mathbf{K}^{\mathrm{T}}\right)^{-1}\left(\mathbf{k}_0-\mathbf{K}\mathbf{N}^{-1}\mathbf{n}\right)\right]}_{\mathbf{w}}\\ &\quad+\hat{\mu}_2\underbrace{\mathbf{N}^{-1}\left[\mathbf{K}^{\mathrm{T}}\left(\mathbf{K}\mathbf{N}^{-1}\mathbf{K}^{\mathrm{T}}\right)^{-1}\mathbf{K}\mathbf{N}^{-1}\mathbf{M}\hat{\boldsymbol{\xi}}-\mathbf{M}\hat{\boldsymbol{\xi}}\right]}_{\mathbf{z}}=\mathbf{w}+\hat{\mu}_2\mathbf{z}\,,\end{aligned} \tag{20}$$

where **w**, **z** are vector functions to describe the estimated parameter vector in terms of $\hat{\mu}_2$. Note that both **w**, **z** depend on $\hat{\boldsymbol{\xi}}$.

Now, by taking the quadratic constraint into consideration, the estimates of Eq. (20) is inserted back into Eq. (10):

$$\begin{aligned}&\left(\mathbf{w}+\hat{\mu}_2\mathbf{z}\right)^{\mathrm{T}}\mathbf{M}\left(\mathbf{w}+\hat{\mu}_2\mathbf{z}\right)-\alpha=\mathbf{0}\\ \Rightarrow&\left(\mathbf{z}^{\mathrm{T}}\mathbf{M}\mathbf{z}\right)\hat{\mu}_2^2+\left(2\mathbf{w}^{\mathrm{T}}\mathbf{M}\mathbf{z}\right)\hat{\mu}_2+\left(\mathbf{w}^{\mathrm{T}}\mathbf{M}\mathbf{w}-\alpha\right)=\mathbf{0}\,.\end{aligned} \tag{21}$$

By this quadratic equation one can select a meaningful solution $\hat{\mu}_2$, and compute Eqs (20) and (21) iteratively in analogy of the unweighted constrained TLS solution proposed by *Schaffrin and Felus (2009)* until the desired solution is found. In the designed computation process the two real values of $\hat{\mu}_2$, as obtained by Eq. (21), yield two different values of the cost function, and the $\hat{\mu}_2$ corresponding to the smaller variance component is chosen. The convergence of the iterative computation of the unknown parameter vector will be investigated in the future. Here it must be noted that this Lagrange multiplier is computed via using the algorithm of *Mahboub and Sharifi (2013 a,b)*. This also means that the two real values of $\hat{\mu}_2$ will yield two estimated parameter vectors obtained by Eq. (20); then, one can simply choose the parameter vector, which corresponds to the smaller of the two variance components as computed by Eq. (22). The variance component can be obtained by

$$\hat{\sigma}_0^2 = \frac{\left(\mathbf{y} - \mathbf{A}\hat{\xi}\right)^{\mathrm{T}} \left(\hat{\mathbf{B}}\mathbf{Q}_{ll}\hat{\mathbf{B}}^{\mathrm{T}}\right)^{-1} \left(\mathbf{y} - \mathbf{A}\hat{\xi}\right)}{n - u + m + 1} \tag{22}$$

for the constrained case, which is similar to the variance component estimator in *Mahboub and Sharifi (2013a)*.

4. THE CONSTRAINED WTLS ALGORITHMS

Approaches such as ours, *Mahboub (2012)* and *Mahboub and Sharifi (2013 a,b)* are not STLS techniques although they produce an optimal STLS solution. In other words, the STLS approaches are based on imposing given external conditions, and consequently, are usually confusing, see for example weighted structured TLS (*Markovsky and Van Huffel, 2006*) while our approach automatically produces a structured TLS solution. Also it is noted that currently, we usually implement STLS techniques for post adjustment and updating our previous results, e.g. *Schaffrin et al. (2012)*. It is reasonable that one adjusts the structured EIV models using our WTLS approach since they are considerably more simple, efficient and exact.

Summarizing the formulae introduced in the last section, the algorithms for the unconstrained and the constrained WTLS solution are designed as Algorithms 1–4 for different cases as follows:

Algorithm 1: WTLS solution without constraints

1. Begin with the initial values of the parameter vector $\hat{\xi}^0 = \left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1} \mathbf{A}^{\mathrm{T}}\mathbf{y}$;
2. Define $\mathbf{Q}_{ll} = \mathbf{S}\mathbf{Q}\mathbf{S}^{\mathrm{T}}$, and compute for $i \in N$

$$\hat{\mathbf{B}}^i = \left[\left(\hat{\xi}^{i-1}\right)^{\mathrm{T}} \otimes \mathbf{I}_n, -\mathbf{I}_n\right],$$

$$\hat{\mathbf{E}}_A^i = \mathrm{Ivec}\left(\hat{\mathbf{e}}_A^i\right) = \mathrm{Ivec}\left(-\begin{bmatrix}\mathbf{Q}_{AA} & \mathbf{Q}_{Ay}\end{bmatrix}\left(\hat{\mathbf{B}}^i\right)^{\mathrm{T}}\left(\hat{\mathbf{B}}^i\mathbf{Q}_{ll}\left(\hat{\mathbf{B}}^i\right)^{\mathrm{T}}\right)^{-1}\left(\mathbf{y} - \mathbf{A}\hat{\xi}^{i-1}\right)\right).$$

Note that Ivec in order to retrieve the $n \times u$ matrix $\hat{\mathbf{E}}_A^{i-1}$ from the vector $\hat{\mathbf{e}}_A^{i-1}$ denotes the appropriate inverse operator of the operator vec.

$$\mathbf{N}^i = \left(\mathbf{A} - \hat{\mathbf{E}}_A^i\right)^{\mathrm{T}} \left(\hat{\mathbf{B}}^i \mathbf{Q}_{ll} \left(\hat{\mathbf{B}}^i\right)^{\mathrm{T}}\right)^{-1} \left(\mathbf{A} - \hat{\mathbf{E}}_A^i\right),$$

$$\mathbf{n}^i = \left(\mathbf{A} - \hat{\mathbf{E}}_A^i\right)^{\mathrm{T}} \left(\hat{\mathbf{B}}^i \mathbf{Q}_{ll} \left(\hat{\mathbf{B}}^i\right)^{\mathrm{T}}\right)^{-1} \left(\mathbf{y} - \hat{\mathbf{E}}_A^i \hat{\boldsymbol{\xi}}^i\right),$$

$$\hat{\boldsymbol{\xi}}^i = \left(\mathbf{N}^i\right)^{-1} \mathbf{n}^i .$$

3. End when $\left\|\hat{\boldsymbol{\xi}}^i - \hat{\boldsymbol{\xi}}^{i-1}\right\| < \varepsilon$ for a small threshold ε, $\hat{\boldsymbol{\xi}}_{\mathrm{STLS}} := \hat{\boldsymbol{\xi}}^i$.

The STLS solution is based on the Gauss Newton approach, which provides a linear convergence rate. The STLS solution is identical to the WTLS solution proposed by *Fang (2011)* and *Snow (2012)*.

Algorithm 2: STLS solution with linear and quadratic constraints

1. Begin with the initial values of the parameter vector, for example

$$\begin{bmatrix} \hat{\boldsymbol{\xi}}^0 \\ \hat{\boldsymbol{\mu}}_1^0 \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{\mathrm{T}}\mathbf{A} & \mathbf{K}^{\mathrm{T}} \\ \mathbf{K} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}^{\mathrm{T}}\mathbf{y} \\ \mathbf{k}_0 \end{bmatrix}.$$

2. Define $\mathbf{Q}_{ll} = \mathbf{SQS}^{\mathrm{T}}$, and compute for $i \in N$:

$$\mathbf{w}^i = \left(\mathbf{N}^i\right)^{-1} \left(\mathbf{n}^i + \mathbf{K}^{\mathrm{T}} \left(\mathbf{K}\left(\mathbf{N}^i\right)^{-1} \mathbf{K}^{\mathrm{T}}\right)^{-1} \left(\boldsymbol{\kappa}_0 - \mathbf{K}\left(\mathbf{N}^i\right)^{-1} \mathbf{n}^i\right)\right),$$

$$\mathbf{z}^i = \left(\mathbf{N}^i\right)^{-1} \left(\mathbf{K}^{\mathrm{T}} \left(\mathbf{K}\left(\mathbf{N}^i\right)^{-1} \mathbf{K}^{\mathrm{T}}\right)^{-1} \mathbf{K}\left(\mathbf{N}^i\right)^{-1} \mathbf{M}\hat{\boldsymbol{\xi}}^{i-1} - \mathbf{M}\hat{\boldsymbol{\xi}}^{i-1}\right),$$

$\mathbf{N}^i$ and $\mathbf{n}^i$ have the identical formulae as in Algorithm 1.

By solving the following quadratic equation

$$\left[\left(\mathbf{z}^i\right)^{\mathrm{T}} \mathbf{M}\mathbf{z}^i\right]\left(\hat{\mu}_2^i\right)^2 + \left[2\left(\mathbf{w}^i\right)^{\mathrm{T}} \mathbf{M}\mathbf{z}^i\right]\hat{\mu}_2^i + \left[\left(\mathbf{w}^i\right)^{\mathrm{T}} \mathbf{M}\mathbf{w}^i - \alpha\right] = \mathbf{0},$$

two values $\left(\hat{\mu}_2^i\right)_{1,2}$ can be obtained.

If both are real values, the two corresponding estimates for the parameter vectors are obtained through

$$\left(\hat{\boldsymbol{\xi}}^i\right)_{1,2} = \mathbf{w}^i + \left(\hat{\mu}_2^i\right)_{1,2} \mathbf{z}^i,$$

which yield the two corresponding variance component values

$$\left(\hat{\mathbf{B}}^i\right)_{1,2} = \left[\left(\hat{\xi}^i\right)_{1,2}^{\mathrm{T}} \otimes \mathbf{I}_n, -\mathbf{I}_n\right]$$

$$\left(\sigma_0^2\right)_{1,2} = \frac{\left(\mathbf{y} - \mathbf{A}\left(\hat{\xi}^i\right)_{1,2}\right)^{\mathrm{T}} \left(\left(\hat{\mathbf{B}}^i\right)_{1,2} \mathbf{Q}_{ll} \left(\hat{\mathbf{B}}^i\right)_{1,2}^{\mathrm{T}}\right)^{-1} \left(\mathbf{y} - \mathbf{A}\left(\hat{\xi}^i\right)_{1,2}\right)}{n - u + m + 1}.$$

Then the parameter corresponding to the smaller cost function is selected as $\hat{\xi}^i$.

3. End when $\left\|\hat{\xi}^i - \hat{\xi}^{i-1}\right\| < \varepsilon$ for a small threshold ε, the linearly and quadratic constrained STLS: $\hat{\xi}_{\mathrm{LQCSTLS}} := \hat{\xi}^i$.

The following Algorithm 3 and Algorithm 4 are simplified, based on Algorithm 2.

Algorithm 3: STLS solution with linear constraints only

1. Begin the initial value of the parameter vector, for example

$$\begin{bmatrix} \hat{\xi}^0 \\ \hat{\boldsymbol{\mu}}_1^0 \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{\mathrm{T}}\mathbf{A} & \mathbf{K}^{\mathrm{T}} \\ \mathbf{K} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}^{\mathrm{T}}\mathbf{y} \\ \mathbf{k}_0 \end{bmatrix}.$$

2. Define $\mathbf{Q}_{ll} = \mathbf{SQS}^{\mathrm{T}}$, and compute for $i \in N$:

$$\hat{\xi}^i = \left(\mathbf{N}^i\right)^{-1} \mathbf{n}^i + \left(\mathbf{N}^i\right)^{-1} \mathbf{K}^{\mathrm{T}} \left(\mathbf{K}\left(\mathbf{N}^i\right)^{-1} \mathbf{K}^{\mathrm{T}}\right)^{-1} \left(\mathbf{k}_0 - \mathbf{K}\left(\mathbf{N}^i\right)^{-1} \mathbf{n}^i\right),$$

$\mathbf{N}^i$ and $\mathbf{n}^i$ have the identical formulae as in Algorithm 1.

3. End when $\left\|\hat{\xi}^i - \hat{\xi}^{i-1}\right\| < \varepsilon$ for a small threshold ε, the linearly constrained STLS:

$$\hat{\xi}_{\mathrm{LCSTLS}} := \hat{\xi}^i = \hat{\xi}_{\mathrm{STLS}} - \left(\mathbf{N}^{i-1}\right)^{-1} \mathbf{K}^{\mathrm{T}} \left(\mathbf{K}\left(\mathbf{N}^{i-1}\right)^{-1} \mathbf{K}^{\mathrm{T}}\right)^{-1} \left(\mathbf{K}\hat{\xi}_{\mathrm{STLS}} - \mathbf{k}_0\right).$$

Note that the end formula indicates the relationship between the linear constrained STLS and the STLS.

Algorithm 4: STLS solution with quadratic constraint only

1. Begin with the initial values of the parameter vector $\hat{\xi}^0 = \left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1} \mathbf{A}^{\mathrm{T}}\mathbf{y}$.

2. Define $\mathbf{Q}_{ll} = \mathbf{SQS}^{\mathrm{T}}$, and compute for $i \in N$:

$$\mathbf{w}^i = \left(\mathbf{N}^i\right)^{-1} \mathbf{n}^i, \quad \mathbf{z}^i = -\left(\mathbf{N}^i\right)^{-1} \mathbf{M}\hat{\xi}^{i-1},$$

$\mathbf{N}^i$ and $\mathbf{n}^i$ have the identical formulae as in Algorithm 1.

By solving the corresponding quadratic equation (same as in Algorithm 2), and then the parameter corresponding to the smaller cost function is selected as $\hat{\xi}^i$.

3. End when $\left\|\hat{\xi}^i - \hat{\xi}^{i-1}\right\| < \varepsilon$ for a small threshold ε, the quadratic constrained STLS: $\hat{\xi}_{\mathrm{QCSTLS}} := \hat{\xi}^i$.

The derived algorithms can solve the STLS problem with both linear and quadratic constraints although the exact convergence properties are still to be investigated. The Algorithm 2 can be regarded as a generalization of *Schaffrin and Felus (2009)* in the sense that the weight information, respectively the structure of the design matrix is taken into consideration. If convergence is achieved, the algorithms provide estimates at the stationary point, which satisfies the first-order necessary conditions Eqs (6)–(10). The second order sufficient conditions is already satisfied, see the Hessian matrix in terms of the random error vector (*Schaffrin and Wieser, 2008*).

5. APPLICATIONS

The following applications demonstrate the use and efficiency of the designed algorithms. The first two applications show a straight line fitting example and a geodetic resection for positioning, and Application 3 presents a 2-D rigid transformation problem.

Application 1: TLS with linear and quadratic constraints

Although in the following example the coefficient matrix is not structured (i.e. $\mathbf{Q}_{ll} = \mathbf{Q}$), the method will still be beneficial when compared with existing constrained TLS method. The input data presented in *Schaffrin and Felus (2009)* for the example is written matrix-wise as follows

$$\mathbf{A} = \begin{bmatrix} -0.5 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 6 \\ 3 \\ 4 \\ 10 \end{bmatrix}, \quad \mathbf{Q}_{ll} = \mathbf{Q} = \mathbf{I}_{16},$$

$$\text{linear constraint: } \begin{bmatrix} -2 & 0 & 3 \end{bmatrix} \boldsymbol{\xi} - 16 = 0,$$

$$\text{quadratic constraint: } \boldsymbol{\xi}^{\mathrm{T}} \operatorname{Diag}\left(\left[\left(\frac{1}{12}\right)^2, \left(\frac{1}{8}\right)^2, \left(\frac{1}{12}\right)^2 \right] \right) \boldsymbol{\xi} - 1 = 0,$$

where "Diag" denotes the operator that converts a vector into a diagonal matrix, with the vector's elements representing the diagonal entries of the matrix (see *Schaffrin and Wieser, 2008*). The magnitude of the stopping criterion is selected as $\varepsilon = 10^{-10}$ for all examples in this paper.

After implementing Algorithms 1–4, the results are presented in Table 1 in the four cases, in which the unconstrained, the linearly constrained, quadratically constrained STLS solution and the STLS solution with both types of constraints are considered. Here, the same formulae for setting the initial values were used as *Schaffrin and Felus (2009)*, which are differently given in these cases. *Schaffrin and Felus (2009)* proposed the mixed TLS solution based on the Euler Lagrange theorem. After solving the Lagrange multiplier

Table 1. Numerical result of Application 1, $\hat{\xi} = \begin{bmatrix} \hat{\xi}_1 & \hat{\xi}_2 & \hat{\xi}_3 \end{bmatrix}^{\mathrm{T}}$.

Estimates	Unconstrained TLS		TLS with Linear Constraint		TLS with Quadratic Constraint		TLS with Linear and Quadratic Constraints	
	Proposed Method	SF2009	Proposed Method	SF2009	Proposed Method	SF2009	Proposed Method	SF2009
$\hat{\xi}_1$	4.68316	4.68316	2.36823	2.36819	4.97386	4.97385	2.59730	2.59729
$\hat{\xi}_2$	6.24535	6.24535	5.69850	5.69844	6.46567	6.46567	6.23045	6.23044
$\hat{\xi}_3$	5.13041	5.13041	6.91215	6.91213	5.01993	5.01993	7.06486	7.06486
Iterations	22	---	10	12	22	8	11	15
TSSR	0.18400	0.18400	0.21284	0.21284	0.18477	0.18477	0.21854	0.21854
Variance Component	0.18400	0.18400	0.10642	0.10642	0.09239	0.09239	0.07285	0.07285

SF2009 - *Schaffrin and Felus (2009)*, TLS - Total Least Squares, *TSSR* - total sum of squared residuals

by an arranging quadratic equation, the unknown parameters can be determined. Note that the comparison of the iterations makes sense only if the identical initial values are used.

As portrayed in Table 1, the parameter estimates are different from the results presented in *Schaffrin and Felus (2009)* at the level of 10^{-5}, because the magnitude of the stopping criteria is chosen differently. If the same magnitude $\varepsilon = 10^{-10}$ of the stopping criteria is chosen, the parameter estimates are identical by using both methods, which indicates that the proposed methods are able to provide an exact solution. The iterations of the proposed methods are presented in comparison with the iterations by applying the method of *Schaffrin and Felus (2009)*. The comparison shows that the proposed methods might be better in considering the convergence behavior if the problem includes linear constraints. Furthermore, the proposed solutions do not need to solve by the quadratic eigenvalue problem at every iteration as presented in *Sima et al. (2004)*, which would definitely make it a large computational expense. However, their method is based on the eigenvalue decomposition, and thus does not depend on the initial value making it a robust method, which might just provide a practical initial value for our iteration process. After comparing the parameter estimates, the last two rows of Table 1 show us that the value of the total sum of squared residuals (*TSSR*, see *Schaffrin et al., 2012*) and variance component coincides with the results presented in *Schaffrin and Felus (2009)*.

Application 2: Linearly constrained TLS with large perturbations

In the following typical surface-reconstruction problem, the data show large perturbations (see *Schaffrin and Felus, 2009*). After choosing the linearly constrained LS solution for the initial values, according to the following dataset as done by *Schaffrin and Felus (2009)*, Algorithm 3 was performed to present the estimates in Table 2.

Table 2. Numerical results of the Application 2, $\hat{\boldsymbol{\xi}} = \begin{bmatrix} \hat{\xi}_1 & \hat{\xi}_2 \end{bmatrix}^{\mathrm{T}}$.

Estimates	TLS with Linear Constraint
$\hat{\xi}_1$	−0.13225
$\hat{\xi}_2$	−5.99190
Iterations (proposed method)	24
Iterations (*Schaffrin and Felus, 2009*)	227
TSSR (total sum of squared residuals)	143.60377
Variance component	20.51482

$$\mathbf{A} = \begin{bmatrix} -10 & -9 & -11 & -10 & 9 & 10 & 11 & 10 \\ -5 & -1 & 2 & 7 & -6 & -2 & 1 & 6 \end{bmatrix}^{\mathrm{T}},$$

$$\mathbf{y} = \begin{bmatrix} 4 & -9 & 9 & -5 & 5 & -8 & 10 & -4 \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{Q}_{ll} = \mathbf{Q} = \boldsymbol{I}_{24},$$

with the linear constraint $\begin{bmatrix} 0.75 & -0.1 \end{bmatrix} \boldsymbol{\xi} - 0.5 = 0$.

Although the calculated parameter estimates correspond with the results presented in *Schaffrin and Felus (2009)*, the iterations of the proposed methods are significantly less than their iterations in a case with large perturbations. In comparison with other constrained ‘closed form’ TLS methods such as that by *van Huffel and Vandewalle (1991, p. 276)*, this proposed method for the linearly constrained case also shows a high degree of efficiency when treating the dataset, both in terms of level of convergence behavior and computational expense in each iteration; cf. the comparison between the ‘closed-form’ solution and the method by *Schaffrin and Felus (2009)*.

Application 3: Two-dimensional rigid transformation

The proposed methods could only be compared with the methods presented in *Schaffrin and Felus (2009)* in regard to the unweighted case, since their method is not able to handle both the structured and the weighted TLS case with linear and quadratic constraints. In many practical examples, some elements of the design/coefficient matrix are fixed and should not be modified and this approach cannot deal with these cases, see *Mahboub and Sharifi (2013a)*. Therefore, in this work, a two-dimensional rigid transformation is tested in order to present the applicability of the proposed methods.

The data in the example originate from *Felus and Burtch (2009)*. Four control points are identified and recorded in the two coordinate systems. Considering all 4 control points, the functional model of the whole system for the rigid transformation should read as a quadratically constrained STLS, namely the EIV model of a similarity transformation presented in *Mahboub and Sharifi (2013 a,b)* subject to $\boldsymbol{\xi}^{\mathrm{T}} \mathbf{M} \boldsymbol{\xi} = 1$, where $\mathbf{M} = \mathrm{Diag}\left(\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}\right)$. The unknown parameter vector is defined as $\boldsymbol{\xi} = \begin{bmatrix} \cos\alpha & \sin\alpha & \Delta x & \Delta y \end{bmatrix}^{\mathrm{T}}$ with rotation angle α and translation parameters Δx and Δy.

Table 3. Coordinates of source system and target system for Application 3 (the dataset was presented in *Felus and Burtch 2009*).

Point No.	*x* (Source)	*y* (Source)	*X* (Target)	*Y* (Target)
1	30	40	290	150
2	100	40	420	80
3	100	130	540	200
4	30	130	390	300

If the measurements are assumed as independently and identically distributed, the description of the cofactor matrix $\mathbf{Q}_{ll}$ can be obtained by error propagation as follows:

$$\mathbf{Q}_{ll} = \mathbf{SQS}^{\mathrm{T}},$$

where

$$\mathbf{S} = \begin{bmatrix} \mathbf{C}_A & \mathbf{0}_{32\times 8} \\ \mathbf{0}_{8\times 8} & \mathbf{I}_8 \end{bmatrix} = \begin{bmatrix} \begin{matrix} \mathbf{I}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_4 \\ \mathbf{0} & \mathbf{I}_4 \\ -\mathbf{I}_4 & \mathbf{0} \\ \multicolumn{2}{c}{\boldsymbol{0}_{16\times 8}} \end{matrix} & \mathbf{0}_{32\times 8} \\ \boldsymbol{0}_{8\times 8} & \boldsymbol{I}_8 \end{bmatrix}, \quad \mathbf{Q} = \mathbf{I}_{16}\ .$$

The matrix $\mathbf{S}$ reflects the functional relationship between the vectorized matrix $\mathrm{vec}([\mathbf{A}, \mathbf{y}])$ with a dimension of 40 × 1 and the functionally independent observation vector assigned by the source coordinates *x* (1, …, 4), *y* (1, …, 4) and the target coordinates *X* (1, …, 4), *Y* (1, …, 4) with a dimension of 16 × 1. It must be emphasized here again that the above cofactor matrix is a singular correlated matrix. Directly inverting this cofactor matrix, however, does not take place in the design of our algorithms.

In this example, the structure of the coefficient matrix should be preserved by the error propagation strategy proposed in this paper. The results of this example for equally weighted case of all observations by implementing Algorithm 4 are presented in Table 4. We also performed the rigid transformation by using the algorithm of *Mahboub and Sharifi (2013b)* in order to confirm. In fact, we simplified their algorithm to suit the rigid transformation problem which only requires quadratic constraint since the algorithm of *Mahboub and Sharifi (2013b)* contains both linear and quadratic constraints. The algorithm of *Mahboub and Sharifi (2013b)* provides the identical results to that of my proposed method when a proper initial value was used in their algorithm. For confirmation of our results, not only the estimated parameter vector and the Lagrange multiplier but also both the constraints and the constrained nonlinear normal equations are computed.

As seen in Table 4, the constraint $\hat{a}^2 + \hat{b}^2 = 1$ is strictly satisfied, yet the translations are larger than the results in the *x* and *y* orientations presented in *Felus and Burtch (2009)*

Table 4. Parameter estimates of the rigid transformation in an equally weighted case and its validations. *TSSR* - total sum of squared residuals.

$\hat{\xi}$ (Eq. (1))	$\hat{\xi}^{\mathrm{T}}\mathbf{M}\hat{\xi}$ (Eq. (10))	$\mathbf{N}\hat{\xi} - \mathbf{n} + \hat{\mu}_2\mathbf{M}\hat{\xi}$ (Eq. (17))	$\hat{\mu}_2$ (Eq. (21))	*TSSR* (Eq. (15))
0.810728 0.585423 307.541719 151.640630	1	-4.00×10^{-10} -4.07×10^{-10} -1.14×10^{-13} 5.68×10^{-13}	11118.75	8.163065×10^{3}

through using a similarity transformation model. These larger translations might be explained by ignoring the z coordinates in this example and using a different transformation model. Furthermore, the constrained nonlinear normal equations are fulfilled numerically.

In order to show the advantages of the proposed algorithm, we perform the example of rigid transformation through a stochastic model which contains cross-covariances. Here, we assume that the stochastic model is expressed as the following toeplitz structure:

$$D\left(\begin{bmatrix} \mathbf{x}_{\text{source}} \\ \mathbf{y}_{\text{source}} \\ \mathbf{x}_{\text{target}} \\ \mathbf{y}_{\text{target}} \end{bmatrix}\right) = \sigma_0^2\left[\left(q_{\text{toeplitz}}\right)_{ij}\right],$$

with $\left(q_{\text{toeplitz}}\right)_{ij} = 1 - \left(|i-j|/16\right)$.

The structure of the covariance matrix means that all the observations are correlated, and thus the cross covariances appear in this problem. When the cross-covariances are considered, the other algorithms cannot treat the rigid transformation problem. The adjustment results are given in Table 5 by implementing Algorithm 4.

Table 5 indicates that considering a fully populated variance covariance matrix, estimates of the parameter vector and Lagrange multiplier differ from those estimated in the equally weighted case. In order to confirm the results, the constraints and the constrained normal equation representing the first order necessary condition were also

Table 5. Parameter estimates of the rigid transformation in the fully weighted case and its validations. *TSSR* - total sum of squared residuals.

$\hat{\xi}$ (Eq. (1))	$\hat{\xi}^{\mathrm{T}}\mathbf{M}\hat{\xi}$ (Eq. (10))	$\mathbf{N}\hat{\xi} - \mathbf{n} + \hat{\mu}_2\mathbf{M}\hat{\xi}$ (Eq. (17))	$\hat{\mu}_2$ (Eq. (21))	*TSSR* (Eq. (15))
0.801263 0.598312 280.748599 165.572622	1	1.31×10^{-10} 0 0 5.68×10^{-14}	130400.00	1.004793×10^{5}

computed in this case and presented in the second and third columns of Table 5. The total sum of the squared weighted residuals (TSSWR) is much larger than the *TSSR* presented in Table 4 because the weights are distinct in the both cases.

6. CONCLUSIONS AND FURTHER STUDIES

In summary, a constrained WTLS method was presented to estimate the parameters within a structured EIV model with linear and quadratic constraints. The proposed structure identifies functionally independent random errors and takes functional relationships among them into account. The WTLS method developed here can imprint these linear functional relationships between unrepeated elements of the coefficient matrix. Meanwhile, this strategy is readily compatible in cases where linear and quadratic constraints are incorporated. In the first two applications the designed algorithms produced exactly the same outputs when compared to the existing constrained TLS methods, especially in a case with large perturbations. Another application was concerned with a rigid transformation and indicated that the proposed method is able to adjust the EIV model with constraints appropriately, referring to the unknown parameters and the elements within the structured coefficient matrix simultaneously. In the future, a quality description including a cofactor matrix of the estimated parameters and an unbiased variance component estimate ought to be investigated.

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