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More axiomatics for the Hirsch index

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Abstract The Hirsch index is a number that synthesizes a researcher's output. It is defined as the maximum number h such that the researcher has h papers with at least h citations each. Woeginger (Math Soc Sci 56: 224–232, 2008a; J Informetr 2: 298–303, 2008b) suggests two axiomatic characterizations of the Hirsch index using monotonicity as one of the axioms. This note suggests three characterizations without adopting the monotonicity axiom.

Keywords Hirsch index · Axiomatic characterization · Publications · Citations · Research quality

Introduction

This paper offers three axiomatic characterizations of the Hirsch (2005) index; see Wikipedia (2008) for a discussion of advantages and criticisms of the Hirsch index. The three differ from Woeginger's (2008a) characterization in requiring fewer axioms (three instead of five) and in dispensing with the axiom on which Woeginger's result hinges conceptually: monotonicity (more citations or papers do not lower the index).

Definitions and axioms

Let $\mathbb{N}$ be the set of non-negative integers and R the set of non-negative real numbers. Members of $\mathbb{N}$ represent both the number of papers of a given researcher and the number of citations that a paper can receive. Define X to be the set of all vectors $x = (x_1, x_2,\ldots,x_n)$ such that $n \in \mathbb{N}\backslash\{0\}$ and $x_1 \geq x_2 \geq \ldots \geq x_n$. For $x \in X$: (i) d_x is the number of components of vector x (the dimension or size of x); (ii) c_x is the number of components of vector x

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different from 0; (iii) for $i \in \{1,\ldots,d_x\}$, x_i is the ith component of vector x and stands for the total number of citations of paper i; and (iv) $x^\Sigma = x_1 + x_2 + \cdots + x_{d_n}$ is the sum of the d_x components of x (the weight of x). With $\varnothing$ designating the empty vector (the no paper case), a researcher's output will be represented by a member of $D = X \cup \{\varnothing\}$. For $x = \varnothing$ the convention is that $c_x = d_x = min\{x_1,\ldots,x_{d_n}\} = 0$.

For $x \in X$ and $y \in X$: (i) the distance $\delta(x, y)$ between $x \in X$ and $y \in X$ is defined as $\delta(x, y) = max\{x^\Sigma, y^\Sigma\} - min\{x^\Sigma, y^\Sigma\}$; and (ii) $x \geq y$ holds if, and only if, $d_x \geq d_y$ and, for all $i \in \{1,\ldots,d_y\}$, $x_i \geq y_i$. With respect to the empty vector $\varnothing$: (i) for all $x \in X$, $\delta(x, \varnothing) = \delta(\varnothing, x) = x^\Sigma$; and (ii) for all $x \in X$, $x \geq \varnothing$. Define $D_0 = \{x \in D$: $d_x = 0\} = \{\varnothing\}$ and, for $n \in \mathbb{N}\backslash\{0\}$, $D_n = \{x \in D$: $d_x = n\}$.

Definition 1 A research output index (or index, for short) is a mapping f: $D \rightarrow R$.

Woeginger (2008a, p. 225) defines an (impact) index as a mapping f: $D \rightarrow \mathbb{N}$ satisfying the monotonicity property MON and such that, for all $x \in X$ with $c_x = 0$, $f(x) = 0$.

MON. For all $x \in D$ and $y \in D$, $x \geq y$ implies $f(x) \geq f(y)$.

The definition of an index as an integer-valued mapping is restrictive because it excludes reasonable indices like the average citation index. In addition, assuming $f(x) = 0$ when $c_x = 0$ and $d_x \geq 1$ is also restrictive because an index need not always be interpreted as an impact index: viewed as a research output index, it is not unreasonable to attribute value to the production of papers and make $f(0,\ldots,0) \neq 0$. Finally, Woeginger (2008a, p. 227) stresses that his axioms should be interpreted within the context of MON. Though it is difficult to question MON as a desirable property of an index, it may be worth approaching the characterization of the Hirsch index without constraining the choice of axioms by their connection with MON.

Definition 2 The Hirsch index is the research output index h such that $f(\varnothing) = 0$ and, for all $x \in X$, $h(x) = max\{n \in \{0, 1,\ldots,c_x\}$: $x_n \geq n\}$.

A1. For all $x \in X$, if $c_x = d_x$ then $min\{min\{x_1,\ldots,x_{d_x}\},d_x\} \leq f(x) \leq d_x$.

A1 sets upper and lower bounds to the index in the case in which all the papers are cited: on the one hand, the index cannot be greater than the number d_x of papers; and, on the other, the index is, as long as this is consistent with the previous constraint, not smaller than the smallest number of citations. A1 establishes that the index is bounded above by size and bounded below by the smallest magnitude between size and the minimum contribution to the weight of the output.

A2. For all $n \in \mathbb{N}$, $x \in D_n$ and $y \in D_{n+1}$, if $y \geq x$ and $f(y) > f(x) = max\{f(z)\}_{z \in D_n}$ then $\delta(x, y) > c_x$.

Suppose x is an output with size n reaching the maximum index that size n allows and that x is subsequently expanded by gaining weight (the number of citations of existing papers) or size (by adding another paper, possibly receiving some citation). Suppose this output expansion generates an increase of the index. By A2, the weight necessary to achieve this must be higher than the number c_x of cited papers in x; that is, if the maximum index reachable in D_n requires all papers to be cited, the new output y must have more than n citations more than x. Roughly speaking, if more citations and one more paper rise the index of an output already achieving the maximum index in the domain of outputs with n papers then more than n citations must have been necessary. This suggests that, once the maximum index in a size category has been reached, a further increase in the index by jumping to the next size category demands adding at least the equivalent to one citation to each cited paper.

It may appear that A2 brings an index very close to the Hirsch index. Nonetheless, A2 does not imply MON: the index $f(x) = 1/(1 + h(x))$ satisfies A2 but not MON.

A2 can be generalized to a family of axioms of the sort "if $y \geq x$ and $f(y) > f(x)$ then $\delta(x, y) > c(x, y)$", for any given $c{:}D \times D \to R$. For instance, the use of the constant function $c(x, y) = 0$ seems to point to indices in which each citation counts, as occurs, for instance, with the index generating the average number of citations.

A2$_1$. For all $n \in \mathbb{N}$, $x \in D_n$ and $y \in D_{n+1}$, if $y \geq x$ and $max\{f(z)\}_{z \in D_{n+1}} = f(y) > f(x)$ then $\delta(x, y) > c_x$.

A2$_1$ is a version of A2 in which it is not the initial output x that is required to reach the highest index within the set of outputs of its size but the final output y.

A2$_2$. For all $n \in \mathbb{N}$, $k \in \mathbb{N}\backslash\{0\}$, $x \in D_n$ and $y \in D_{n+k}$, if $y \geq x$ and $max\{f(z)\}_{z \in D_{n+k}} = f(y) > f(x) = max\{f(z)\}_{z \in D_n}$ then $\delta(x, y) > kc_x$.

A2$_2$ is less general than A2 in forcing both inputs to reach the maximum index in their respective category sizes but is more general in relating several sizes. In this respect, A2$_2$ is, in a way, a transitive version of A2: if, under the given constraints, going from size n to size $n + 1$ takes more than n citations, then going from size n to $n + k$ must take more than kn citations. The results presented next give an impression that, to a certain extent, A2, A2$_1$ and A2$_2$ are exchangeable conditions, with A2$_1$ and A2$_2$ being closer substitutes for each other than A2. For $n \in \mathbb{N}\backslash\{0\}$ and $x \in D_n$, $x_{-n} = (x_1,\ldots,x_{n-1})$ is the member of D_{n-1} obtained from x by deleting the last component x_n of x.

A3. For all $n \in \mathbb{N}\backslash\{0\}$ and $x \in D_n$, if $f(x) \neq max\{f(y)\}_{y \in D_n}$ then $f(x) = f(x_{-n})$.

By A3, if an output without minimum size is not achieving the maximum index corresponding to its size then losing one paper should not affect the index. A3 can be viewed as a weak version of paper monotonicity, because it identifies a situation in which having one paper more does not lower the index: when the addition of another paper does not make the resulting output attain the maximum index associated with its size, then the paper is worthless in the sense that its presence or absence does not modify the index. Even seen as a monotonicity property, A3 is weaker than MON, that expresses both paper and citation monotonicity.

A4. For all $x \in X$, and letting $n = d_x$, if $f(x) = f(x_1,\ldots, x_{n-1})$ then, for all k such that $0 \leq k \leq x_n$, $f(x_1,\ldots,x_{n-1}) = f(x_1,\ldots,x_{n-1}, k)$ and $f(x) = f(x_1,\ldots,x_n, k)$.

A4 is a sort of independence condition: if adding a paper with r citations does not alter the index, then adding another paper with r or fewer citations produces the same effect in both the initial output and in the one obtained after including the paper with r citations. In consequence, if a certain change does not affect a given output then a smaller change never affects a larger output.

Results

Remark 3 The Hirsch index satisfies A1, A2, A2$_1$, A2$_2$, A3 and A4.

A1 is an immediate implication of the definition of the Hirsch index. Notice that, for all $n \in \mathbb{N}$, $max\{h(y)\}_{y \in D_n} = n$. Concerning A2, if $n \in \mathbb{N}$, $x \in D_n$, $y \in D_{n+1}$, $y \geq x$ and $h(y) > h(x) = max\{h(z)\}_{z \in D_n}$ then $h(x) = n$ and $h(y) \geq n + 1$, so paper $n + 1$ must receive at least $n + 1$ citations in y, which implies $\delta(x, y) > n = c_x$. As for A2$_1$, if $n \in \mathbb{N}$, $x \in D_n$, $y \in D_{n+1}$, $y \geq x$ and $max\{h(z)\}_{z \in D_{n+1}} = h(y) > h(x)$ then $h(y) = n + 1$ and $h(x) \leq n$, so paper $n + 1$ must receive at least $n + 1$ citations in y, which implies $\delta(x, y) > n \geq c_x$. With respect to A2$_2$, if $n \in \mathbb{N}$, $x \in D_n$, $y \in D_{n+k}$, $y \geq x$ and $max\{h(z)\}_{z \in D_{n+k}} = h(y) > h(x) = max\{h(z)\}_{z \in D_n}$ then $h(y) = n + k$ and $h(x) = n$, so papers $n + 1,\ldots,n + k$

must each receive at least $n + k$ citations in y. Therefore, $\delta(x, y) \geq k(n + k) > kn \geq kc_x$. As regards A3, it follows from $f(x) \neq max\{h(y)\}_{y \in D_n}$ that $x_n < n$. This makes the number x_n of citations of the last paper irrelevant to compute $h(x)$ and, accordingly, $h(x) = h(x_{-n})$. With respect to A4, $h(x) = h(x_1,\ldots,x_{n-1})$ means that $x_n \leq h(x_1,\ldots,x_{n-1})$. Hence, adding to both $(x_1,\ldots,x_{n-1})$ and x another paper having at most x_n citations cannot increase the Hirsch index.

Proposition 4 With $\alpha \in \{1, 2\}$, an index f satisfies A1, A2$_\alpha$ and A3 if, and only if, f is the Hirsch index.

Proof "⇐"Remark 3. "⇒"With $\alpha \in \{1, 2\}$, let f be an index satisfying A1, A2$_\alpha$ and A3. Step 1: f agrees with the Hirsch index on D_0. Since the only member of D_0 is $x = \emptyset$ and since $d_x = min\{x_1, \ldots, x_{d_x}\} = 0$, by A1, $f(\emptyset) = 0 = h(\emptyset)$.

Step 2: f agrees with the Hirsch index on D_1. Let $x \in D_1$. Case 1: $x_1 \geq 1$. By A1, $f(x) = 1$. Case 2: $x_1 = 0$. Case 2a: $f(x) \neq max\{f(z)\}_{z \in D_1}$. Since $x \in D_1$, $x_{-1} = \emptyset$. By A3, $f(x) = f(x_{-1}) = 0 = h(x)$. Case 2b: $f(x) = max\{f(z)\}_{z \in D_1}$. Let $y = \emptyset$. By step 1, $f(y) = max\{f(z)\}_{z \in D_0} = 0$. Case 2b1: A2$_1$ holds. Then $y \in D_0$, $x \in D_1$, $max\{f(z)\}_{z \in D_1} = f(x)$, $x \geq y$ and $\delta(y, x) = 0 \leq c_y = 0$. By A2$_1$, $f(x) \leq f(y) = 0$. Since $f(x) \geq 0$ by definition of index, $f(x) = 0 = h(x)$. Case 2b2: A2$_2$ holds. Then $y \in D_0$, $x \in D_1$, $f(x) = max\{f(z)\}_{z \in D_1}$, $f(y) = max\{f(x)\}_{z \in D_0}$, $x \geq y$ and $\delta(y, x) = 0 \leq c_y = 0$. By A2$_2$ when $k = 1$, $f(z) \leq f(y) = 0$. Hence, $f(x) = 0 = h(x)$.

Step 3: for $n \in \mathbb{N}\backslash\{0, 1\}$, f agrees with the Hirsch index on D_n. Choose $n \in \mathbb{N}\backslash\{0, 1\}$ and, by steps 1 and 2, suppose that, for all $k \in \{0, 1,\ldots,n - 1\}$, f agrees with the Hirsch index on D_k. To prove that f agrees with the Hirsch index on D_n, choose $x \in D_n$. Let $h = h(x)$. Case 1: $h = n$. This means that, for all $i \in \{1,\ldots,n\}$, $x_i \geq n$. Hence, $c_x = d_x = n$ and, by A1, $f(x) = d_x = n = h$. Case 2: $h < n$. By the induction hypothesis, $f(x_{-n}) = h(x_{-n})$. As $h(x) = h < n$, it follows that $x_n \leq h$ and, thus, $h(x_{-n}) = h(x)$. In sum, $f(x_{-n}) = h$.

Case 2a: $f(x) \neq max\{f(z)\}_{z \in D_n}$. By A3, $f(x) = f(x_{-n}) = h = h(x)$. Case 2b: $f(x) = max\{f(z)\}_{z \in D_n}$. Let $k \in \{2,\ldots,n\}$ and $y \in D_k$ satisfy, for all $i \in \{1,\ldots,k\}$, $y_i \geq k$. By A1, $f(y) \geq min\{min\{y_1,\ldots,y_k\}, k\} = k$. The Hirsch index is such that, for all $r \in \mathbb{N}$, $max\{h(z)\}_{z \in D_r} = r$. Given $f(y) \geq k$, by the induction hypothesis, $f(v) = max\{f(z)\}_{z \in D_k}$ implies $f(v) = k$. As a consequence, for all $k \in \{2,\ldots,n\}$,

$$max\{f(z)\}_{z \in D_k} = k. \tag{1}$$

Case 2b1: $\alpha = 1$. By (1), $max\{f(z)\}_{z \in D_n} = f(x)$ implies $f(x) > f(x_{-n})$. As a result, $x_{-n} \in D_{n-1}$, $x \in D_n$, $x \geq x_{-n}$ and $max\{f(z)\}_{z \in Dn} = f(x) > f(x_{-n})$ imply, by A2$_1$, $\delta(x_{-n}, x) > c_{x_{-n}} \geq h$. But $\delta(x_{-n}, x) = x_n$ and, since $h(x_{-n}) = h$, $x_n \leq h$: contradiction.

Case 2b2: $\alpha = 2$. Let $v \in D_h$ satisfy, for all $i \in \{1,\ldots,h\}$, $v_i = x_i$. By A1, $f(v) = h$. By (1), $f(v) = max\{f(z)\}_{z \in D_h}$. Let $r = n - h$. For $t \in \{1,\ldots,r\}$, let $x^t \in D_{h+t}$ satisfy, for all $i \in \{1,\ldots,h + t\}$, $x^t_i = x_i$. It follows from $h(x) = h$ that, for all $i \in \{1,\ldots,h + t\}$, $x_i \leq h = c_v$. Given this, the fact that $x^r = x$ implies $\delta(v, x) \leq rh \leq rc_v$. Summarizing, $v \in D_h$ and $x \in D_{h+r}$ are such that $x \geq v$, and $\delta(v, x) \leq rc_x$. By A2$_2$, $f(x) \leq f(v)$. Hence, $f(x) \leq f(v) = h < n$, which contradicts $f(x) = max\{f(z)\}_{z \in Dn} = n$. □

Remark 5 Neither A2$_1$ nor A2$_2$ can be replaced by A2 in Proposition 4: an index f satisfying A1, A2 and A3 need not be the Hirsch index, as Example 6 proves.

Example 6 Let f be the index such that $f(3, 1, 1) = 3$ and, for all $x \in D\backslash\{(3, 1, 1)\}$, $f(x) = h(x)$. Whereas f satisfies A1, A2 and A3, it is not the Hirsch index.

Proposition 7 An index f satisfies A1, A2 and A4 if, and only if, f is the Hirsch index.

Proof "⇐"Remark 3. "⇒" Let f be an index satisfying A1, A2 and A4. Step 1: f agrees with the Hirsch index on D_0. Since the only member of D_0 is $x = \emptyset$ and since $c_x = d_x = min\{x_1, \ldots, x_{d_x}\} = 0$, by A1, $f(\emptyset) = 0 = h(\emptyset)$. Step 2: f agrees with the Hirsch index on D_1. Let $x \in D_1$. By A1, $min\{x_1, 1\} \leq f(x) \leq 1$. Thus, $x_1 \geq 1$ implies $f(x) = 1 = h(x)$. If $x_1 = 0$ then let $y = \emptyset$. By step 1, $f(y) = max\{f(z)\}_{z \in D_0} = 0$. In addition, $x \geq y$ and $\delta(y, x) = 0 < c_y = 0$. By A2, $f(x) \leq f(y) = 0$. By definition of index, $f(x) \geq 0$. In sum, $f(x) = 0 = h(x)$.

Step 3: for $n \in \mathbb{N}\backslash\{0, 1\}$, f agrees with the Hirsch index on D_n. Choose $n \in \mathbb{N}\backslash\{0, 1\}$ and, by steps 1 and 2, suppose that, for all $k \in \{0, 1,\ldots,n - 1\}$, f agrees with the Hirsch index on D_k. To prove that f agrees with the Hirsch index on D_n, choose $x \in D_n$. Let $h = h(x)$. Case 1: $h = n$. This means that, for all $i \in \{1,\ldots,n\}$, $x_i \geq n$. Hence, $c_x = d_x = n$ and, by A1, $f(x) = d_x = n = h$. Case 2: $h < n$. Let $v \in D_h$ satisfy, for all $i \in \{1,\ldots,h\}$, $v_i = x_i$. By A1, $f(v) = h$. The Hirsch index is such that, for all $r \in \mathbb{N}$, $max\{h(z)\}_{z \in D_r} = r$. By A1, the induction hypothesis and $f(v) = h, max\{f(z)\}_{z \in D_h} = h$. Let $r = n - h$. For $t \in \{1,\ldots,r\}$, let $x^t \in D_{h+t}$ satisfy, for all $i \in \{1,\ldots,h + t\}$, $x^t_i = x_i$. It follows from $h(x) = h$ that, for all $i \in \{h + 1,\ldots,n\}$, $x_i \leq h$. Define w to be the member of D_{h+1} such that $w_{h+1} = h$ and, for all $i \in \{1,\ldots,h\}$, $w_i = v_i$. Then $v \in D_h$, $w \in D_{h+1}$, $w \geq v$, $f(v) = max\{f(z)\}_{z \in D_h}$ and $\delta(w, v) = h = c_v$. Therefore, by A2, $f(w) \leq f(v) = h$. By A1, $f(w) \geq h$. Consequently, $f(w) = h = f(v)$. Given this, by A4, $f(v) = f(x^1)$. This result, by A4, yields $f(x^1) = f(x^2)$. By repeated application of A4, for all $t \in \{1,\ldots, r - 1\}$, $f(x^t) = f(x^{t+1})$. Summing up, $h = f(v) = f(x^1) = \cdots = f(x^r) = f(x)$. □

Remark 8 Examples 9, 10 and 11 prove that no axiom in Propositions 4 and 7 is redundant.

Example 9 Let f be the index such that, for all $x \in D$, $f(x) = 1 + h(x)$. Then f satisfies A2, $A2_1$, $A2_2$, A3 and A4; does not satisfy A1; and is not the Hirsch index.

Example 10 Let f be the index such that, for all $x \in D$, $f(x) = d_x$. Then f satisfies A1, A3 and A4; satisfies neither of A2, $A2_1$ and $A2_2$; and is not the Hirsch index.

Example 11 Let f be the index such that, for all $x \in D$, $f(x) = h(x) - 1$ if $min\{x_1, \ldots, x_{d_x}\} < h(x) < d_x$ and $f(x) = h(x)$ otherwise. Then f satisfies A1, A2, $A2_1$ and $A2_2$; satisfies neither A3 nor A4; and is not the Hirsch index.

Concluding comments

Woeginger (2008b, p. 301) provides another characterization of the Hirsch index, on the domain of integer-valued indices, in which monotonicity is still assumed and an interesting symmetry axiom is postulated. For $x = (x_1,\ldots,x_n) \in D$, Woeginger defines the reflection $R(x)$ of x to be the vector $(y_1,\ldots,y_k)$ such that $k = x_1$ and y_i is the number of components in x whose value is not smaller than i. For instance, if $x = (7, 2, 2, 1, 0)$ then $R(x) = (4, 3, 1, 1, 1, 1, 1)$. The symmetry axiom holds that the value of the index should be preserved under reflections: $f(x) = f(R(x))$. As a result, papers and citations are exchangeable variables through reflection.

One of the referees recommends mentioning Quesada (2008) as another characterization of the Hirsch index relying as well on monotonicity. This paper axiomatizes the Hirsch index, on the domain of real-valued indices, using monotonicity and another two axioms

(Woeginger 2008b assumes six). The first axiom strengthens A1 by requiring that $min\{min\{x_1,\ldots,x_{d_x}\},c_x\} \leq f(x) \leq min\{max\{x_1,\ldots,x_{d_x}\},d_x\}$. The second axiom can be viewed as another monotonicity-type property and bears some resemblance to A4: if $f(x_1,\ldots,x_n) = f(y_1,\ldots,y_m)$ and $f(x_1,\ldots,x_n, a) > f(x_1,\ldots,x_n)$ then $f(y_1,\ldots,y_m, a) > f(y_1,\ldots,y_m)$, provided that $(y_1,\ldots,y_m, a)$ is a well-defined output. This says that if the index does not distinguish between two outputs and the addition of another paper to one output causes an increase in the index then the same qualitative effect should arise from the addition of the same paper to the second output.

The resulting characterization seems to indicate that the Hirsch index can be obtained by postulating sufficiently strong monotonicity requirements and by imposing appropriate bounds to that monotonicity. Propositions 4 and 7 can be seen as obtained from the strategy of weakening monotonicity and, in exchange, adopting independence conditions stating when the index should remain unaltered: whereas A1 is the axiom setting the bounds, the A2 axioms express a necessary condition for the index to be monotonic in a particular case and A3 and A4 are independence axioms identifying changes in a research output that should not affect the index.

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