Effective calculation of gravity effects of uniform triangle polyhedra

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ABSTRACT

Uniform tetrahedra are commonly used elementary bodies for gravity calculations from which arbitrary polyhedra can be composed. A simple derivation of the gravity effect is presented for the apex P of the tetrahedron expanded from P to an arbitrarily oriented plane triangle. Integration of its potential effect in a rotated coordinate system applies vector algebra and renders the anomalous potential depending on the distance of P over the triangle plain and a function of the triangle coordinates. Partial differentiation by moving P infinitesimally in z-direction leads to two terms, a simple and a complex one; they can be understood as describing the same difference from two points of view: leaving P at the apex of the changed polyhedron or moving P off the unchanged polyhedron. Both views imply the same shape change and the sum over the polyhedron is thus numerically equal. Hence we need to calculate only the one of the terms of the differential which is simpler. The calculation of the gravity effect is numerically simplified and more stable. This has been tested for many models and is demonstrated by two examples.

Keywords: tetrahedron, gravity effect, computation

1. INTRODUCTION

Polyhedron approximation is an efficient approach to calculating the gravity effect Δg of arbitrarily shaped three-dimensional uniform bodies (*Pohánka, 1988*; *Holstein et al. 1999*; *Holstein 2002a,b*; *Jacoby and Smilde, 2009*). Polyhedra may be composed of triangles which suspend tetrahedra from the observation point *P*. The tetrahedron is thus the elementary body for gravity calculation, i.e., its effect at the apex at *P*. A simple method of deriving the equations is presented and tested by examples. It applies imagination rather than abstract arithmetic and has thus didactic advantages.

The derivation starts with the potential effect ΔU written in coordinates rotated into a system with one axis normal to the plane triangle, similar to the approach of *Götze and Lahmeyer (1988)*. Calculation of $\Delta g = \partial U / \partial z$ as the partial vertical derivative after *z*, i.e. two terms.

The first term has the simple geometrical meaning of removing *P* by *dz* from the unchanged tetrahedron. The other describes the effect of changing the basic tetrahedron by infinitesimally moving its apex *P* by *dz*. The two terms describe exactly the same differential shape of the tetrahedron, however with different interpretations. Numerically they differ generally, depending on the 3D orientation of the individual triangle 123 (Fig. 1). Due to the identical differential geometries, the relationships of the polyhedral triangles on the fore- and backsides are such that the individual differences cancel out in the sum total, such that for calculating the polyhedral gravity effect only the simpler term may be used.

The simpler first term is then efficiently evaluated numerically with practical advantages. The FORTRAN program developed (*Çavşak, 1992*) includes, as an option, the evaluation of both terms, in order to check the calculation. The application to polyhedra is briefly described. They are defined by plane triangles, i.e., by triangulation which is carried out by the code. The numerical quality of the calculations is demonstrated with polyhedral examples.

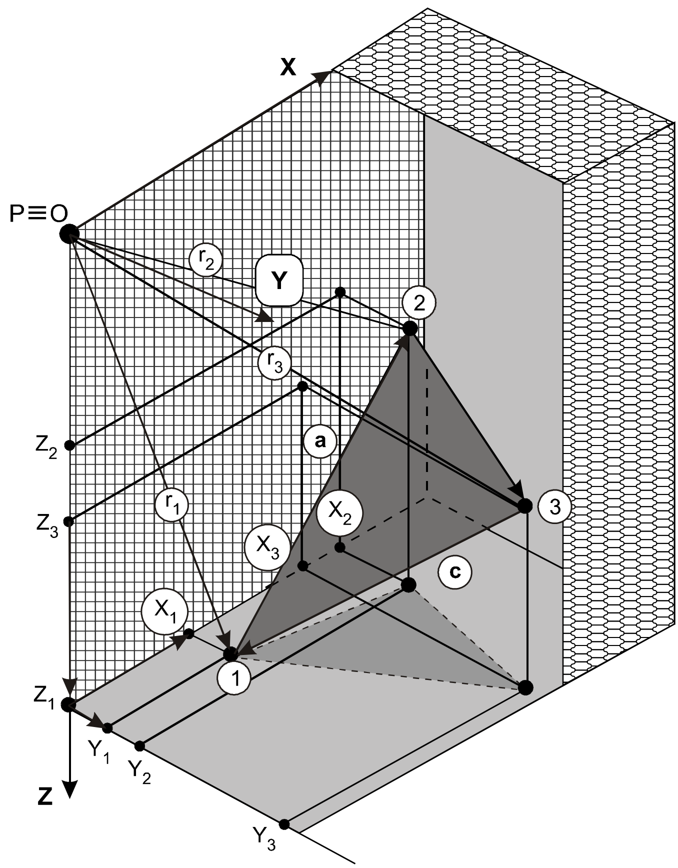

Fig. 1. Illustration of the parameters describing the tetrahedron T_O (O-1-2-3), arbitrarily oriented in earth-bound x, y, z coordinates; triangle t_O or 1-2-3 projected onto bottom x-y plane.

2. THE TETRAHEDRON CONTRIBUTION TO THE GRAVITATIONAL POTENTIAL ΔU AND TO THE GRAVITY EFFECT ΔG

Fig. 1 shows the basic uniform tetrahedron, called T_O, expanded from the "observation" point P at O to an arbitrarily oriented planar triangle, T_O, or 1-2-3 in an earth-oriented Cartesian coordinate system (x, y, z) with origin O and z pointing vertically downward. This does not reduce generality. First, the gravitational potential ΔU of the tetrahedron is derived for its apex P, then the gravity effect, Δg, is obtained as the vertical component of the potential gradient, by vertical differentiation of ΔU.

Integration of the tetrahedral potential effect, ΔU, in arbitrary orientation is awkward, but the orientation is irrelevant for the potential. Therefore a suitable coordinate transformation is carried out (Fig. 2): the ξ, η, ζ system is defined such that the triangle is in the $\xi - \eta$ plane and one edge (1-2) is parallel to ξ. The coordinate transformation is carried out by vector operations. With $\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$ and $\vec{c} = c_x \vec{i} + c_y \vec{j} + c_z \vec{k}$,

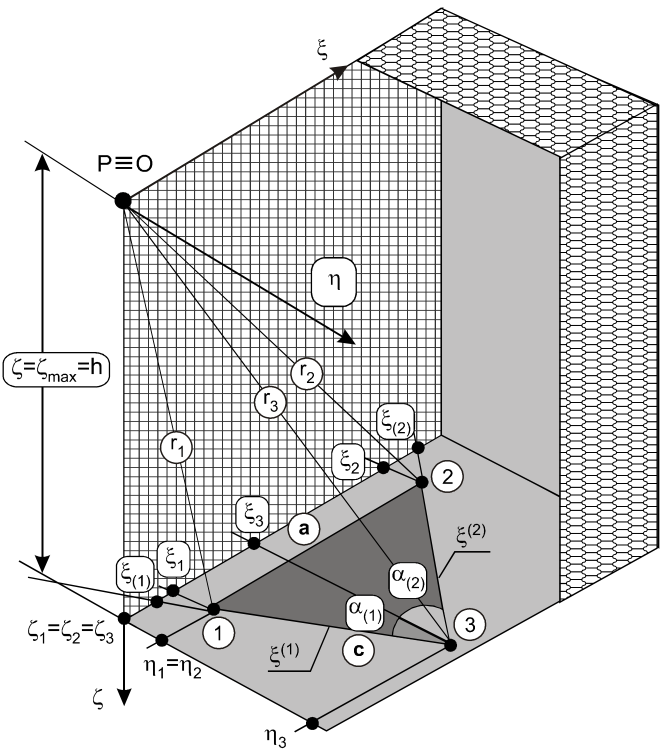

Fig. 2. Explanation of the nomenclature used in describing the tetrahedron, T_O, after coordinate transformation from x, y, z to the system ξ, η, ζ (see text) with the triangle, t_O, lying in the bottom ξ-η plane.

$\vec{\zeta} = \vec{c} \times \vec{a}$, $\vec{\eta} = \vec{\zeta} \times \vec{c}$, the unit vectors are $\hat{\xi} = \hat{a} = \vec{a}/|\vec{a}| = \hat{\xi}_x \vec{i} + \hat{\xi}_y \vec{j} + \hat{\xi}_z \vec{k}$ and analogously $\hat{\eta} = \vec{\eta}/|\vec{\eta}| = \hat{\eta}_x \vec{i} + \hat{\eta}_y \vec{j} + \hat{\eta}_z \vec{k}$ and $\hat{\zeta} = \vec{\zeta}/|\vec{\zeta}| = \hat{\zeta}_x \vec{i} + \hat{\zeta}_y \vec{j} + \hat{\zeta}_z \vec{k}$.

The triangle corner 1 can be written as $\xi_1 = X_1 \hat{\xi}_x + Y_1 \hat{\xi}_y + Z_1 \hat{\xi}_z$, $\eta_1 = X_1 \hat{\eta}_x + Y_1 \hat{\eta}_y + Z_1 \hat{\eta}_z$, $\zeta_1 = X_1 \hat{\zeta}_x + Y_1 \hat{\zeta}_y + Z_1 \hat{\zeta}_z$ and corners 2 and 3 are analogous (see Figs. 1 and 2).

The integration is carried out first along infinitesimal oblique pyramids expanded from $P \equiv O$ to the $d\xi d\eta$ quadrangles:

$$\int_{\zeta=0}^{h} r^{-1} dV \text{ , where } dV = d\xi' d\eta' d\zeta = \frac{\zeta^2}{h^2} d\xi d\eta d\zeta \text{ ,}$$

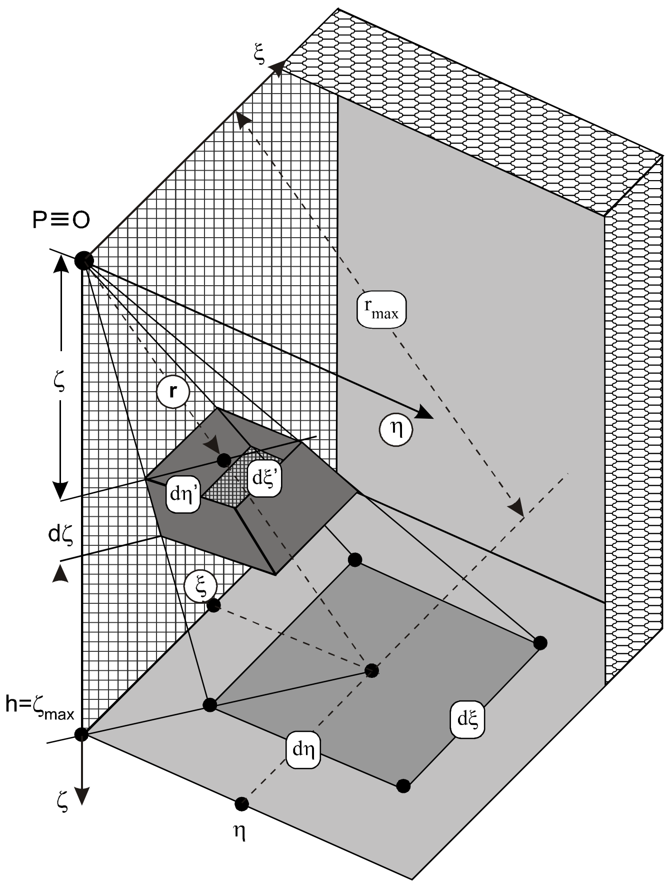

Fig. 3. Schematic illustration of application of the theorem of intersecting lines to the oblique pyramid infinitesimal element (exaggerated) in integratig ΔU.

applying $h=\zeta_{max}$ and the theorem of intersecting lines (see Fig. 3; for more details see Appendix A). This renders

$$\Delta U = \frac{\rho}{h}\int_{\eta_1}^{\eta_3}\int_{\xi^{(1)}}^{\xi^{(2)}}\int_{\zeta=0}^{h}\frac{\zeta d\zeta d\xi d\eta}{\left(\xi^2+\eta^2+h^2\right)^{1/2}} = \frac{\rho h}{2}\int_{\eta_1}^{\eta_3}\int_{\xi^{(1)}}^{\xi^{(2)}}\frac{d\xi d\eta}{\left(\xi^2+\eta^2+h^2\right)^{1/2}} . \qquad (1)$$

Integration for ΔU continues between the sides $\xi^{(1)} = \xi_{(1)} + \eta\tan\alpha_{(1)}$ and $\xi^{(2)} = \xi_{(2)} + \eta\tan\alpha_{(2)}$ and η is integrated from $\eta_1 = \eta_2$ to η_3 (see Fig. 2).

The result can be written in the form $\Delta U = hY/2$, where Y is a complicated expression (see Appendix A) which can be schematically written as

$$Y = F_{32} - F_{22} - \left(F_{31} - F_{11}\right),$$

with the first index, $i = 1, 2, 3$ referring to the triangle corners, and $(k) = (1),(2)$, referring to the sides $\xi^{(1)}$ and $\xi^{(2)}$, respectively, with $\alpha_{(k)}$ and $\xi_{(k)}$. The functions have the form:

$$\begin{aligned} F_{ik} &= \eta_i \ln\left(\xi_i + r_i\right) + \xi_{(k)}\cos\alpha_{(k)} \ln\left(r_i + \frac{\eta_i}{\cos\alpha_{(k)}} + \xi_{(k)}\sin\alpha_{(k)}\right) \\ &\quad + h\arctan\left(\frac{h^2\tan\alpha_{(k)} - \xi_{(k)}\eta_i}{hr_i}\right) . \end{aligned} \qquad (2)$$

The gravity effect is obtained by vertical (z) differentiation of $\Delta U = hY/2$:

$$\Delta g = \frac{1}{2}\frac{\partial\left(hY\right)}{\partial z}$$

and because both h and Y depend on z, with the product rule,

$$\Delta g = \frac{1}{2}\left(Y\frac{\partial h}{\partial z} + h\frac{\partial Y}{\partial z}\right) . \qquad (3)$$

$\Delta U = hY/2$ is calculated for the apex of the given tetrahedron T_O, to be differentiated by infinitesimally shifting the observation point P by dz. As Y is written in the rotated (ξ, η, ζ) system, a coordinate transformation of the expression to the terrestrial (x, y, z) system is first carried out. However, it leads to a rather complex expression (Appendix B).

As explained in the Introduction, the two terms of Eq.(3) describe two views of the same operation of differentiating the effect, Y, implying two definitions of the basic tetrahedron T_O with the apex P. Either P is moved infinitesimally to P' from the given fixed T_O (this may be the customary view), or the tetrahedron T_O is changed to T'_O by its apex P moving to P', such that the tetrahedron effect is altered from Y to Y'. Eq.(3) describes the total effect Δg as the sum of one half of each of the two terms. In both views the difference $d\Delta U$ between the effects before and after the infinitesimal shift of P can be interpreted as the effects of the geometrical identical infinitesimal differential tetrahedra at the sides between T_O and T'_O defined by P and P'. Y changes with dz due to changing the tetrahedron T_O of height h to T'_O of height $h + dh$ by the infinitesimal differential volume, where $dh = dz \cos\alpha$, $\alpha = (\boldsymbol{h}, \boldsymbol{z})$ *is* the angle between $\boldsymbol{h}$ and $\boldsymbol{z}$. The original effect is hY, and $d\Delta U = dz\, Y\, \partial h/\partial z$ describes the resulting change from ΔU_O at P to $\Delta U_O + d\Delta U$ at P'. The effect of the given tetrahedron, T_O, at the two points of observation is compared, and $h\, \partial y/\partial z$ is the effect of the infinitesimal masses at the sides of the individual tetrahedron. As argued above, for individual tetrahedral the two terms $Y\, \partial h/\partial z$ and $h\, \partial y/\partial z$ are generally different, but for the whole polyhedron the differences cancel out, because of the geometrically identical infinitesimal changes.

3. APPLICATIONS

The first term, $Y\, \partial h/\partial z$ is much easier to calculate than the second term, because geometrically $\partial h/\partial z = \hat{\zeta}_z$, which is simply the z-component of the ζ unit vector, and Y has already been calculated. This has the intellectual beauty of simplicity and is a little faster and numerically more stable than the complicated term would be. The differentiation of the second term, $h\, \partial y/\partial z$, has been carried out and the numerical equality of both terms has been demonstrated. It suffices to calculate the first term of $\partial \Delta U/\partial z$.

As an example, gravity is calculated for an earth fitting uniform sphere approximated by 2070720 triangles with a density $\rho = 5505$ kg/m^3 (Fig. 4). The triangles have angular dimensions of 0.25° in latitude and longitude (27.8 km at the equator and ~20 km longitudinally in mid-latitudes). Both terms of equation (3) render identical results within 12 decimals (980337.9 mGal), and they are compared to an analytical calculation (Newton's law) for a point mass (volume multiplied by ρ) at the sphere centre (980341.5 mGal). The relative difference is about $\sim 5 \times 10^{-5}$ partly due to geometrical details of the "observation point" and the spherical volume, as the plain triangles cut off small spherical calottes such that the volume of the sphere is reduced relatively by about 10^{-6}. A second example is shown in Fig. 5; it is simpler but has no axis of symmetry between P and body shape. It is a tetrahedron of density 200 kg/m^3 and the dimensions shown in the figure.

An arbitrary surface of a uniform three-dimensional body can be approximated to any degree of detail by a polyhedron of plane triangles defining the basic tetrahedra. The parameterisation is flexible and efficient. The effects Δg and ΔU of the polyhedron are the sums of all the basic tetrahedral effects. With a consistently defined sequence of

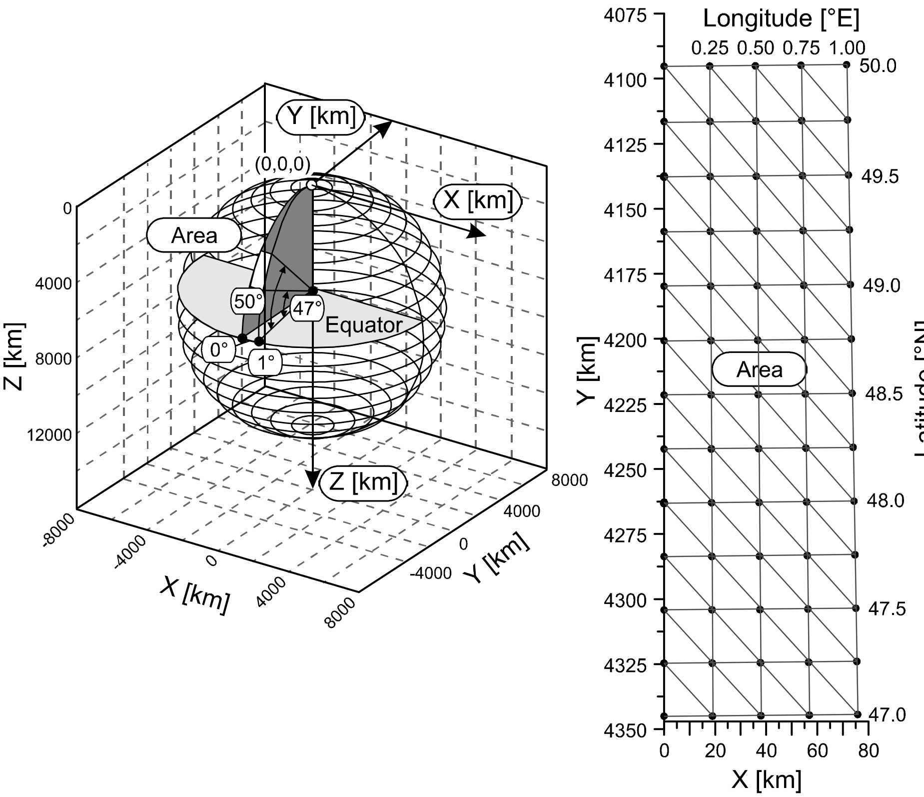

Fig. 4. Example of an earth fitting sphere approximated by 2070720 triangles with a constant density $\rho = 5505$ kg/m^3. The triangles cannot be shown for the whole sphere but the parametrization is illustrated by a strip of 3° lat. × 1° lon.

computational steps, the partial effects are automatically calculated with the correct sign, i.e. positive for "far-side" triangles and negative for "near-side" triangles; "near" and "far" signify the geometrical relation of the observation point *P* and the polyhedron. The far-side basic tetrahedral effects are added, while the near-side tetrahedral effects are subtracted such that only the effects of the intervening polyhedron remain. The calculations are also correct if *P* is enclosed in a polyhedron.

The routine can be profitably applied to local and global gravity modelling, e.g. of triangulated topography, any internal discontinuities or any other geodetic, geophysical and geological problem. Examples of applying the algorithm are the comparison of 2D and 3D spherical models of mid-ocean ridges by *Çavşak (2008)* and the inversion of gravity anomalies over spreading oceanic ridges *Jacoby et.al. (2005).*

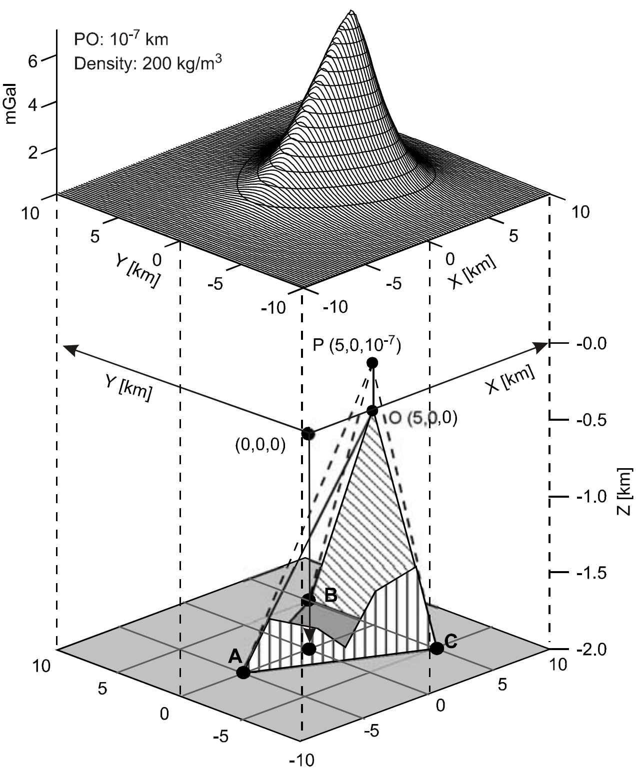

Fig. 5. Model of a tetrahedron-shaped geological body with no symmetry with the observation point P and the calculated gravity effect on the top. *PO* is the distance of the observation point *P* from the origin of the coordinate system (*O*).

APPENDIX A
INTEGRATION OF ΔU

Beginning with the integral, from Eq.(1)

$$\int_{\xi^{(2)}}^{\xi^{(1)}} \frac{d\xi}{\left(\xi^2+\eta^2+h^2\right)^{1/2}} = \ln\left[\xi+\left(\xi^2+\eta^2+h^2\right)^{1/2}\right]\Bigg|_{\xi^{(1)}=\xi_{(1)}+\eta\tan\alpha_{(1)}}^{\xi^{(2)}=\xi_{(2)}+\eta\tan\alpha_{(2)}} , \qquad \text{(A.1)}$$

and with $\eta_1 = \eta_2$ (Fig. 2) for $\xi^{(2)}$, we can write

$$\Delta U^{\xi^{(2)}} = \frac{1}{2}\left(G\rho h\right)\int_{\eta_1}^{\eta_3} \ln\left[\left(\xi_{(2)}+\eta\tan\alpha_{(2)}\right)+\sqrt{\left(\xi_{(2)}+\eta\tan\alpha_{(2)}\right)^2+\eta^2+h^2}\right]\mathrm{d}\eta , \qquad \text{(A.2)}$$

with $\eta = \eta_3$, $\xi^{(1)} = \xi^{(2)} = \xi_3$, $r_3 = \sqrt{\xi_3^2 + \eta_3^2 + h^2}$ and $\tan\alpha_{(2)} = \dfrac{\xi_3 - \xi_2}{\eta_3 - \eta_2}$.

Eq.(2) can be written (*Chapman, 1979*; *Çavşak, 1992*):

$$F_{32}\left(\eta_3, \xi^{(2)}\right) = \eta_3 \ln\left(\xi_3 + r_3\right) + \xi_{(2)} \cos\alpha_{(2)}$$
$$\ln\left(r_3 + \frac{\eta_3}{\cos\alpha_{(2)}} + \xi_{(2)} \sin\alpha_{(2)}\right) + h \arctan\left(\frac{h^2 \tan\alpha_{(2)} - \xi_{(2)}\eta_3}{h\, r_3}\right). \tag{A.3}$$

For $\xi^{(1)}$ the solution is analogous.

For $\eta = \eta_1$, $\xi^{(1)} = \xi_1$ and $\xi^{(2)} = \xi_2$, $r_1 = \sqrt{\xi_1^2 + \eta_1^2 + h^2}$, $r_2 = \sqrt{\xi_2^2 + \eta_2^2 + h^2}$ and $\tan\alpha_{(1)} = \dfrac{\xi_1 - \xi_3}{\eta_1 - \eta_3}$ (see Fig. 2), where

$$\xi_1 = X_1\hat{\xi}_x + Y_1\hat{\xi}_y + Z_1\hat{\xi}_z, \quad \xi_2 = X_2\hat{\xi}_x + Y_2\hat{\xi}_y + Z_2\hat{\xi}_z, \quad \xi_3 = X_3\hat{\xi}_x + Y_3\hat{\xi}_y + Z_3\hat{\xi}_z,$$
$$\eta_1 = \eta_2 = X_1\hat{\eta}_x + Y_1\hat{\eta}_y + Z_1\hat{\eta}_z, \quad \eta_3 = X_3\hat{\eta}_x + Y_3\hat{\eta}_y + Z_3\hat{\eta}_z, \tag{A.4}$$

we can formulate

$$Y = F_{32}\left(\eta_3, \xi_3\right) - F_{22}\left(\eta_1, \xi_2\right) - F_{31}\left(\eta_3, \xi_3\right) + F_{11}\left(\eta_1, \xi_1\right) \tag{A.5}$$

and with the height of the tetrahedron $h = \zeta_1 = \zeta_2 = \zeta_3 = X_1\hat{\zeta}_x + Y_1\hat{\zeta}_y + Z_1\hat{\zeta}_z$ (see Fig. 2)

$$\Delta U = \frac{1}{2} G\rho\left(hY\right). \tag{A.6}$$

APPENDIX B
CALCULATION OF THE GRAVITY EFFECT

In Eq.(3) of the paper, the complicated term $Y' = \partial Y / \partial z$ takes the form $Y' = F'_{32} - F'_{22} - \left(F'_{31} - F'_{11}\right)$

$$F'_{ik} = \hat{\eta}_z \ln\left(\xi_i + r_i\right) + \frac{\eta_i}{\xi_i + r_i}\left(\hat{\xi}_z + \frac{\xi_i\hat{\xi}_z + \eta_i\hat{\eta}_z + h\hat{h}_z}{r_i}\right)$$
$$+\left(\hat{\xi}_z \cos\alpha_{(k)} - \hat{\eta}_z \sin\alpha_{(k)}\right)\ln\left(r_i + \frac{\eta_i}{\cos\alpha_{(k)}} + \xi_{(k)} \sin\alpha_{(k)}\right) \tag{B.1}$$
$$+\xi_{(k)} \cos\alpha_{(k)}\left(\frac{1}{subst_1}\right) + \hat{h}_z \arctan\left(\frac{h^2 \tan\alpha_{(k)} - \xi_{(k)}\eta_i}{h\, r_i}\right) + h\frac{subst_2}{subst_3},$$

where

$$subst_1 = \left(r_i + \frac{\eta_i}{\cos\alpha_{(k)}} + \xi_{(k)} \sin\alpha_{(k)} \right)$$

$$\times \left(\frac{\xi_i \hat{\xi}_z + \eta_i \hat{\eta}_z + h \hat{h}_z}{r_i} + \frac{\hat{\eta}_z}{\cos\alpha_{(k)}} + \hat{\xi}_z \sin\alpha_{(k)} - \hat{\eta}_z \frac{\sin^2\alpha_{(k)}}{\cos\alpha_{(k)}} \right),$$

$$subst_2 = \left\{ 2h\hat{h}_z \cdot \tan\alpha_{(k)} - \left[\left(\hat{\xi}_z - \tan\alpha_{(k)} \hat{\eta}_z \right) \eta_i + \xi_{(k)} \hat{\eta}_z \right] \right\} h\, r_i$$

$$- \left(h^2 \tan\alpha_{(k)} - \xi_{(k)} \eta_i \right) \left(\hat{h}_z r_i + h \frac{\xi_i \hat{\xi}_z + \eta_i \hat{\eta}_z + h \hat{h}_z}{r_i} \right),$$

$$subst_3 = \left(h r_i \right)^2 + \left(h^2 \tan\alpha_{(k)} - \xi_{(k)} \eta_i \right)^2 .$$

With $h = \zeta_1 = X_1 \hat{\zeta}_x + Y_1 \hat{\zeta}_y + Z_1 \hat{\zeta}_z$, $\hat{\zeta}_z = \partial h / \partial z$, we obtain the traditional form $\Delta g = \partial \Delta U / \partial z = \frac{1}{2} \left(\hat{\xi}_z Y + Y' h \right)$, but, as argued in the paper, for the whole polyhedron

$$\sum_{i=1}^{n} \hat{\zeta}_{z_i} Y_i = \sum_{i=1}^{n} Y_i' h_i , \qquad \Delta g = \sum_{i=1}^{n} \hat{\zeta}_{z_i} Y_i . \tag{B.2}$$

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