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Impact measures: What are they?

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Abstract
A mathematical, axiomatic definition for the hitherto vague notion of "impact measure" is proposed. For this four conditions are defined on a given set of (rank-frequency) functions (of which the aim is to measure impact). The most typical condition explains how an impact measure should behave on the most productive sources appearing in this function (i.e. in the left side of this rank-frequency function). An overview of "classical" impact measures is provided and it is proved (in most but not in all cases) that they satisfy these conditions for impact measures. This approach can be compared with (but is different from) the approach in econometrics where one defines what concentration is for a (rank-frequency) function. In this way I embed the important notion of impact into the important Lorenz theory.

Keywords Impact · Axiomatic approach · h-Index · g-Index · Pointwise defined measures · Truncated average · Truncated number of items · A-index

Introduction

Most scientists have some idea about what is meant by "impact" and "impact measures" but few can give a precise definition. Harter and Nisonger (1997) published one of the few attempts to define an impactful journal, namely one that publishes many articles and receives many citations. This idea was made concrete as follows: the journal impact factor in the year 1994 (their example) is equal to the number of citations received by the journal in 1994 to articles published in 1992 and 1993. They keep the same publication window as the standard Garfield–Sher impact factor (JIF), not to give an undue advantage to older, established journals. This definition was mainly given to argue that the famous Journal impact factor (JIF) was a misnomer as it is a relative indicator, which 'punishes' a journal for publishing many articles.

In my opinion, bibliometricians, preferring a strictly logical and mathematical definition of impact and, of an impact measure, have not yet come up with a suitable definition. That is precisely what I will try to do.

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In this contribution, I define a measure m as a function defined on a set of functions (its domain) with values in the positive real numbers $\mathbf{R}^+$ (its codomain). These functions represent source-item relations (Egghe, 2005) such as publications as sources and number of received citations over a given period as items. Which type of functions m can be said to measure impact?

To tackle this problem, I present the following plan of attack:

- First I describe characteristic properties of concentration measures. Although these are not the same as impact measures, they are used as a source of inspiration for thinking toward impact measures.
- Then I show similarities and—especially—differences between the two notions.
- I elaborate on these proposed "axioms" for impact measures.
- I continue by examining well-known measures which have been used to measure impact and investigate if they meet these conditions. It is shown that most of them, but not all, are indeed acceptable impact measures in the framework of my conditions.
- I finally present my conclusions and suggestions for further research.

As stated above, I first consider a somewhat similar idea as the notion of impact, namely that of concentration or inequality, leading to concentration measures. In the discrete case, both theories consider arrays (vectors) $X = (x_1, x_2, \ldots)$ ranked in decreasing order, i.e., X is a decreasing function. Here x_j is a non-negative number representing the number of items 'produced' by the source at rank j. In the continuous case, one considered continuous functions $y = Z(x)$, with $x \in [0, T]$ or $x \in \mathbf{R}^+$. In the two cases, one assumes that the functions X or Z are monotone. I will moreover assume that they are decreasing. In this way, the most productive sources come first.

Now a function m is a concentration measure if it meets the following three requirements (Allison, 1978).

(i) $m(C) = 0$, with C a constant array or a constant function;
(ii) $m(aX) = m(X)$, with $a > 0$;
(iii) If Y is derived from X via an elementary transfer (see further), then $m(Y) > m(X)$.

Condition (i) is evident as a constant has no inequality. Condition (ii), known as scale invariance, expresses that only the distribution function matters. Finally, condition (iii) states that if $X = (x_1, x_2, \ldots, x_N)$ and $Y = (x_1, \ldots, x_{i-1}, x_i + a, x_{i+1}, \ldots, x_{j-1}, x_j - a, x_{j+1}, \ldots, x_N)$ with $a > 0$, then $m(Y) > m(X)$. Expressed in terms of monetary units, it states that if one takes from a poorer one and gives to an already richer one, then concentration (here of wealth) increases. I note that array Y must be decreasing and hence Y might have to be re-arranged.

What can be learned from this example? When studying impact measures condition (i) becomes

(I) $m(X) = 0$ if and only if $X = 0$.

This already is a difference with concentration measures as a strictly positive, constant situation must have a positive impact.

Condition (ii) makes no sense for impact measures. If $a > 1$ then, at least intuitively, aX must have a higher impact than X. More generally I will require for an impact measure m that

(II) Y ≥ X ⇒ m(Y) ≥ m(X) and Y = X ⇒ m(Y) = m(X).

Here Y ≥ X means: $y_j \geq x_j$, for all j = 1, …, N. Hence, when the situation in Y is more productive than the situation in X, one should also have a higher value for the impact measure m: m(Y) ≥ m(X). In the sequel I will refine the order ≥ to be able to describe impact in a finer way (it will be denoted as ≪), see next section. Clearly, conditions (I) and (II) are necessary conditions for impact, but these conditions are certainly not sufficient, otherwise, the constant function zero, m(X) = 0 would be a valid impact measure.

In concentration theory the third condition provides sufficiency. I will formulate a third condition for a measure to be an impact measure in the next section. Next it is checked if well-known indicators such as the h-index, the g-index, the R-index, the average, and the total number of items (Rousseau et al., 2018) meet these three conditions and hence can be considered to be bona fide impact measures.

Development and statement of the third condition for impact measures

Point (iii) from concentration provides a start in the direction of what I want to hold for an impact measure. Point (iii) considers sources with a high number of items. These receive even more items at the expense of a lower-ranked source. Impact measures should, in my opinion, concentrate on sources with a high number of items, and may neglect sources with a low number. The exact meaning of the words ‘high’ and ‘low’ will depend on the used measure. Representing mentally a source-item relation as a graph the sources with a high number of items are situated in the lower ranks (because arrays X or functions Z are decreasing).

In the following, only continuous functions will be used, omitting the discrete case (arrays). This provides nice and pure arguments omitting possible discrete aberrations. For the moment, the number of sources is kept fixed as T. Hence functions are defined on the interval [0, T]. Note though that the theory works equally well for functions defined on]0, T].

Let $\boldsymbol{U_T}$ = {Z ‖ Z: [0, T] → $\mathbf{R}^+$, continuous and decreasing}; different subsets $\boldsymbol{Z_T} \subset \boldsymbol{U_T}$ will be considered.

Inequality between functions Y, Z in $\boldsymbol{U_T}$ is expressed in the following two ways:

$$\mathrm{Y} \ \geq \ \mathrm{Z}, \text{ if for all x in } [0, \mathrm{T}] : \mathrm{Y(x)} \ \geq \ \mathrm{Z(x)}$$

For a given set B ⊂ [0, T] I say that Y ≫ Z on B if Y(x) > Z(x) on B.

Note that in the context of the first inequality Y > Z means that Y ≥ Z and there exists a point x_0 such that $Y(x_0) > Z(x_0)$. Inequality ≫ is different and much more demanding since I require a strict inequality in every element of B. For this reason, I used a double inequality sign, ≫. Finally, the standard notations ≤ and < for the inequality between numbers will be used, trusting the reader to make a distinction between inequality of functions and inequality of numbers.

From now on in this section I keep T fixed and hence simply write $\boldsymbol{Z}$ for $\boldsymbol{Z_T}$. I propose now the following requirement for impact measures:

(III.1) ∀Z ∈ $\boldsymbol{Z}$, $\exists a_Z \in]0, T[$ such that (Y ∈ $\boldsymbol{Z}$ and Y ≫ Z on [0, a_Z] implies that m(Y) > m(Z)).

The following requirement can also be formulated:

(III.2) $\forall Z \in \boldsymbol{Z}$, $\exists b_Z \in]0, T[$ such that ($Y \in \boldsymbol{Z}$ and $Y \ll Z$ on $[0, b_Z]$ implies that $m(Y) < m(Z)$).

I already note that the h-index and the g-index meet requirements (I), (II), (III.1), and (III.2), and this with $a_Z = b_Z = h_Z$ for the h-index and $a_Z = b_Z = g_Z$ for the g-index. This follows from their definitions and the continuity of Z. I return to this point later and will give complete proofs of these statements.

I observe that the coefficient of variation V, defined as the standard deviation divided by the mean is a concentration measure, but it is not an impact measure as it does not meet the second requirement (II). For a constant function $Y = K > 0$, defined on an interval [0, T], $V = 0$, but any non-constant continuous, decreasing function X for which $0 < X(x) < Y(x)$ on [0, T] has a V-value strictly larger than 0. Conversely, the h-index meets the requirements (I), (II), (III.1), and (III.2) but it is not a concentration measure because adding an item to the richest source never changes the h-index (except for the null case), see also (Egghe, 2009).

It may be said that (III.1) and (III.2) express the distinguishing power, i.e., evaluating power of impact, of the functions m with respect to the lower abscissa values of the functions in $\boldsymbol{Z}$.

Proposition 1 *If* $Z \in \boldsymbol{Z}$ *meets requirement* (*III*.1) *or* (*III*.2) *then*

$$Y \gg Z \text{ on } B = [0, T] \Longrightarrow m(Y) > m(Z).$$

Proof This is clear as $Y \gg Z$ on [0, T] implies that $Y \gg Z$ on $[0, a_Z]$ and similarly on $[0, b_Y]$. □

Although (III.1) and (III.2) seem to be equivalent, this is actually not the case. Concretely, it is not true that for any $\boldsymbol{Z}$ and any m, meeting (III.1) implies meeting (III.2) or meeting (III.2) implies meeting (III.1). I will provide an example of a function m which meets (III.1) and not (III.2), and similarly, another function, on another $\boldsymbol{Z}$, which meets (III.2) and not (III.1).

A. An example of a function m that meets (I), (II), and (III.1) but not (III.2).

Let $T > 0$ be fixed and consider the function $y = Z(x) \Longleftrightarrow \frac{x}{T} + \frac{y}{T} = 1$, for $x \in [0, T]$.

For $0 < R < S < T$ I define the functions $y = Y_{R,S} \Longleftrightarrow$

$$\begin{cases} \frac{x}{S} + \frac{y}{S} = 1, & \text{if } x \in [0, R] \\ y = S - R, & \text{if } x \in]R, T] \end{cases}$$

Function Z and functions $Y_{R,S}$ are illustrated in Fig. 1.

The set $\boldsymbol{Z}$ is defined here as $\{Z, 0\} \cup \{Y_{R,S}\}$, where the second set refers to all functions $Y_{R,S}$ as defined above. I consider the measure m defined on $\boldsymbol{Z}$ as

$$m(Y) = \int_0^T Y(s)ds$$

It is clear that this function m meets requirements (I) and (II) on $\boldsymbol{Z}$. I first determine R such that $m(Y_{R,S}) > m(Z) = \frac{T^2}{2}$ (this property will be needed further on). It is easily seen that $m(Y_{R,S}) = (T - R)(S - R) + R(S + S - R)/2 = TS - TR + R^2 - R^2/2 = TS - TR + R^2/2$, which I require to be larger than $T^2/2$. Defining the quadratic function $f_S(R)$ as $R^2 - 2TR + 2TS - T^2$, I have to determine for which R this function is larger than or equal to

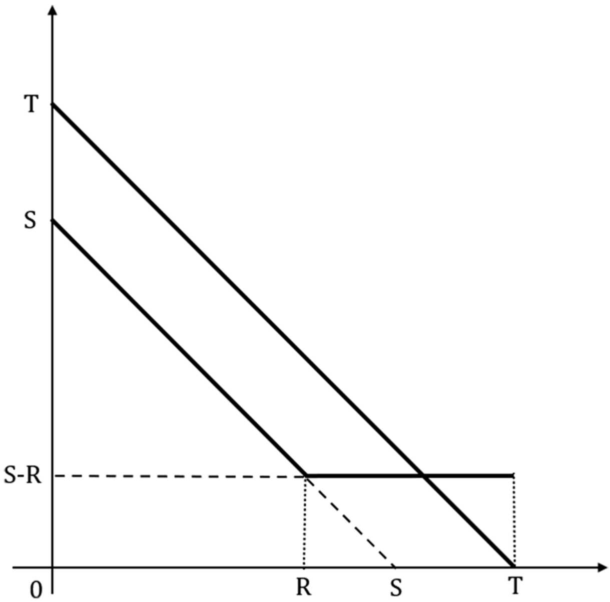


Fig. 1 Graphs of the functions Z and $Y_{R,S}$ for the values $0<R<S<T$, with T fixed

zero. This function is zero if $R = \frac{2T \pm \sqrt{4T^2 - 4(2TS - T^2)}}{2}$. Only the minus sign is meaningful as R < T. Taking the minus sign yields: $R = T - \sqrt{2T(T-S)}$, where I note that the expression under the square root sign is strictly positive. In order to have a strict inequality, I take R somewhat smaller, namely: $R = \left(T - \sqrt{2T(T-S)}\right)\frac{S}{T}$. This is the R-value I will use further on. Note though that I have to check if $0<R<S$ because every valid R must meet this inequality.

(a) $R>0$ if $T - \sqrt{2T(T-S)} > 0 \Longleftrightarrow T^2 > 2\,T(T-S) \Longleftrightarrow S > T/2$.
(b) $R<S$ if $\left(T - \sqrt{2T(T-S)}\right)\frac{S}{T} < S$ which is clearly correct if $S<T$.

Now I consider a subset of **Z**, denoted $\boldsymbol{Z}_{\#}$ and defined as: $\boldsymbol{Z}_{\#} = \{Z, 0\} \cup \left\{ Y_S; \frac{T}{2} < S < T \right\}$, where Y_S is short for $Y_{S,R}$, with R as determined above. I prove now that m meets the requirement (III.1) on $\boldsymbol{Z}_{\#}$.

Requirement (III.1) is checked for the functions 0, Z and Y_S.

For the null function 0 I take, for example, $a_0 = T/2$ and from $Y \gg 0$ on [0, T/2] I find that $m(Y) > m(0) = 0$.

For the function Z I see that there never exists a function Y and a point a such that $Y \gg Z$ on [0,a] (because $Y_S(0) = S < T = Z(0)$). Hence for Z requirement (III.1) is void and hence logically always correct.

Now I take a fixed value S, denoted as S_1 and hence the corresponding R_1. I define $a_1 = T - \frac{S_1 - R_1}{2} < T$. Then, cf. Fig. 1, the abscissa of the intersection of $y = S_1 - R_1$ with Z.

= the abscissa of the intersection of Y_{S_1} with Z.
$= T - (S_1 - R_1) < a_1 = T - \frac{S_1 - R_1}{2}$.

The functions 0 and Z are not larger than Y_{S_1} on [0, a_1] so I do not have to check anything for them.

Consider now a function $Y_{S_2} \gg Y_{S_1}$ on [0, a_1]. Then, by the construction of the functions Y_S, $S_2 > S_1$. As a_1 is strictly larger than the abscissa of the intersection point of Y_{S_1} with Z, $Y_{S_2} \gg Y_{S_1}$ in that point. As $S_2 > S_1$ and $S_2 < T$, I see that the function Y_{S_2} is horizontal in that intersection point. From this, I derive that $S_2 - R_2 > S_1 - R_1$ and hence $Y_{S_2} \gg Y_{S_1}$ on [0, T]. I conclude that $m(Y_{S_2}) > m(Y_{S_1})$ proving that m meets condition (III.1) on $\boldsymbol{Z}_{\#}$.

Next, I prove that m does not meet the requirement (III.2).

Consider again the function $Z \in \boldsymbol{Z}_{\#}$. Then for all a in]0, T[there exists a function $Y_S \ll Z$ on [0,a]. Indeed, it suffices to take T – S + R > a, which is possible as a < T and $\lim_{S\to T}(T - S + R) = \lim_{S\to T}\left(T - S + \left(T - \sqrt{2T(T-S)}\right)\frac{S}{T}\right) = T$. Now, for this function Y_S, I have $m(Y_S) > T^2/2 = m(Z)$. This shows that m on $\boldsymbol{Z}_{\#}$ does not meet the requirement (III.2).

B. An example of a function m which meets (I), (II) and (III.2) but not (III.1).

Fix T > 0 and let 0 < R < S < T, 0 < P < Q $\le \frac{2P+T}{3}$ < T. Then I take $\boldsymbol{Z} = \{Z_S, 0\} \cup \{Z_{P,Q,R,S};\text{for all} P, Q, R, S \text{ as defined below}\}$.

For x ∈ [0, T]: I set $Z_S(x) = S$, and

$$Z_{P,Q,R,S}(x) = \begin{cases} S, & \text{if } x \in [0, P] \\ \left(\frac{S-R}{P-Q}x + \frac{RP-SQ}{P-Q}\right), & \text{if } x \in [P, Q] \\ R, & \text{if } x \in [Q, T] \end{cases}$$

The function $Z_{P,Q,R,S}$ is illustrated in Fig. 2.

The measure m is again defined as $m(Z) = \int_0^T Z(t)dt$, with $Z \in \boldsymbol{Z}$.

It is obvious that m meets requirements (I) and (II). I show now that m on $\boldsymbol{Z}$ meets requirement (III.2).

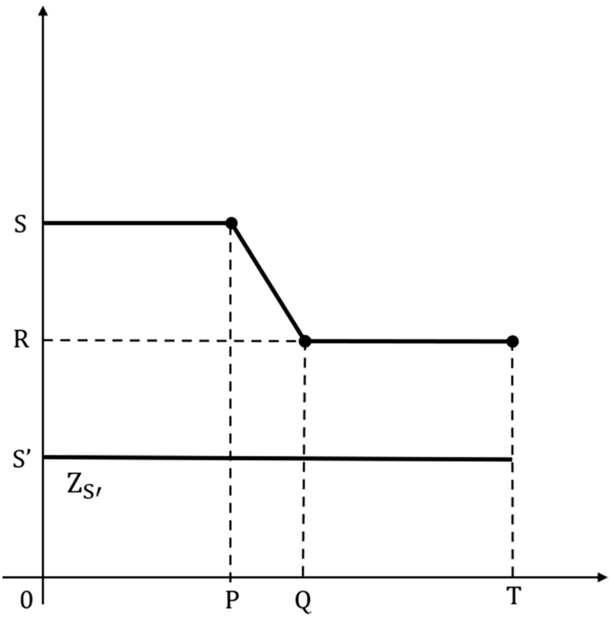


Fig. 2 An illustration of a function $Z_{S'}$ and a function $Z_{P,Q,R,S}$

For Z = $Z_{P,Q,R,S}$ I take $0 < b_Z = (P+T)/2 < T$. Then I have for all functions Y ∈ ***Z***, with Y ≪ Z on [0, b_Z] that Y ≪ Z on [0, T], as $(2P+T)/3 < b_Z$, and hence m(Y) < m(Z).

For $Z = Z_S$ I take $b_Z = T/2 < T$ and note that for Z = 0 the condition is void.

Next, I show that m on ***Z*** does not meet requirement (III.1).

Let $Z = Z_S$ and for every a ∈]0, T[: consider $Y_a = Z_{P',Q',R',S'}$ with P′ = a, Q′ = P′ + δ, S′ = S + δ, R′ = δ, with δ > 0 (to be determined). It is clear that $Y_a \gg Z_S$ on [0,a]. In order to obtain $m(Y_a) < m(Z_S)$ I need: $(S+\delta)\cdot a + S\delta/2 + \delta^2 + (T-a-\delta).\delta < ST$, or: $aS + \delta(T + S/2) < ST$, or $0 < \delta < \frac{2S(T-a)}{S+2T}$, what is possible as a < T.

Proposition 2 (i) *If* $sup_{Z\in\mathbf{Z}}(a_Z) < T$, *and if a measure m on* ***Z*** *meets requirement* (*III*.1) *then it also meets requirement* (*III*.2) *on* ***Z***.

(*ii*) *If* $sup_{Z\in\mathbf{Z}}(b_Z) < T$, *and if a measure m on* ***Z*** *meets requirement* (*III*.2) *then it also meets requirement* (*III*.1) *on* ***Z***.

Proof (i) Let $a = sup_{Z\in\mathbf{Z}}(a_Z)$, with 0 < a < T. Then I take, for all Z in ***Z***: $b_Z = a$. If then Y, Z ∈ ***Z***, with Y ≪ Z on $[0,b_Z] = [0,a] \supset [0,a_Y]$, I conclude by property (III.1) that m(Y) < m(Z), proving property (III.2). The proof of part (ii) is similar.

I further note that the requirements in Proposition 2 are always met if ***Z*** is a finite set. □

Proposition 3 *If for a measure m*, *requirements* (*III*.1) *and* (*III*.2) *hold*, *then I can take for every Z in* **Z**: $a_Z = b_Z$.

Proof It is obvious that if requirement (III.1) holds on $[0,a_Z]$ and if $a_Z \le a < T$, then requirement (III.1) also holds on [0, a]. Similarly, if requirement (III.2) holds on $[0,b_Z]$ and if $b_Z \le b < T$, then requirement (III.1) also holds on [0,b]. Hence taking, for every Z in ***Z***: $c_Z = \max(a_Z,b_Z)$ I see that requirements (III.1) and (III.2) both hold on [0, c_Z].□

Requirement (III.1) as well as (III.2) can both be considered as very natural properties for an impact measure. Hence, I am now looking for a property that includes both.

I use the following properties:

(III) $\forall X \in \mathbf{Z}$, $\exists a_X \in]0,T[$ such that: for all Y, Z in ***Z***, (Y ≫ Z on [0, $\min(a_Y,a_Z)$]) implies that m(Y) > m(Z).

This expression is trivially equivalent with:

(IIIa) $\forall X \in \mathbf{Z}$, $\exists a_X \in]0,T[$ such that: for all Y, Z in ***Z***, (Y ≪ Z on [0, $\min(a_Y,a_Z)$]) implies that m(Y) < m(Z).

Besides (III) I also consider (III′):

(III′) $\forall X \in \mathbf{Z}$, $\exists a_X \in]0,T[$ such that: for all Y, Z in ***Z***, (Y ≫ Z on [0, $\max(a_Y,a_Z)$]) implies that m(Y) > m(Z).

The purpose is that I want to find out which of the two, (III) or (III′), is the most appropriate to characterize the notion of impact. The answer is given in the next theorem.

Theorem 1 (III) $\Longleftrightarrow (III.1) \wedge (III.2) \Rightarrow (III.1) \vee (III.2) \Rightarrow (III')$, *and* $(III') \nRightarrow (III.1) \vee (III.2)$

Proof (III) ⇒ (III.1) ∧ (III.2).

This follows immediately as, by (III) and (IIIa): $[0,a_Z] \supset [0, \min(a_Y,a_Z)]$ and $[0,a_Y] \supset [0, \min(a_Y,a_Z)]$.

$$(III.1) \wedge (III.2) \Rightarrow (III)$$

I know already that, for all Z in **Z** I may take $a_Z = b_Z$ (in (III.1) and (III.2)) let now Y ≫ Z on $[0, \min(a_Y,a_Z)]$. Then I have either Y ≫ Z on $[0, a_Z]$ or Y ≫ Z on $[0,a_Y]$. In the first case (III.1) leads to m(Y) > m(Z), in the second case (III.2) yields m(Y) > m(Z). This proves this part and hence the equivalence of the first two expressions.

(III.1) ∧ (III.2) ⇒ (III.1) ∨ (III.2) is just a logical implication.

$$(III.1) \vee (III.2) \Rightarrow \left(III'\right)$$

Take $c_Z = a_Z$ or b_Z depending on which of the two requirements holds. Now, $\forall Y, Z \in$ Z, with Y ≫ Z on $[0, \max(c_Y,c_Z)]$ I have that Y ≫ Z on $[0,c_Y]$ as well as on $[0, c_Z]$. In case (III.1) holds I conclude from the first case that m(Y) > m(Z), if (III.2) holds the second case leads to the same conclusion.

$$\left(III'\right) \nRightarrow (III.1) \vee (III.2)$$

I provide a counterexample. The impact function is still $m(Z) = \int_0^T Z(s)ds$. I already considered a case for which (III.1) holds and (III.2) does not hold. This case is defined in the square [0, T] x [0, T]. I also have an example where (III.2) holds and (III.1) does not, this time defined in [0, T] x]0, T[.

Let ***Y*** be the set of functions used for the first case, and ***Z*** the set of functions used for the second case. I add the value T to each function Z in ***Z***, leading to $\boldsymbol{Z}_{(+T)} = \{Z + T: Z \in \boldsymbol{Z}\}$. Then m still meets (III.2) while it does not meet requirement (III.1) on $\boldsymbol{Z}_{(+T)}$. Put $\boldsymbol{Z}^* = \boldsymbol{Y} \cup Z_{(+T)}$. *I* note that every function in $\boldsymbol{Z}_{(+T)}$ is situated above every function in ***Y***. Then m meets (III.1) hence (III′) on ***Y*** and m meets (III.2) hence (III′) on $\boldsymbol{Z}_{(+T)}$. Hence m meets (III′) on ***Z****.

Now, m does not meet (III.2) on ***Y***, hence also not on ***Z**** and m does not meet (III.1) on $\boldsymbol{Z}_T$, hence also not on ***Z****. This ends the proof of part (e), and hence of Theorem 1. □

From Theorem 1 I conclude that (III) is the property I am after. Hence I conclude that an impact measure m on the set $\boldsymbol{U} = \{Z \parallel Z: [0, T] \to \mathbf{R}^+$, continuous and decreasing}, or a subset ***Z*** of ***U*** is a function.

$m: \boldsymbol{Z} \subset \boldsymbol{U} \to \mathbf{R}^+: Z \to m(Z)$.

which meets the following three requirements:

(I) m(Z) = 0 if and only if Z = 0.
(II) Y ≥ X ⇒ m(Y) ≥ m(X) and Y = X ⇒ m(Y) = m(X).
(III) $\forall X \in \boldsymbol{Z}, \exists a_X \in \,]0, T[$ such that: for all Y, Z in **Z**, (Y ≫ Z on $[0, \min(a_Y,a_Z)]$ implies that m(Y) > m(Z)).

A simple example: the continuous equivalent of the number of items, e.g. citations, of the largest source, e.g. article, namely m(Z) = Z(0).

I check the three requirements:

(I) m(Z) = 0 if and only if Z = 0, obviously;

(II) $Y \geq X \Rightarrow m(Y) = Y(0) \geq m(X) = X(0)$ and $Y = X \Rightarrow m(Y) = Y(0) = X(0) = m(X)$.

(III) $\forall X \in \boldsymbol{Z}, \exists a_X \in]0, T[$ such that: for all Y, Z in Z (Y ≫ Z on [0, min(a_Y,a_Z)] implies that m(Y) > m(Z)). Taking all a_X = T/2, I see that Y ≫ Z on [0, T/2] implies that Y(0) > Z(0), or m(Y) > m(Z). Obviously, here I may take a_X equal to any number strictly between 0 and T. Then inf(a_X) = 0.

Proposition 4 *If* ***Z*** *consists of functions ending with an interval where the function is zero, then requirements* (*III*) *and* (*III*.1) *are equivalent.*

Proof I have to show that these situations always meet requirement (III.2). Consider a function Z in ***Z*** such that Z = 0 on [n_Z,T] ($0 \leq n_Z < T$). Now I take $b_Z = (T + n_Z)/2$ and consider a function Y in ***Z*** such that Y ≪ Z on [0, b_Z]. As such a function Y does not exist (III.2) is valid. □

A similar observation can be made related to (III.1). Let M be a fixed strict positive number. If ***Z*** consists of functions Z for which Z(0) = M then all these functions Z meet requirement (III.1). Indeed, given Z, take any a_Z with $0 < a_Z < T$. Then I need a function Y ≫ Z on [0, a_Z], but such functions do not exist as Y(0) = Z(0) = M.

These observations show that (III.1) as well as (III.2) are important.

I stated earlier that I want to focus on the left-hand side of the function graph. That is what requirement (III) does. In this context, I note that if two different functions Y and Z in ***Z*** have the same value in zero, Y(0) = Z(0) = n (let us assume that ***Z*** = {Y,Z}), then condition (III) becomes empty and hence such functions always meet the requirement (III). Yet, such functions do not necessarily meet requirement (II). Assume that $0 < c_Z < c_Y < T_Z < T_Y < T$ and define the functions Y and Z as follows:

$$Y(x) = \begin{cases} n & x \in [0, c_Y] \\ -\frac{n}{(T_Y - c_Y)}(x - T_Y) & x \in [c_Y, T_Y] \\ 0 & x \in [T_Y, T] \end{cases}$$

and

$$Z(x) = \begin{cases} n & x \in [0, c_Z] \\ -\frac{n}{(T_Z - c_Z)}(x - T_Z) & x \in [c_Z, T_Z] \\ 0 & x \in [T_Z, T] \end{cases}$$

Then clearly Y ≥ Z on [0, T]. Yet, with $m(X) = \frac{X(0)}{T_X}$, X in {Y,Z} I have $m(Y) = \frac{n}{T_Y} < \frac{n}{T_Z} =$ m(Z), contradicting requirement (II). Of course, (II) does not imply (III). To see this it suffices to take m(X), X in {Y,Z}, equal to a constant C > 0.

Condition (IV) for a variable number of sources

Until now I have kept $T > 0$ fixed, working on subsets of $\boldsymbol{U_T}$. I now consider the union $\boldsymbol{U} = \mathrm{U}_{T>0}\, \boldsymbol{U_T} = \mathrm{U}_{T>0}\left\{Z|| \text{ Z: } [0, \mathrm{T}] \rightarrow \boldsymbol{R}^{+}, \text{ continuous and decreasing}\right\}$.

Assume that I take $W > T$ and I extend functions Z defined on [0, T] and with $Z(T) = 0$, on]T, W] by taking the value zero on this interval. Now I will require that a measure m does not increase strictly. More precisely: consider $\boldsymbol{U}$ and let a measure m be defined on $\boldsymbol{Z} \subset \boldsymbol{U}$. Then I formulate requirement (IV):

(IV). Let $Z \in \boldsymbol{U}$, with $\mathrm{dom}(Z) = [0, T]$ and $Z(T) = 0$. Then I take $W > T$, and define the function Y_Z as

$$\mathrm{Y_Z(x)} = \begin{cases} Z(x), & \text{for } x \in [0, T] \\ 0, & \text{for } x \in [T, W] \end{cases}$$

It is clear that $Y_Z \in \boldsymbol{U}$. Then I require that $m(Y_Z) \leq m(Z)$.

I first give an example of a measure m that does not meet requirement (IV). Let $m(Z) = T \cdot Z(0)$, defined for any function Z in $\boldsymbol{U}$. For fixed T this is an impact measure, as it meets requirements (I), (II), and (III). Yet, with Y_Z as defined above I have:

$$m\left(Y_Z\right) = W \cdot Y_Z(0) = W \cdot Z(0) = \frac{W}{T} m(Z) > m(Z).$$

Condition (IV) leads to a simple classification of impact measures. A distinction can be made between:

Type (IV.1): functions that meet the requirement $m(Y_Z) = m(Z)$, for all Z in U.

Type (IV.2): functions that meet the requirement $m(Y_Z) \leq m(Z)$ and for which $m(Y_Z) < m(Z)$ for at least one Z in $\boldsymbol{U}$.

Note that for $Z = 0$, $m(Z) = m(Y_Z) = 0$.

Intuitively (details follow later) it can be said that the h-index is of type IV.1, while an average is of type IV.2.

I conclude this investigation by recalling the definition of an impact measure m defined on $\boldsymbol{U} = U_{T>0}\, \boldsymbol{U_T} = U_{T>0}\left\{Z|| \text{ Z: } [0, \mathrm{T}] \rightarrow \boldsymbol{R}^{+}, \text{ continuous and decreasing}\right\}$.

The function m: $\boldsymbol{U} \rightarrow \mathbf{R}^{+}$ is an impact measure if it meets the following four requirements.

Restricted to any $Z_T \subset \boldsymbol{U}_T$ it meets the following three requirements:

(I) $m(Z) = 0$ if and only if $Z = 0$.

(II) $Y \geq X \Rightarrow m(Y) \geq m(X)$ and $Y = X \Rightarrow m(Y) = m(X)$

(III) $\forall X \in \boldsymbol{Z_T} \subset \boldsymbol{U_T}, \exists a_X \in]0, T[$ such that: for all Y, Z in $\boldsymbol{Z_T}$, ($Y \gg Z$ on $[0, \min(a_Y, a_Z)]$) implies that $m(Y) > m(Z)$.

Moreover, on $\boldsymbol{U}$ it meets the extra requirement:

(IV) Let Z in $\boldsymbol{U}$, with $\mathrm{dom}(Z) = [0, T]$ and $Z(T) = 0$. If now $W > T$, then for Y_Z defined as

$$Y_Z(\mathrm{x}) = \begin{cases} Z(x), & \text{for } x \in [0, T] \\ 0, & \text{for } x \in [T, W] \end{cases}$$

$$\mathrm{m}\left(\mathrm{Y_Z}\right) \leq \mathrm{m(Z)}.$$

Similarly, I defined measures m_T which are only defined on subsets $\boldsymbol{Z_T}$ of $\boldsymbol{U_T}$. Such measures are required to meet (I), (II), (III) on a given domain [0, T].

I note that if a measure m is an impact measure on a set $\boldsymbol{Z}$ then it is, trivially, also an impact measure on any subset of $\boldsymbol{Z}$. Similarly if a measure is an impact measure on a set $\boldsymbol{Z}$, consisting of functions defined on [0, T], then it is also an impact measure on the set $\boldsymbol{Z*}$ which consists of the functions in Z, restricted to the interval $[0,T_0]$, with $T_0 < T$.

This ends my introduction of the definition of an impact measure. In the next part, I will study some examples.

A study of well-known measures and their impact properties

Before I consider some well-known measures I introduce another tool for my study.

Given an impact measure m and a set $\boldsymbol{Z}$ I known that the set of all numbers a ∈]0,T[(I omit the index Z) for which (III.1) is valid is an interval, open on the right-hand side, ending in T. This interval, to which I now add T, is denoted as $C(m_Z)$. I set $\inf(C(m_Z)) = c(m_Z)$. Here $\inf(C(m_{Z)})$ denotes the infimum of the set $C(m_Z)$, being the greatest element in $\mathbf{R}^+$ that is less than or equal to all elements of $C(m_Z)$). Since 0 is a lower bound, this infimum clearly exists.

Similarly, I know that the set of all numbers b ∈]0,T[(again the index Z is omitted) for which (III.2) is valid is an interval, open on the right-hand side, ending in T. This interval, to which too I add T, is denoted as $D(m_Z)$. I set $\inf(D(m_Z)) = d(m_Z)$.

I next show that these infima are actually minima.

Before starting the proof I recall that earlier I observed that it is possible to develop my theory for functions Z defined on]0,T] (and not [0, T]). In that case it is possible that $\inf(C(m_Z)) = 0$, leading to $C(m_Z) =]0,T]$. Then the infimum is not a minimum.

Proposition 5 $C(m_Z) = [c(m_Z), T]$.

$D(m_Z)) = [d(m_Z), T]$.

Proof I already know that

$$\left]c(m_Z), T\right] \subset C(m_Z) \subset \left[c(m_Z), T\right]$$

If $c(m_Z)$ does not belong to $C(m_Z)$ then there exists Y in $\boldsymbol{Z}$, such that Y≫Z on [0, $c(m_Z)$] and $m(Y) \le m(Z)$. As Y and Z are continuous and $c(m_Z) < T$ (by (III.1)) I know that there exists $a \in \left]c(m_Z), T\right[$ such that Y ≫ Z on [0,a], with $m(Y) \le m(Z)$. This is in contradiction with (III.1), hence $c(m_Z) \in C(m_Z)$ and thus $C(m_Z) = [c(m_Z), T]$.

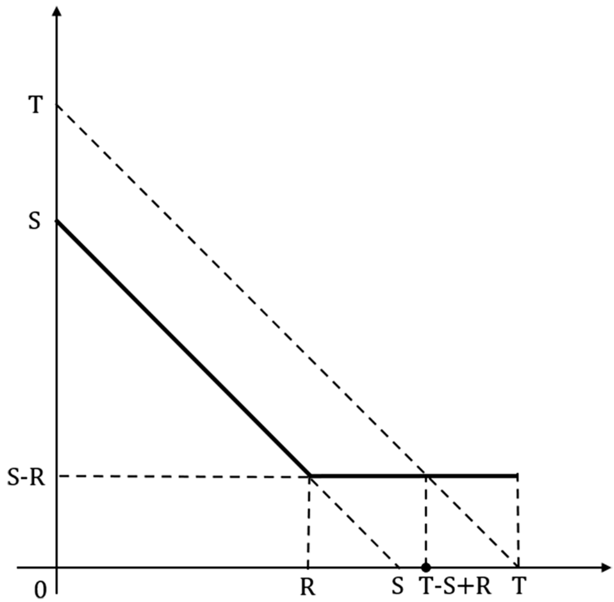
Similarly, the second equation can be proven.

The value T is included in these intervals because, if (III.1), resp. (III.2), are valid then $Y \ll Z$ on [0, T] implies that $m(Y) < m(Z)$.

Next I wonder if (III) implies $c(m_Z) = d(m_Z)$. Surprisingly, this equality does not always hold as shown by the following example.

Example. An example of a measure m defined on a set $\boldsymbol{Z}$ such that for all Z in $\boldsymbol{Z}$ (III.1) and (III.2) hold, but $c(m_Z) \neq d(m_Z)$.□

Fig. 3 The function y = Z(x) (bold), used in the example showing that $c(m_Z) > d(m_Z)$



Remark I note that this example is not in contradiction with Proposition 3.

For 0 < R < S < T, T fixed I define the function y = Z(x), see Fig. 3, with
$$\begin{cases} \frac{x}{S} + \frac{y}{S} = 1, & \text{if } x \in [0, R] \\ y = S - R, & \text{if } x \in]R, T] \end{cases}$$
The function Z as defined above, actually depend on R, S, and T, leading to infinitely many functions Z. For simplicity I do not mention these parameters but keep in mind that T is fixed, but R and S are variable with 0 < R < S < T.

Now I set $\boldsymbol{Z} = \{Z; y = Z(x)\}$ and $m(Z) = \int_0^T Z(x)dx = (S - R)T + R^2/2$.

(a) For all Z in ***Z***, (III.2) holds with $d(m_Z) \le R$.

Proof of (a): for all Z in ***Z*** and Y in ***Z*** with Y ≪ Z on [0,R], I see that Y ≪ Z on [0, T] and hence m(Y) < m(Z). This shows that R plays the role of b_Z so that (III.2) holds and clearly $d(m_Z) \le R$.

(b) For all Z in ***Z***, (III.1) holds and $R < c(m_Z) \le T - S + R$.

Proof of the first inequality in (b). I take one function Z in ***Z*** (parameters R < S) and take Y in ***Z*** (with parameters R′ and S′) with Y⋙Z on [0,R] and m(Y) ≤ m(Z) such that (III.1) does not hold for a = R. Note that as Y⋙Z on [0,R], Y(0) = S′ > Z(0) = S. I write S′ = S + δ, where I will determine δ later. Take R′ = S < S′. Then $m(Y) = (S' - R')T + (R'^2)/2 = (S + \delta - S)T + S^2/2 = \delta T + S^2/2$. I now require that this expression is strictly smaller than $(S - R)T + R^2/2$.

Hence I need: $0 < \delta < \frac{(S-R)T + (R^2 - S^2)/2}{T} = \frac{(S-R)T - (S-R)(S+R)/2}{T} = \frac{(S-R)(2T-S-R)}{2T}$. As 0 < R < S < T this expression is strictly positive. So I take $\delta = \frac{(S-R)(2T-S-R)}{4T} > 0$.

Proof that (III.1) holds and of the second inequality in $R < c(m_Z) \le T - S + R$.

Let Z in ***Z*** with parameters R and S as above and consider any function Y in ***Z*** (with parameters R′ and S′) such that Y⋙Z on [0, T − S + R]. As S′ < T the graph of such a function Y must be horizontal in T − S + R. As now Y(T − S + R) > Z(T − S + R) I see that Y ≫ Z on [0, T] and thus that m(Y) > m(Z). This shows that (III.1) holds and $c(m_Z) \leq T - S + R$.

From parts (a) and (b) I see that for all Z in ***Z***: $c(m_Z) > d(m_Z)$. This completes this first example. Now I give another example where $d(m_Z) > c(m_Z)$.

An example for which for all Z in ***Z***: d(mZ) > $c(m_Z)$.

For 0 < R < S = (R + T)/2 < T, T fixed I define the function y = Z(x), see Fig. 4, with

$$y = \begin{cases} S - R, & 0 \leq x \leq R \\ S - x, & R \leq x \leq S \\ 0, & S \leq x \leq T \end{cases}$$

As for the previous example, I note that the function Z depends on R and T, leading to infinitely many functions Z. For simplicity I do not mention these parameters but keep in mind that T is fixed, but R is variable 0 < R < T and S = (R + T)/2.
Now I set ***Z*** = {Z; y = Z(x)} and $m(Z) = \int_0^T Z(x)dx = R(T - R)/2 + (T - R)^2/8$.

(a) For all Z in ***Z***: (III.1) holds with $c(m_Z) \leq R$.
If $Z_{R'}$ is in ***Z*** with $Z_{R'} \gg Z_R$ on [0,R] then $Z_{R'}(0) = (T - R')/2 > Z_R(0) = (T - R)/2$. Hence R′ < R. Yet, then also S′ = (R′ + T)/2 < (R + T)/2 = S, which implies that it is impossible that $Z_{R'} \gg Z_R$ on [0,R]. This implies that (III.1) holds with $c(m_Z) \leq R$.

(b) For all Z in ***Z***: (III.2) does not hold with $b_Z = R$.
For Z_R (Z with parameter R) I choose $Z_{R'}$ such that $Z_{R'} \ll Z_R$ on [0,R] (their existence is shown further). Then R′ > R. I denote R′ as R + δ (δ > 0). Then $m(Z_{R'}) = R'(T - R')/2 + (T - R')^2/8 = (R + \delta)(T - R - \delta)/2 + (T - R - \delta)^2/8$. I require that this is strictly larger than $m(Z_R) = R(T - R)/2 + (T - R)^2/8$.
This holds if $-\delta R/2 + \delta(T - R)/2 - \delta^2/2 - \delta(T - R)/4 + \delta^2/8 > 0$.

or: − R + T − R − δ − T/2 + R/2 + δ/4 > 0.
or: 0 < δ < 2(T-3R)/3.

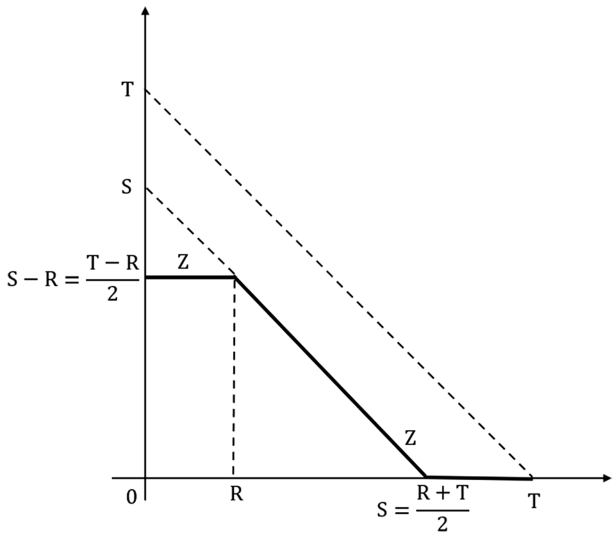


Fig. 4 The function y = Z(x) used in the example showing that $d(m_Z) > c(m_Z)$

This means that I have to take $R < T/3$. In that case Z_R and Z_R' exist with $Z_{R'} \ll Z_R$ on [0,R] and $m(Z_{R'}) > m(Z_R)$.

(c) (III.2) holds for $b_Z = S \in]0,T[$ and hence there exists $d(m_Z) \leq S$ such that $d(m_Z) > R$ (by (a)) $\geq c(m_Z)$ (by (b)).

Indeed, for all Z in ***Z***, Z(S) = 0. Hence there does not exist Y in ***Z*** such that $Y \ll Z$ on [0,S] so that I do not have to check anything.

This ends the proof of the example for which for all Z in ***Z***: $d(m_Z) > c(m_Z)$.

If m is a measure on ***Z*** then I have the following result.

Proposition 6 *Condition (III)* $\Longleftrightarrow$ *for all Z in* ***Z***: $C(m_Z) \cap D(m_Z) \cap]0,T[\neq \emptyset$.

Proof a. (⇒) Let $Z \in$ ***Z***, and let $a_Z \in]0,T[$ for which (III.1) holds. Let similarly let $b_Z \in]0,T[$ for which (III.2) holds. Then $c_Z = \max(a_Z, b_Z) \in C(m_Z) \cap D(m_Z) \cap]0,T[$.

b. (⇐) Let $Z \in$ ***Z***, and let $c_Z \in C(m_Z) \cap D(m_Z) \cap]0,T[$. Then (III.1) holds with $a_Z = c_Z$ and (III.2) holds with $b_Z = c_Z$. Hence (III) holds. □

In the sequel, I will investigate some well-known measures to determine if they meet the requirements for an impact measure. From now on I set ***Z*** = ***U***.

Case (i): The h-index.

Recall the definition of the h-index of a continuous, decreasing function Z, denoted as h_Z:

$x = h(Z)$ (further denoted as h_Z) $\Longleftrightarrow Z(x) = x$.

I note that the h-index of a continuous function Z is well-defined (the definition above is not valid if a function Z would have discontinuities). Moreover, if $Z(T) > T$, I say that the h-index of Z is not defined. Saying that in this case $h_Z = T$—as is sometimes done—does not lead to an impact measure in my sense. Yet, a generalized h-index, h_θ, see (Egghe & Rousseau, 2019), is defined for $\theta \geq \left(\frac{Z(T)}{T}\right)$, the classical h-index being the case $\theta = 1$.

It is clear that the h-index meets the requirements (I) and (IV.1). Consider now requirement (II). Let Y and Z be functions in ***Z***, such that $Y \geq Z$. Then $Y(h_Z) \geq Z(h_Z)$ and hence, as Y is decreasing $h_Y \geq h_Z$. This proves (II).

Now, for (III.1): consider Y and Z functions in ***Z***, such that $Y \gg Z$ on $[0, h_Z]$. Then $Y(h_Z) > Z(h_Z) = h_Z$, and hence, as Y is decreasing, $h_Y > h_Z$. This shows that h meets requirement (III.1) and $c(h_Z) \leq h_Z$. I next show that $c(h_Z) \geq h_Z$.

I take the number a such that $0 < a < h_Z$ and a number δ such that $0 < \delta$. Then the function Y is defined as follows:

$$Y(x) = \begin{cases} Z + \delta, & x \in [0, a] \\ Z(a) + \delta + \frac{\frac{h_Z}{2} - Z(a) - \delta}{h_Z - a}(x - a), & x \in [a, h_Z] \\ \frac{h_Z}{2}, & x \in [h_Z, T] \end{cases}$$

The middle part of the definition of Y(x) is the line connecting the points (a, Z(a) + δ) and (h_Z, $h_Z/2$). This function Y is decreasing and continuous and hence belongs to ***U***. It can be seen that $Y \gg Z$ on [0,a]. Now I show that $h_Y \leq h_Z$ which will lead to the conclusion that (III.1) does not hold in a. Assume that $h_Y > h_Z$, then

$$Y(h_Y) \leq Y(h_Z) = Z(h_Z)/2 < Z(h_Z) = h_Z < h_Y.$$

which contradicts the definition of h_Y. From this result I see that $c(h_Z) = h_Z$ and $C(h_Z) = [h_Z, T]$.

Now I show that the h-index also meets condition (III.2). Let Y, Z in $\mathcal{Z}$ with $Y \ll Z$ on $[0,h_Z]$. Then $Y(h_Z) < Z(h_Z) = h_Z$ and hence $h_Y < h_Z$. This shows that h meets requirement (III.2) and hence taken together with (III.1) h meets requirement (III). I know that $d(h_Z) \le h_Z$. I next consider a function Z which is strictly decreasing, take a number a such that $0 < a < h_Z$ and define Y as follows:

$$Y(x) = \begin{cases} Z(x) - \frac{Z(a)-h_Z}{2}; & x \in [0,a] \\ \frac{Z(a)+h_Z}{2}; & x \in [a,T] \end{cases} \quad (*)$$

Now, Y belongs to $\boldsymbol{U}$, because this function is decreasing, continuous, and positive. I further see that $Y \ll Z$ on [0,a] as $a < h_Z$ and Z is strictly decreasing (so that $Z(a) > h_Z$). I show that $h_Y \ge h_Z$ and hence (III.2) does not hold for a. Assume that $h_Y < h_Z$ then $Y(h_Y) \ge Y(h_Z) = (Z(a) + h_Z)/2 > h_Z > h_Y$, which is a contradiction. Hence I conclude that $d(h_Z) = h_Z = c(h_Z)$ and $D(h_Z) = [h_Z, T]$.

Case (ii): Pointwise defined measures (Egghe, 2021).

I recall the definition from (Egghe, 2021).

A measure m defined on $\boldsymbol{Z}$ is pointwise defined if there exists a function $y = f(\theta,x)$, $\theta > 0$ such that $x = m_\theta(Z)$, denoted as $m_{\theta,Z}$, $\Longleftrightarrow Z(x) = f(\theta,x)$.

Taking, with $\theta = 1$, $f(\theta,x) = x$ I obtain the h-index as studied above. Taking $f(\theta,x) = \theta x$ I obtain the generalized h-index, h_θ, see (Egghe & Rousseau, 2019) and taking $f(\theta,x) = \theta x^p$ ($p > 0$) I find the generalized Kosmulski-indices, with for $\theta = 1$, the original Kosmulski-indices (Kosmulski, 2006).

For pointwise defined measures m the following theorem can be proved. □

Theorem 3 *Let m be a pointwise defined measure where f(θ,x) is increasing in x, meets all the requirements to be an impact measure (I), (II), (III) and (IV) and* $c(m_{\theta,Z}) = m_\theta(Z) \ge d(m_{\theta,Z})$, *with equality if Z is strictly decreasing.*

The proof follows the lines of the proof for the h-index with $y = f(\theta,x)$ replacing $f(x) = x$.

Remark I have shown that for a pointwise defined measure m, and Z strictly decreasing, $c(m_Z) = d(m_Z)$, while for Z decreasing, $c(m_Z) \ge d(m_Z)$. Strict inequality for e.g., $m = h$ occurs if, for all x in $[a, h_Z]$, $Z(x) = h_Z$, with $0 < a < h_Z$. If a is the minimum value with this property then $c(h_Z) = h_Z > d(h_Z) = a$, as follows from the previous proof where I replace h_Z by a and a by b, with $0 < b < a$ in the definition of the function Y (*).

Case (iii): The g-index.

I recall the definition of the g-index for the continuous case. For $Z \in \boldsymbol{Z}$:

$$x = g(Z)(\text{denoted as } g_Z) \iff \int_0^x Z(s)ds = x^2$$

Theorem 4 *The g-index is an impact measure, with* $c(g_Z) = g_Z \ge d(g_Z)$, *hence* $C(g_Z) = [g_Z, T]$ *and, for Z strictly decreasing* $d(g_Z) = g_Z$, *and hence* $D(g_Z) = [g_Z, T]$.

Proof It is obvious that the g-index meets conditions (I) and (IV.1). Now I consider requirement (II). Let $Y, Z \in \boldsymbol{Z}$, with $Y \ge Z$. Then:

$$\int_0^{g_Z} Y(s)ds \geq \int_0^{g_Y} Z(s)ds = \left(g_Z\right)^2$$

This means that $g_Y \geq g_Z$, showing that g meets requirement (II). Next, I consider (III.1). Let Y,Z ∈ ***Z***, with Y≫Z on $[0,g_Z]$. Then, using continuity, I have:

$$\int_0^{g_Z} Y(s)ds > \int_0^{g_Y} Z(s)ds = \left(g_Z\right)^2$$

and hence, by the definition of the g-index, $g_Y > g_Z$. Hence, I also have $c(g_Z) \leq g_Z$. The g-index also meets the requirement (III.2), hence (III) and $d(g_Z) \leq g_Z$. Indeed, let Y,Z ∈ ***Z***, with Y≪Z on $[0,g_Z]$. Again using continuity, I have:

$$\int_0^{g_Z} Y(s)ds < \int_0^{g_Y} Z(s)ds = \left(g_Z\right)^2$$

and thus $g_Y < g_Z$. This also leads to $d(g_Z) \leq g_Z$.

The proofs that $c(g_Z) = g_Z$ and that for strictly decreasing Z, $d(g_Z) = g_Z$ follows in a similar way as for h. The latter will be shown now.

Let $0 < a < g_Z$ and define for $0 < \delta < Z(a)$ (δ will be determined further on):

$$Y(x) = \begin{cases} Z(x) - \delta, & x \in [0, a] \\ Z(a) - \delta, & x \in [a.T] \end{cases}$$

Then Y ∈ ***U*** and Y≪Z on [0,a]. I will show that there exists δ such that $g_Y > g_Z$, hence $d(g_Z) \geq g_Z$ (leading to equality). As Z is strictly decreasing I have $h_Z < g_Z$, I may obviously assume that $a > h_Z$. Now, for x > a, I have:

$$\int_0^x Y(s)ds = \int_0^a Y(s)ds + \int_a^x Y(s)ds = \int_0^a (Z(s) - \delta)ds + \int_a^x (Z(a) - \delta)ds = \int_0^a Z(s)ds + xZ(a) - aZ(a) - \delta x.$$

Then $x = g_Y \iff \int_0^a Z(s)ds + xZ(a) - aZ(a) - \delta x = x^2$

$$\iff x^2 + x(\delta - Z(a)) + aZ(a) - \int_0^a Z(s)ds = 0$$

Solving this quadratic equation yields:

$$g_Y = x = \frac{(Z(a) - \delta) \pm \sqrt{(Z(a) - \delta)^2 - 4\left(aZ(a) - \int_0^a Z(s)ds\right)}}{2}$$

Z is strictly decreasing and continuous, hence $\int_0^a Z(s)ds > aZ(a)$, so that only the plus-sign is meaningful ($g_Y > 0$). The problem is now reduced to finding δ > 0 such that

$$g_Y = x = \frac{(Z(a) - \delta) + \sqrt{(Z(a) - \delta)^2 - 4\left(aZ(a) - \int_0^a Z(s)ds\right)}}{2} > g_Z$$

This inequality is equivalent with:

$$\sqrt{4\left(\int_0^a Z(s)ds - aZ(a)\right) + (Z(a) - \delta)^2} > 2g_Z + \delta - Z(a)$$

I first show that the right-hand side of this inequality is positive. I know that $Z(a) < h_Z < a < g_Z$. Hence $2g_Z + \delta - Z(a)$ is certainly positive. This allows us to square the above inequality, leading to:

$$4\left(\int_0^a Z(s)ds - aZ(a)\right) + (Z(a) - \delta)^2 > \left(2g_Z + \delta - Z(a)\right)^2$$
$$\Leftrightarrow \int_0^a Z(s)ds - aZ(a) > \left(g_Z\right)^2 + \delta g_Z - g_Z Z(a)$$

From this inequality I see that I can find the required $\delta > 0$ if the following inequality holds:

$$\int_0^a Z(s)ds - aZ(a) > \left(g_Z\right)^2 - g_Z Z(a) = \int_0^{g_Z} Z(s)ds - g_Z Z(a)$$

As $a < g_Z$, this inequality holds if:

$$\int_a^{g_Z} Z(s)ds < \left(g_Z - a\right)Z(a)$$

This inequality follows from the fact that Z is continuous and strictly decreasing. So, with such a δ I have $g_Y > g_Z$ which contradicts (III.2) for $a = g_Z$ and hence $d(g_Z) \geq g_Z$. I conclude that $d(g_Z) = g_Z$. □

A similar proof shows that the generalized g-index g_θ (Egghe & Rousseau, 2019) is a valid impact measure.

Case (iv): The R-index.

For Z in ***U***, the continuous R-index, $R > 0$, is defined as (Egghe & Rousseau, 2008; Jin et al., 2007):

$$R^2(Z) = \left(R_Z\right)^2 = \int_0^{h_Z} Z(s)ds$$

where h_Z is the h-index of Z.

Theorem 5 *The R-index is an impact measure and* $c(R_Z) = h_Z \geq d(h_Z)$; *hence* $C(R_Z) = [h_Z, T]$. *For Z strictly decreasing I have* $d(R_Z) = h_Z$ *and hence* $D(R_Z) = [h_Z, T]$. *This is the first case of a measure m for which* $c(m_Z) \neq m_Z$.

Proof The R-index clearly meets requirements (I) and (IV.1). To prove (II) I consider Y, Z in ***Z***, with $Y \geq Z$ and hence $h_Y \geq h_Z$, using the fact that h is an impact measure. Hence:

$$R_Y^2 = \int_0^{h_Y} Y(s)ds \geq \int_0^{h_Z} Z(s)ds = R_Z^2$$

For (III.1) I again take Y,Z in Z, now with $Y \gg Z$ on $[0, h_Z]$. As h meets (III) it follows that $h_Y > h_Z$. Hence, using the continuity of Y and Z I have:

$$R_Y^2 = \int_0^{h_Y} Y(s)ds > \int_0^{h_Z} Y(s)ds > \int_0^{h_Z} Z(s)ds = R_Z^2$$

Hence (III.1) holds for R and $c(R_Z) \leq h_Z$.

To prove (III.2) I take Y, Z in **Z**, and $Y \ll Z$ on $[0, h_Z]$. As h satisfies requirement (III) I obtain $h_Y < h_Z$ and hence, using the continuity of Y and Z:

$$R_Y^2 = \int_0^{h_Y} Y(s)ds < \int_0^{h_Z} Y(s)ds < \int_0^{h_Z} Z(s)ds = R_Z^2$$

This shows that (III.2) holds and $d(R_Z) \leq h_Z$. This proves (III). To finish the proof Theorem 5 I still have to show that $c(R_Z) = h_Z$ and, for strictly decreasing functions Z, that $d(R_Z) = h_Z$. This can be done using the same functions as those used in the proof for h and will not be repeated here. □

Similar results hold for the generalized R-index R_θ defined as:

$$R_\theta^2(Z) = \int_0^{h_{\theta,Z}} Z(s)ds$$

This is left to the reader.

Next, I study the truncated average.

Case (v): The truncated average μ_θ.

Given Z in **Z**, I define the θ-truncated average, with $0 < \theta \leq T$, as.

$\mu_\theta(Z)$, denoted as $\mu_{\theta,Z} = \frac{1}{\theta}\int_0^\theta Z(s)ds$. If $\theta = T$ then I have the usual average on the interval [0, T].

Theorem 6 *The truncated average is an impact measure if $\theta < T$. If $\theta = T$ (the usual average) it only meets requirement (I), (II) and (IV.2). For $\theta < T$ I have $c(\mu_{\theta,Z}) = \theta \geq d(\mu_{\theta,Z})$ and hence $C(\mu_{\theta,Z}) = [\theta, T]$, while $C(\mu_Z) = \{T\}$. If $\theta < T$ and Z is strictly decreasing then $d(\mu_{\theta,Z}) = \theta$ and hence $D(\mu_{\theta,Z}) = [\theta, T]$. If $\theta = T$, then $D(\mu_Z) = \{T\}$.*

Before starting the proof I like to point out that all c- and d-values are independent of the function Z.

Proof It is clear that all μ_θ, $0 < \theta \leq T$ meet requirements (I) and (IV.2). Also (II) is satisfied: for Y,Z in **Z**, $Y \geq Z$ I have:

$$\mu_{\theta,Y} = \frac{1}{\theta}\int_0^\theta Y(s)ds \geq \frac{1}{\theta}\int_0^\theta Z(s)ds = \mu_{\theta,Z}$$

For the proof that the truncated average meets requirement (III.1) I take Y,Z in **Z**, with $Y \gg Z$ on $[0, \theta]$. Then, by the continuity of Y and Z I have that

$$\mu_{\theta,Y} = \frac{1}{\theta}\int_0^\theta Y(s)ds > \frac{1}{\theta}\int_0^\theta Z(s)ds = \mu_{\theta,Z}$$

showing that (III.1) holds for $0 < \theta < T$ and hence $c(\mu_{\theta,Z}) \leq \theta T < T$. If $\theta = T$ I only have $\mu_Y > \mu_Z$ for $a_Z = T$, which is not sufficient to prove (III.1). To prove (III.2) I take Y,Z in **Z**, with $Y \ll Z$ on $[0, \theta]$. Then, again using continuity I have

$$\mu_{\theta,Y} = \frac{1}{\theta}\int_0^\theta Y(s)ds < \frac{1}{\theta}\int_0^\theta Z(s)ds = \mu_{\theta,Z}$$

Showing that (III.2) holds for $0<\theta<T$ and $d(\mu_{\theta,Z})\leq\theta T$. Again, if $\theta=T$ I only have $\mu_Y < \mu_Z$ for $b_Z=T$, which is not what is needed for (III.2).

In a similar way as for the other measures I can show that $c(\mu_{\theta,Z})=\theta$, for $0\leq\theta\leq T$, leading to $C(\mu_{\theta,Z})=[\theta,T]$ for $0<\theta<T$ and $C(\mu_Z)=\{T\}$. Moreover, for Z strictly decreasing, $d(\mu_{\theta,Z})=\theta$ (a proof follows), hence $D(\mu_{\theta,Z})=[\theta,T]$ for $0<\theta<T$ and $D(\mu_Z)=\{T\}$.

Proof that for Z strictly decreasing, $d(\mu_{\theta,Z})=\theta$.

Assume that $0<a<\theta$. Then I define, for all Z in **Z**, Z strictly decreasing a function Y as follows:

$$Y = \begin{cases} Z-\delta, & on[0,a] \\ Z(a)-\delta, & on[a,T] \end{cases}$$

with $0<\delta<Z(a)$ to be determined further on. It can be seen that $Y\in \mathbf{Z}$ and $Y \ll Z$ on [0,a]. Then

$$\begin{aligned}\mu_{\theta,Y} &= \frac{1}{\theta}\int_0^\theta Y(s)ds = \frac{1}{\theta}\left(\int_0^a Y(s)ds + \int_a^\theta Y(s)ds\right) \\ &= \frac{1}{\theta}\left(\int_0^a (Z(s)-\delta\right)ds + \int_a^\theta (Z(a)-\delta)ds) \\ &= \frac{1}{\theta}\left(\left(\int_0^a Z(s)ds\right) - \delta a + (\theta-a)Z(a) - (\theta-a)\delta\right)\end{aligned}$$

I require that this expression is strictly larger than $\mu_{\theta,Z} = \frac{1}{\theta}\int_0^\theta Z(s)ds$.

This inequality holds if $\int_a^\theta Z(s)ds < (\theta-a)Z(a) - \delta\theta$. I can find such a $\delta>0$, actually infinitely many, because Z is strictly decreasing and continuous and hence

$$\int_a^\theta Z(s)ds < (\theta-a)Z(a)$$

This shows that $d(\mu_{\theta,Z})\geq\theta$, and conclude that $d(\mu_{\theta,Z})=\theta$. □

I have shown that μ_θ, with $0<\theta<T$ is an impact measure, while this is not true for the classical average μ. This average does not even meet requirement (III′) as $c(\mu_Z)=d(\mu_Z)=T$.

I realize that this is a rather surprising result. Hence I provide a simple discrete example.

Let X = (10, 10, 10, 5) and Y = (11, 11, 11, 1) then $Y \gg X$ on the index set {1, 2, 3}. The role of T is here played by the number 4. Yet $\mu_Y=34/4<35/4=\mu_X$. Such an example cannot be given for μ_θ, $0<\theta<T$, θ a natural number, because for such θ I always have $\mu_{\theta,Y}=11>10=\mu_{\theta,Z}$.

I recall that in my view, the notion of impact always relates to what happens on the left-hand side and never on the complete interval [0, T].

Case (vi): The truncated total number of items.

The truncated total number of items is defined, for $0<\theta<T$ and Z in **Z** as:

$$I_{\theta,Z} = \int_0^\theta Z(s)ds$$

This is the total number of items in the θ most important sources. For θ = T I have the total number of items, denoted as I.

Theorem 7 *If* $0 < \theta < T$, *then* $I_{\theta,Z}$ *is an impact measure. If* $\theta = T$ *then it only meets requirements* (*I*), (*II*) *and* (*IV*.1). *For* $0 < \theta < T$ *I have*: $c(I_{\theta,Z}) = \theta \geq d(I_{\theta,Z})$ *and hence* $C(I_{\theta,Z}) = [\theta, T]$. *Yet*, $C(I_Z) = \{T\}$ *so that* I_Z *is not an impact measure. If Z is strictly decreasing and* $0 < \theta < T$, *then* $d(I_{\theta,Z}) = \theta$ *and* $D(I_{\theta,Z}) = [\theta, T]$, *while for* $\theta = T$, $D(I_Z) = \{T\}$.

The proof follows completely the proof of Theorem 6 and hence is omitted.

The example after Theorem 6 can also be used here. I showed that while Y ≫ X on {1,2,3} and with {1,2,3,4} in the role of [0, T]: $I_Y = 34 < 35 = I_Z$, yet with θ = 1 I have $I_{\theta,Y} = 11 > 10 = I_{\theta,Z}$, with θ = 2 I have $I_{\theta,Y} = 22 > 20 = I_{\theta,Z}$, and for θ = 3 I have $I_{\theta,Y} = 33 > 30 = I_{\theta,Z}$.

Case (vii): Percentiles.

Definition Let Z be in ***Z*** defined on [0, T], then a 100 θ% percentile (0 < θ < 1) for Z is defined as:

$$P_{\theta,Z} = Z(\theta T).$$

Theorem 8 *The* 100 *θ% percentile* $P_{\theta,Z}$ *is an impact measure and* $c(P_{\theta,Z}) = \theta T \geq d(P_{\theta,Z})$ *and hence* $C(P_{\theta,Z}) = [\theta T, T]$. *If Z is strictly decreasing then also* $d(P_{\theta,Z}) = \theta T$ *and* $D(P_{\theta,Z}) = [\theta T, T]$.

Proof Clearly $P_{\theta,Z}$ always meets requirements (I), (II), and (IV.2). To show (III.1) I consider Y, Z in ***Z*** with Y ≫ Z on [0, θT]. Then $Y(\theta T) = P_{\theta,Y} > P_{\theta,Z} = Z(\theta T)$, which proves (III.1) with $c(P_{\theta,Z}) \leq \theta T$. For (III.2) I take Y,Z in ***Z*** with Y ≪ Z on [0, θT]. Then, clearly, $Y(\theta T) = P_{\theta,Y} < P_{\theta,Z} = Z(\theta T)$, which proves (III.2) with $d(P_{\theta,Z}) \leq \theta T$. Similarly, as for the previous measures, I have that $c(P_{\theta,Z}) = \theta T$ and, if Z is strictly decreasing, $d(P_{\theta,Z}) = \theta T$. This equality is shown now. Let Z be given and let 0 < a < θT. Then I define for 0 < δ < Z(a):

$$Y(x) = \begin{cases} Z(x) - \delta, & x \in [0, a] \\ Z(a) - \delta, & x \in [a, T] \end{cases}$$

As a < θT and Z is strictly decreasing I have Z(a) > Z(θT). Next, I choose δ such that

$$Z(a) > \delta = \frac{Z(a) - Z(\theta T)}{2} > 0$$

Then, as θT > a: Y(θT) = Z(a) – δ = (Z(a) + Z(θT))/2 > Z(θT), and hence $P_{\theta,Y} > P_{\theta,Z}$. This proves that (III.2) does not hold for a. Consequently $d(P_{\theta,Z}) \geq \theta T$, leading to the required result that $d(P_{\theta,Z}) = \theta T$. □

Case (viii): The A-index.

The A-index.

For Z in ***Z***, the A-index (Egghe & Rousseau, 2008; Jin, 2006), denoted as A_Z is in the continuous case defined as:

$$A_Z = \frac{1}{h_Z} \int_0^{h_Z} Z(s)ds$$

It is known (Egghe & Rousseau, 2008) that in the discrete case this is not a good index because it is possible that for the arrays X_1 and X_2 $h(X_1) > h(X_2)$, while $A(X_1) < A(X_2)$. A simple example given in (Egghe & Rousseau, 2008) consists of taken $X_1 = (10,2)$ and $X_2 = (10,1)$. I will next show that its continuous version is not a good measure either, i.e., is not an impact measure in my sense. □

Theorem 9 *For all strictly decreasing Z in* $\mathbf{Z}$, $C(A_Z) = D(A_Z) = \emptyset$, *hence* $c(A_Z)$ *and* $d(A_Z)$ *do not exist. Hence the A-index does not meet requirements (II) and (III) and cannot be considered an impact measure in my sense.*

Proof It suffices to show that $T \notin C(A_Z)$ and $T \notin D(A_Z)$ Indeed, consider Z in $\mathbf{Z}$, strictly decreasing, and $0 < \delta < h_Z$ (where the exact value of δ is determined further on). Then I define Y as follows:

$$Y(x) = \begin{cases} Z(x) - \delta, & x \in [0, h_z] \\ \max(Z(x) - \delta, 0), & x \in [h_Z, T] \end{cases}$$

I see that $Y \ll Z$ on $[0, h_Z]$ and hence $h_Y < h_Z$. Now $A_Y = \frac{1}{h_Y}\int_0^{h_Y} Y(s)ds = \frac{1}{h_Y}\int_0^{h_Y}(Z(s) - \delta)ds = \frac{1}{h_Y}\int_0^{h_Y} Z(s)ds - \delta$.

Now I note that the function J: $x \to \frac{1}{x}\int_0^x Z(s)ds$ ($Z > 0$ fixed) is continuous and strictly decreasing. This implies that the expression $\frac{1}{h_Y}\int_0^{h_Y} Z(s)ds - \frac{1}{h_Z}\int_0^{h_Z} Z(s)ds$ is strictly positive. Now I define δ as $min\left(\frac{1}{2}\left(\frac{1}{h_Y}\int_0^{h_Y} Z(s)ds - \frac{1}{h_Z}\int_0^{h_Z} Z(s)ds\right), h_Z\right)$. Then I have:

$$\begin{aligned} A_Y &\geq \frac{1}{h_Y}\int_0^{h_Y} Z(s)ds - \frac{1}{2}\left(\frac{1}{h_Y}\int_0^{h_Y} Z(s)ds - \frac{1}{h_Z}\int_0^{h_Z} Z(s)ds\right) \\ &= \frac{1}{2}\left(\frac{1}{h_Y}\int_0^{h_Y} Z(s)ds + \frac{1}{h_Z}\int_0^{h_Z} Z(s)ds\right) > \frac{1}{h_Z}\int_0^{h_Z} Z(s)ds = A_Z, \end{aligned}$$

where I have again used the fact that the function I is strictly decreasing. Thus $T \notin \left(C(A_Z)\right) \cup D(A_Z))$ and hence $C(A_Z) = D(A_Z) = \emptyset$. This proves that the A-index does not meet the requirement (III).

The A-index does not even meet requirement (II). Indeed, I know that $Y \leq Z$ on [0, T] because, for $x \in]h_Z, T]$, $\max(Z(x) - \delta, 0)$ is equal to $Z(x) - \delta < Z(x)$ if $Z(x) > \delta$, and if $Z(x) \leq \delta$, then $Y(x) = 0 < Z(x)$, unless possibly for $x = T$, as Z is strictly decreasing and then $Y(x) = Z(x)$. In any case $Y \leq Z$ while $A_Y > A_Z$, contradicting (II). □

Conclusions and suggestions for further research

In this article, I defined natural properties for a measure m to be considered an impact measure. Essentially these requirements are:

(a) m must have distinguishing power on the left-hand side of the graph of the functions Z in $\mathbf{Z}$ (the sources with the most impact) for which it is assumed to measure impact. In this I keep the domain [0, T] fixed.
(b) Increasing the domain for functions by empty sources may never lead to an increase in impact.

I showed that most well-known functions used to measure impact are also impact measures in my sense, be it with some restrictions. For the average, only a truncated version meets my requirements and a similar remark holds for the number of items.

Our investigations lead to two classifications of impact measures. The first classification makes use of the property (III). Let m and n be two impact measures in my sense and assume that for a certain function Z in **Z**, $c(m_Z) \neq c(n_Z)$. I assume that $c(m_Z) < c(n_Z)$ then for a in$]c(m_Z), c(n_Z)[$ there exists $Y \gg Z$ on [0,a] with $m(Y) > m(Z)$ and $n(Y) \leq n(Z)$. In this case, m and n cannot be considered to be "equivalent" (in the sense of not acting in the same way on all functions). I suggest as a topic for further research an investigation of the equivalence (or not) of well-known measures such as the h, g, and the R index.

The second classification uses (IV). Property (IV.1) refers to those measures which stay invariant when adding empty sources, while property (IV.2) refers to measures that may become smaller when adding empty sources. This property too may be investigated further.

I am convinced that my approach solves a problem that was not fully recognized before. Yet, as this is the first time an attempt is made to define the meaning of an impact measure, I admit that it is always possible to propose another set of conditions.

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