GRAVITATIONAL ATTRACTION AND POTENTIAL OF SPHERICAL SHELL WITH RADIALLY DEPENDENT DENSITY

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ABSTRACT

Solutions to the direct problem in gravimetric interpretation are well-known for wide class of source bodies with constant density contrast. On the other hand, sources with non-uniform density can lead to relatively complicated formalisms. This is probably why analytical solutions for this type of sources are rather rare although utilization of these bodies can sometimes be very effective in gravity modeling. I demonstrate an analytical solution to that problem for a spherical shell with radial polynomial density distribution, and illustrate this result when applied to a special case of 5th degree polynomial. As a practical example, attraction of the normal atmosphere is calculated.

Keywords: gravimetry, direct problem, variable density, truncated spherical shell

1. INTRODUCTION

Analytical solutions for gravitation attraction have been presented for variety of bodies of simple geometries. These solutions usually assume constant density distribution, but the case of non-constant density can also be found in the literature (e.g. *Holstein, 2003*; *García-Abdeslem, 2005*).

The gravitational effect of spherical shell with constant density was derived by various authors and at least 11 (!) independent solutions have been published; in addition, I know about one formula which remained unpublished. Summary of the formulae was provided elsewhere (*Karcol, 2010*). One may suppose that there exist even more expressions. Among the published ones, *Cassinis et al. (1937)*, *Talwani (1973)* and *Mikuška et al. (2006)* can be mentioned as prominent examples. It can be shown, however, that their results are all the same, even if they do not appear so at the first view. The differences are mainly notational and algorithmic organization. In fact all of them are only special types of one general expression (*Karcol, 2010*).

The above authors set the shell density as constant in their formulations. However, the task becomes more complicated if the density is a function of depth or altitude (i.e. radial distance of the mass element from the origin of spherical coordinate system). In the case of complete spherical shell (a “hollow sphere”), the problem is reduced to simple computation of its mass. On the other hand, in the case of truncated spherical shell, the situation is much more complicated. In this paper I derive the solution for the gravitational potential and gravitational attraction of a circularly truncated spherical shell with polynomial density in the radial direction, i.e. in the distance from the origin of coordinate system. I believe that such a solution is presented herewith for the first time.

There can be several applications of such a solution, such as compartment-by-compartment calculation of the Earth's atmosphere effect when the topography is taken into consideration (*Mikuška et al., 2008*), compartment-by-compartment calculation of the bathymetric correction or estimating the effects of deep seated density inhomogeneity effects on a planetary scale. For instance, in the case of (normal) atmosphere, we can well describe its density distribution by a polynomial of 5th degree of variable ρ and directly calculate its gravitation effect, as carried out below.

In this article, Greek letters will refer to parameters of the body, while Latin letters refer to the calculation point.

The calculation point *P* is situated on the rotational axis of the truncated spherical shell which, at the same time, represents the vertical axis of the adopted spherical coordinate system (Fig. 1). In fact, this special position results in a very important simplification to the problem. For instance, it is well known that obtaining an analytical (closed-form) solution for a point situated apart from that axis is not possible because of the need for elliptic integrals (need of the numerical integration or series expansion, e.g. *Heck and Seitz, 2007*).

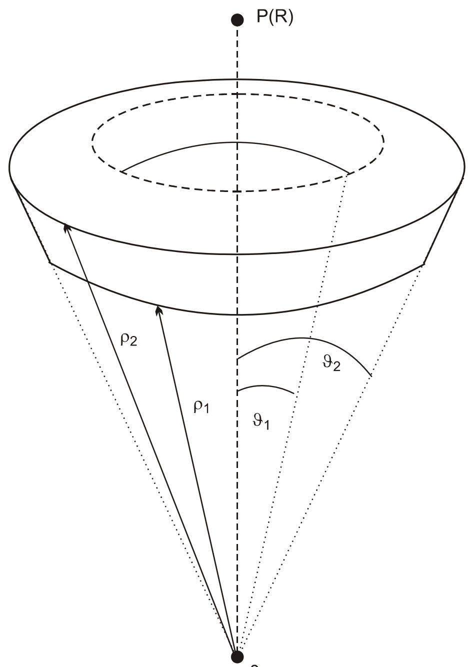

Fig. 1. Geometry of an annular spherical shell.

2. FORMULATION OF THE PROBLEM

2.1. Gravitational Attraction

In a Cartesian coordinate system, the vertical derivative V_z of the gravitational potential V at the calculation point $P\left(x,y,z\right)$ is:

$$V_z\left(x,y,z\right) = -G\int\limits_{\xi}\int\limits_{\eta}\int\limits_{\zeta} \frac{\sigma\left(\xi,\eta,\zeta\right)\left(\zeta - z\right)}{\left[\left(\xi - x\right)^2 + \left(\eta - y\right)^2 + \left(\zeta - z\right)^2\right]^{3/2}} d\xi\, d\eta\, d\zeta \,,$$

where G is the Newtonian gravitational constant for which I adopted in this paper the value 6.673×10^{-11} $\mathrm{m^3kg^{-1}s^{-2}}$ (*NIMA, 2000, Table 3.5, p.3-8*), and $\sigma\left(\xi,\eta,\zeta\right)$ is the density of the body as a function of position. The leading negative sign arises from the upward orientation of the vertical z-axis. Transformation to the spherical coordinates $\left(\rho,\vartheta,\varphi\right)$, accepting a density variation only along the radial direction, gives for the calculation point $P\left(R,0,0\right)$ (later in the text denoted simply as $P\left(R\right)$):

$$V_R\left(R,0,0\right) = -G\int\limits_{\rho}\int\limits_{\vartheta}\int\limits_{\varphi} \frac{\sigma\left(\rho\right)\left(\rho\cos\vartheta - R\right)\rho^2\sin\vartheta}{\left[\rho^2 - 2R\rho\cos\vartheta + R^2\right]^{3/2}} d\rho\, d\vartheta\, d\varphi \,. \qquad (1)$$

The density is described as an n-th degree polynomial of the variable ρ: $\sigma\left(\rho\right) = \sum\limits_{k=0}^{n} a_k \rho^k$, where a_k are the constant polynomial coefficients. If we set the boundaries of longitudinal variable φ to $\varphi_1 = 0$ and $\varphi_2 = 2\pi$, an axial symmetry is obtained for the shell and then the radial derivative V_R is numerically equal to the gravitational effect. Moreover, in problems requiring only the vertical component of gravitation, we can use general boundaries of variable φ, as in compartment-by-compartment calculation of atmospheric effect when Earth topography is considered.

Integration with respect to the variable φ gives a simple multiplicative factor of 2π (or by term $\left(\varphi_2 - \varphi_1\right)$ if the shell is incomplete). Integration with respect to variable ϑ gives, on replacing $\sigma\left(\rho\right)$ with its polynomial form:

$$V_R\left(R\right) = 2\pi G\int\limits_{\rho} \left[\frac{\left(\sum\limits_{k=0}^{n} a_k\rho^k\right)\left(\rho^3 - \rho^2 R\cos\vartheta\right)}{R^2\left(\rho^2 - 2R\rho\cos\vartheta + R^2\right)^{1/2}}\right]_{\vartheta_1}^{\vartheta_2} d\rho \,. \qquad (2)$$

We set $\left(\rho^2 - 2R\rho\cos\vartheta + R^2\right)^{1/2} = A$ and rearrange the terms in the integrals, to obtain:

$$V_R\left(R\right) = \frac{2\pi G}{R}\sum_{k=2}^{n+3}\left[\left(\frac{a_{k-3}}{R} - a_{k-2}\cos\vartheta\right)I_k\right]_{\vartheta_1}^{\vartheta_2} , \tag{3}$$

where we have set $a_{-1} = a_{n+1} = 0$ for the out of range coefficients and $I_k = \int\left(\rho^k / A\right)d\rho$. Equality between Eqs.(2) and (3) can be proven by mathematical induction.

The integrals I_k can be solved using a recursive formula of *Gradshteyn and Ryzhik (1980)*:

$$\int\frac{x^m\,dx}{\sqrt{cx^2+bx+a}} = \frac{x^{m-1}\sqrt{cx^2+bx+a}}{mc} - \frac{\left(2m-1\right)b}{2mc}\int\frac{x^{m-1}\,dx}{\sqrt{cx^2+bx+a}} - \frac{\left(m-1\right)b}{2mc}\int\frac{x^{m-2}\,dx}{\sqrt{cx^2+bx+a}} . \tag{4}$$

Actually, we do not explicitly have to derive each integral to its recursive termination. Instead, we can express one integral in terms of the pervious two lower degree integrals. Such an iterative arrangement not only facilitates the derivation but is also very efficiently programmed. Explicit integration could, of course, be carried out to the end, but the result would be rather lengthy.

Our coefficients in Eq.(4) are $a = R^2$; $b = -2R\cos\vartheta$; $c = 1$, so we can write:

$$I_k = \int\frac{\rho^k}{A}d\rho = \frac{1}{k}\left[\rho^{k-1}A + R\cos\vartheta\left(2k-1\right)I_{k-1} - \left(k-1\right)R^2 I_{k-2}\right] . \tag{5}$$

Finally, it leads us to computation of I_0:

$$I_0 = \int_\rho \frac{d\rho}{A} = \ln\left(2A + 2\rho - 2R\cos\vartheta\right) . \tag{6}$$

Combining the above results, we arrive at an expression for the gravitational attraction $V_R\left(R\right)$ of an annular spherical shell whose radial density dependency is described by an n-th degree polynomial, namely:

$$V_R(R)=\frac{2\pi G}{R}\sum_{k=2}^{n+3}\Bigg[\Bigg[\frac{1}{k}\left(\frac{a_{k-3}}{R}-a_{k-2}\cos\vartheta\right)$$

$$\left(\rho^{k-1}A+(2k-1)R\cos\vartheta I_{k-1}-(k-1)R^2 I_{k-2}\right)\Bigg]_{\vartheta_1}^{\vartheta_2}\Bigg]_{\rho_1}^{\rho_2}, \tag{7}$$

where we have set $a_{-1}=a_{n+1}=0$.

Here, we first substitute the boundaries ϑ_i, $i=1,2$ and then calculate the integrals $I_k(\vartheta_i,\rho_i)$ for the variable ρ_i, as shown above.

The attraction of a circular spherical shell represents a special case of Eq.(7), corresponding to $\vartheta_1=0$. We computed this limit for the special positions of calculation point - inside of the layer: $R\in\langle\rho_1;\rho_2\rangle$ or on the upper/lower edge of the layer: $R=\rho_1$ or $R=\rho_2$. After rearranging, we obtain the gravitational attraction V_R for the circular spherical shell as:

$$V_R(R)=\frac{2\pi G}{R}\Bigg[\sum_{k=2}^{n+3}\frac{1}{k}\left(\frac{a_{k-3}}{R}-a_{k-2}\cos\vartheta_2\right)\Big(\rho^{k-1}A_2+(2k-1)R\cos\vartheta_2 I_{k-1}$$

$$-(k-1)R^2 I_{k-2}\Big)-\frac{1}{R}\sum_{k=0}^{n}a_k\,\mathrm{sgn}(\rho-R)\frac{\rho^{k+3}-R^{k+3}}{k+3}\Bigg]_{\rho_1}^{\rho_2}, \tag{8}$$

where $A_2=\left(\rho^2-2R\rho\cos\vartheta_2+R^2\right)^{1/2}$, $a_{-1}=a_{n+1}=0$.

Explicit expressions for the gravitational attraction of an annular and circular spherical shell with 5th degree polynomial density dependency are given in Appendix A. The 5th degree seems to be sufficiently high to deal with most situations of geophysical interest.

2.2. Gravitational Potential

The formulation of our problem for the gravitational potential $V(R)$ of an annular spherical shell will be as follows:

$$V(R)=G\int_\rho\int_\vartheta\int_\varphi\frac{\left(\sum_{k=0}^{n}a_k\rho^k\right)\rho^2\sin\vartheta}{\left[\rho^2-2R\rho\cos\vartheta+R^2\right]^{1/2}}d\rho\, d\vartheta\, d\varphi\,. \tag{9}$$

The integration with respect to the variables φ and ϑ, after slight rearrangement, gives:

$$V(R) = \frac{2pG}{R} \sum_{k=0}^{n} \left[a_k J_{k+1} \right]_{\vartheta_1}^{\vartheta_2} , \tag{10}$$

where $J_{k+1} = \int\limits_{\rho} \rho^{k+1} A \, d\rho$ and $A = \left[\rho^2 - 2R\rho\cos\vartheta + R^2 \right]^{1/2}$.

The general form for this type of integral is (*Gradshteyn and Ryzhik, 1980*):

$$\int x^m \sqrt{cx^2 + bx + a} \, dx = \frac{x^{m-1}}{(m+2)c} \left(cx^2 + bx + a \right)^{3/2} - \frac{(2m+1)b}{2(m+2)c} \int x^{m-1} \sqrt{cx^2 + bx + a} \, dx - \frac{(m-1)a}{(m+2)c} \int x^{m-2} \sqrt{cx^2 + bx + a} \, dx . \tag{11}$$

We will perform the integration with respect to the variable ρ in the same recurrence sense as before and will likewise use the same iterative computing procedure. Our coefficients a, b and c are, as before: are $a = R^2$; $b = -2R\cos\vartheta$; $c = 1$, so Eq.(10) becomes:

$$J_{k+1} = \int\limits_{\rho} \rho^{k+1} A \, d\rho = \frac{1}{k+3} \left(\rho^k A^3 + (2k+3) R \cos\vartheta \, J_k - R^2 k J_{k-1} \right). \tag{12}$$

The term J_0 is equal to I_0 in Eq.(6).

Combining previous results, we finally obtain for the potential of an annular spherical shell with radially dependent polynomial density:

$$V(R) = \frac{2\pi G}{R} \sum_{k=0}^{n} \left[\left[\frac{a_k}{k+3} \left(\rho^k A^3 + R\cos\vartheta (2k+3) J_k - R^2 k J_{k-1} \right) \right]_{\vartheta_1}^{\vartheta_2} \right]_{\rho_1}^{\rho_2} . \tag{13}$$

Similarly, the potential of the circular spherical shell $(\vartheta_1 = 0)$ is:

$$V(R) = \frac{2\pi G}{R} \sum_{k=0}^{n} \left\{ \frac{a_k}{k+3} \left[\left(\rho^k A_2^3 + R\cos\vartheta_2 (2k+3) J_k - R^2 k J_{k-1} \right) - \operatorname{sgn}(\rho - R) \rho^{k+2} \left(\rho - \frac{(k+3)R}{k+2} \right) \right]_{\rho_1}^{\rho_2} \right\} , \tag{14}$$

under conventions identical to Eq.(8).

3. NUMERICAL VERIFICATION OF FORMULA (7)

The attraction of a shell with 5th degree polynomial density in the radial variable ρ calculated by Eq.(7) was tested against an approximated density model, the attraction of which was calculated independently by the formula of *Mikuška et al. (2006)*. For a given 5th degree density polynomial I created a discrete spherically "layered" model, each layer having constant density but different from the next, and I then computed the gravitational effects of both density distributions. I also tried to determine the layering discretization necessary to fit the gravitational attraction from the continuous distribution with sufficient accuracy. This test was done for a 5th degree density polynomial and for several different numbers of layers. The polynomial coefficients are shown in Table 1. The model density lies in the interval $\langle 1000.937; 1063.535 \rangle$ kg m^{-3}.

The parameters of the spherical shell were: $\rho_1 = 500$ m, $\rho_2 = 2500$ m, $\vartheta_1 = 10°$, $\vartheta_2 = 30°$, and the calculation points had their coordinates $R \in \langle 1; 300 \rangle$ m with the step $k = 1$ km. Three "layered" models were used:

a) 20 layers with thickness of 100 m;
b) 200 layers with thickness of 10 m; and
c) 2000 layers with thickness of 1 m.

The calculated differences $\Delta g = g_{poly} - g_{layer}$ between the continuous and layered models at each calculation point were limited to following intervals, respectively:

a) $\Delta g \in \left\langle -9.641004 \times 10^{-4}; 11.744661 \times 10^{-4} \right\rangle$ mGal;

b) $\Delta g \in \left\langle -9.632152 \times 10^{-6}; 11.742583 \times 10^{-6} \right\rangle$ mGal;

c) $\Delta g \in \left\langle -9.6321 \times 10^{-8}; 11.7425 \times 10^{-8} \right\rangle$ mGal,

while $\max\left(g_{poly}\right) = 16.07142$ mGal.

As we can see for this specific example, if the number of layers was increased 10 times, the differences decreased (or accuracy increased) approximately 100 times. In other words, the attractions calculated according to Eq.(7) are numerically almost identical with those calculated according to the formula of *Mikuška et al. (2006)*, provided the number of layers with constant density is sufficiently high.

The second test was as follows: the complete spherical shell was split into two parts. For each part, the computation was performed separately. The sum of both attractions was then confronted with the result for the complete spherical shell, based on simple computation of total mass. The splitting angle was $\vartheta = 45°$ (Fig. 2). Other parameters were: $\rho_1 = 500$ m, $\rho_2 = 1000$ m, $R \in \langle 1; 1500 \rangle$ m, with the step $k = 1$ km. The density polynomial coefficients were the same as in previous example (Table 1). The resulting curves of attraction for both shells and their sum are displayed in Fig. 2. It is clear that the sum is equal to the attraction of the complete shell and is acting as expected: it is zero at all inner points, and it is increasing inside the shell with a peaked maximum at the outer boundary and decreasing by the square of the distance from the origin at external points. The maximum difference between composite and single shells was 5×10^{-13} mGal, which is at the level of computational precision used in the programs.

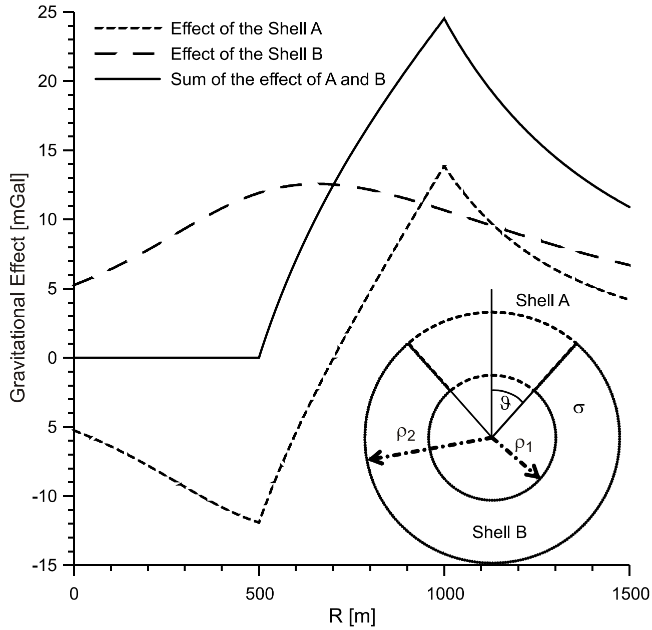

Fig. 2. The model of complete shell divided into two parts (A and B) and gravitational effect of each part as function of the distance *R* from the origin ($\rho_1 = 500$ m, $\rho_2 = 1000$ m, $\vartheta = 45°$. The density σ is described by a 5th degree polynomial with coefficients from Table 1).

4. GRAVITATIONAL ATTRACTION OF THE NORMAL EARTH ATMOSPHERE

The term normal atmosphere (*Ecker and Mittermayer, 1969*) is understood as a simple spherical atmospheric model of dry gas-mixture, stationary with respect to the Earth surface, with density that is continuously decreasing with height. There are several density models, e.g. CIRA61[1] and USSA65[2], which were adopted by *Wenzel (1985)* who calculated the gravitational attraction of the normal atmosphere in spherical approximation. These two density models used a sampling step of 2 km up to the altitude of 50 km. I used the more accurate model of USSA76 (*NOAA, NASA and USAF, 1976*) with sampling step 50 m to calculate coefficients of density polynomial. The upper boundary of this model is 60 km. However, if we restrict ourselves only to terrestrial calculation points it is enough to use the data up to, say, 10 km (more precisely to 8850 m, because of the well-known fact, that the attraction of a complete spherical shell in its inner points is equal to zero). It should by noted that solid particles of the atmosphere, the

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[1] Cospar International Reference Atmosphere 1961

[2] US Standard Atmosphere 1965

Table 1. Coefficients of the tested density polynomial.

Coefficient	Value
a_0	1×10^{3}
a_1	1×10^{-3}
a_2	1×10^{-6}
a_3	1×10^{-9}
a_4	1×10^{-12}
a_5	1×10^{-18}

Table 2. The numerical values of coefficients of the fitting polynomial computed by LSQ method.

Coefficient	Value
a_0	$-3.63047404343979 \times 10^{7}$
a_1	$1.76619445440646 \times 10^{2}$
a_2	$-1.014549089 \times 10^{-4}$
a_3	$2.307001182 \times 10^{-11}$
a_4	$-2.365401534 \times 10^{-18}$
a_5	$9.148879218 \times 10^{-26}$

amount of liquids in it, density variation due to either geographical latitude or season are not considered in this model. The lower boundary of the normal atmosphere neglects the variable topography and is represented by the approximating sphere to which I assigned the radius of 6371001 m (*NIMA, 2000, Table 3.3, p.3–7, radius of sphere of equal volume*). As indicated above, the outer boundary was arbitrarily set to 10000 m.

The density data were approximated by least squares 5th degree polynomial with approximation error less than ± 0.00006 kg/m^3 within the range of $\rho \in \langle 6371001; 6381001 \rangle$ m. The fitting polynomial thus is $\sigma(\rho) = \sum_{k=0}^{5} a_k \rho^k$, where numerical values of the coefficients a_0 to a_5 are shown in Table 2. The density data and the fitting polynomial are displayed in Fig. 3.

It is obvious that the gravitational attraction of the normal atmosphere under spherical approximation will not depend on the “geographical position” of the calculation point; it is only a function of its distance from the origin of coordinate system (as well as model density function). The values of the attraction were calculated with 10 m step from zero height to the altitude of 10000 m (i.e. to $R = 6381001$ m, see Fig. 4). For comparison, the values presented in *Wenzel (1985, p.129, the second column)*, are also displayed.

Wenzel’s attractions (l.c.) are unfortunately presented only with 2 km step. Although he used quite a different way of their evaluation and not an identical density model (namely he adopted an average of CIRA61 and USSA65 densities, continually

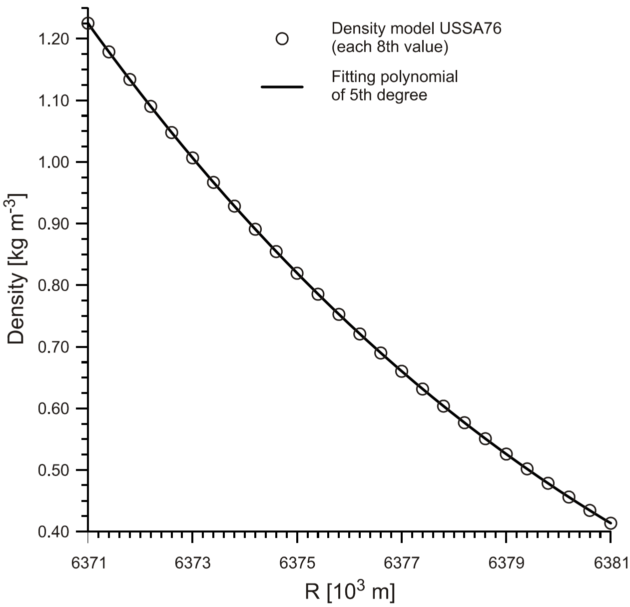

Fig. 3. The density data set with fitting polynomial of 5th degree. *R* is distance from the origin of coordinate system.

approximated by an exponential function within 2-km altitude intervals), it is interesting to note that his result and that one of mine agree very well (the maximum difference being ≈ 0.001 mGal at 10 km altitude).

5. CONCLUSIONS

Single and closed-form analytic expressions for the gravitation (Eq.(7)) and potential (Eq.(13)) effects of spherical shell with radially varying density described by n-th degree polynomial have been derived. The formula for gravitation attraction was numerically tested against a layered application of the simpler constant density formula of *Mikuška et al. (2006)*, and against the outputs of *Wenzel (1985)* whose results have been obtained independently.

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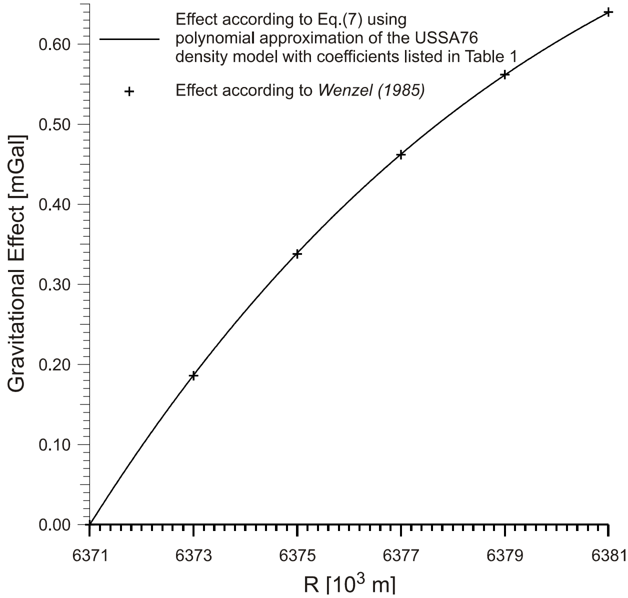

Fig. 4. Gravitational effect of the normal atmosphere. *R* is distance from the origin of coordinate system.

APPENDIX A

Let us set the density distribution as a 5th degree polynomial of variable ρ ($n = 5$) and substitute it into Eq.(7):

$$V_R(R) = \frac{2pG}{R} \sum_{k=2}^{8} \left[\left[\frac{1}{k} \left(\frac{a_{k-3}}{R} - a_{k-2} \cos \vartheta \right) \right. \right.$$

$$\left. \left. \left(\rho^{k-1} A + (2k-1) R \cos \vartheta I_{k-1} - (k-1) R^2 I_{k-2} \right) \right]_{\vartheta_1}^{\vartheta_2} \right]_{\rho_1}^{\rho_2} ,$$

where we have to set $a_{-1} = a_6 = 0$.

If we itemize the previous equation into the respective individual parts, we will, for annular spherical shell, obtain:

$$V_R(R)=\frac{2\pi G}{R}\Bigg\{\left[\left[\frac{a_5}{8R}\left[\rho^7 A+15R\cos\vartheta I_7-7R^2 I_6\right]\right]_{\vartheta_1}^{\vartheta_2}\right]_{\rho_1}^{\rho_2}$$

$$+\left[\left[\frac{1}{7}\left(\frac{a_4}{R}-a_5\cos\vartheta\right)\left(\rho^6 A+13R\cos\vartheta I_6-6R^2 I_5\right)\right]_{\vartheta_1}^{\vartheta_2}\right]_{\rho_1}^{\rho_2}$$

$$+\left[\left[\frac{1}{6}\left(\frac{a_3}{R}-a_4\cos\vartheta\right)\left(\rho^5 A+11R\cos\vartheta I_5-5R^2 I_4\right)\right]_{\vartheta_1}^{\vartheta_2}\right]_{\rho_1}^{\rho_2}$$

$$+\left[\left[\frac{1}{5}\left(\frac{a_2}{R}-a_3\cos\vartheta\right)\left(\rho^4 A+9R\cos\vartheta I_4-4R^2 I_3\right)\right]_{\vartheta_1}^{\vartheta_2}\right]_{\rho_1}^{\rho_2}$$

$$+\left[\left[\frac{1}{4}\left(\frac{a_1}{R}-a_2\cos\vartheta\right)\left(\rho^3 A+7R\cos\vartheta I_3-3R^2 I_2\right)\right]_{\vartheta_1}^{\vartheta_2}\right]_{\rho_1}^{\rho_2} \qquad \text{(A.1)}$$

$$+\left[\left[\frac{1}{3}\left(\frac{a_0}{R}-a_1\cos\vartheta\right)\left(\rho^2 A+5R\cos\vartheta I_2-2R^2 I_1\right)\right]_{\vartheta_1}^{\vartheta_2}\right]_{\rho_1}^{\rho_2}$$

$$+\left[\left[\frac{a_0\cos\vartheta}{2}\left(\rho A+3R\cos\vartheta I_1-R^2 I_0\right)\right]_{\vartheta_1}^{\vartheta_2}\right]_{\rho_1}^{\rho_2}\Bigg\},$$

where

$$I_7=\int\frac{\rho^7}{A}d\rho=\frac{1}{7}\left(\rho^6 A+13R\cos\vartheta I_6-6R^2 I_5\right),$$

$$I_6=\int\frac{\rho^6}{A}d\rho=\frac{1}{6}\left(\rho^5 A+11R\cos\vartheta I_5-5R^2 I_4\right),$$

$$I_5=\int\frac{\rho^5}{A}d\rho=\frac{1}{5}\left(\rho^4 A+9R\cos\vartheta I_4-4R^2 I_3\right),$$

$$I_4=\int\frac{\rho^4}{A}d\rho=\frac{1}{4}\left(\rho^3 A+7R\cos\vartheta I_3-3R^2 I_2\right),$$

$$I_3=\int\frac{\rho^3}{A}d\rho=\frac{1}{3}\left(\rho^2 A+5R\cos\vartheta I_2-2R^2 I_1\right),$$

$$I_2 = \int \frac{\rho^2}{A} d\rho = \frac{1}{2}\left(\rho A + 3R\cos\vartheta I_1 - R^2 I_0\right),$$

$$I_1 = \int \frac{\rho}{A} d\rho = A + R\cos\vartheta I_0 ,$$

$$I_0 = \int \frac{d\rho}{A} = \ln\left|2A + 2\rho - 2R\cos\vartheta\right| ,$$

where $A = \left[\rho^2 - 2R\rho\cos\vartheta + R^2\right]^{1/2}$.

The gravitational attraction of circular spherical shell ($\vartheta_1 = 0$), with the above density distribution will then be:

$$V_R(R) = \frac{2pG}{R}\left[\sum_{k=2}^{n+3}\frac{1}{k}\left(\frac{a_{k-3}}{R} - a_{k-2}\cos\vartheta_2\right)\left(\rho^{k-1}A_2 + (2k-1)R\cos\vartheta_2 I_{k-1} - (k-1)R^2 I_{k-2}\right) - \frac{1}{R}\sum_{k=0}^{n} a_k \operatorname{sgn}(\rho - R)\frac{\rho^{k+3} - R^{k+3}}{k+3}\right]_{\rho_1}^{\rho_2} , \qquad \text{(A.2)}$$

where: $A_2 = \left[\rho^2 - 2R\rho\cos\vartheta_2 + R^2\right]^{1/2}$, $a_{-1} = a_{n+1} = 0$.

The I_k terms in Eq.(A.2) will be almost the same as in Eq.(A.1), the only change is in term A (it has to be replaced by A_2).

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