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The Creative Power of Formal Analogies in Physics: The Case of Albert Einstein

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Abstract In order to show how formal analogies between different physical systems play an important conceptual work in physics, this paper analyzes the evolution of Einstein's thoughts on the structure of radiation from the point of view of the formal analogies he used as "lenses" to "see" through the "black box" of Planck's blackbody radiation law. A comparison is also made with his 1925 paper on the quantum gas where he used the same formal methods. Changes of formal points of view are most of the time taken for granted or passed over in silence in studies on the mathematization of physics as if they had no special significance. Revisiting Einstein's classic papers on the nature of light and matter from the angle of the various theoretical tools he used, namely entropy and energy fluctuation calculations, helps explain why he was in a unique position to make visible the particle structure of radiation and the dual (particle and wave) nature of light and matter. Finally, this case study calls attention to the more general question of the surprising creative power of formal analogies and their frequent use in theoretical physics. This aspect of intellectual creation can be useful in the teaching of physics.

> *He sees exclusively through the lenses of classical thermodynamics.*
> *A. Einstein to W. Julius, 18 December 1911.*

The relationship between mathematics and natural sciences is nowhere more central than in physics. It is thus not surprising that historical and philosophical reflections on this topic have used history of physics as a prime example. Numerous pedagogical studies have also looked at how best to facilitate physics understanding given the obstacles many students face in learning the mathematics necessary for the formulation of most physical laws and theories.[1] There is, however, one aspect of the relations between physics and mathematics that is not enough underlined though it plays a central heuristic role in physical research.

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[1] For recent exemples, see Pereira de Ataíde and Greca (2013), Torregrosa et al. (2006), Quale (2011), Gingras (2001), de Berg (1992).

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This aspect is the use of formal analogies between two different physical systems. I prefer the term “formal” to “mathematical” in order to underline the fact that it is the syntactic form of the equations that are compared and used as a heuristic bridge to understand a new system in the light of a better known one. For two materially and physically very different systems may in fact be very similar and formally identical despite their physical dissimilarities. Their equations may make these underlying similarities visible. The best way to exemplify the manner in which attention to mathematical analogies may suggest new physical properties is to look at particular examples where such a mode of thinking has been successful.

In this paper, we analyze the evolution of Einstein’s thoughts on the structure of radiation from the point of view of the formal analogies he used as “lenses” to “see” through the “black box” of Planck’s blackbody radiation law. A comparison is also made with his 1925 paper on the quantum gas where he used the same formal methods. Changes of formal points of view are most of the time taken for granted or passed over in silence in studies on the mathematization of physics as if they had no special significance. Revisiting Einstein’s classic papers from the angle of the various theoretical tools he used in his major works on radiation, namely entropy and energy fluctuation calculations, helps explain why he was in a unique position to make visible the particle structure of radiation and the dual (particle and wave) nature of light and matter. Our analysis calls attention to the more general question of the surprising creative power of formal analogies and their frequent use in theoretical physics.[2] Attention to this neglected aspect of the epistemology of physics could be useful in pedagogical approaches aiming at a better understanding of the role of mathematics in natural sciences.

1 Entropy as a Probe into the Nature of Radiation

It is well-known that Einstein considered his 1905 paper on light quanta as a “very revolutionary” contribution to physics, as he wrote in May of that year to his best friend Conrad Habicht (Einstein 1995, 20). Although one might think that everything has been said about Einstein’s conception of light quanta,[3] I think that by focusing on the change of formal tools from entropy to energy fluctuation, which happened between 1905 and 1909, one can suggest a new answer to an intriguing question: Why did Einstein first limit himself to the Wien’s approximation, instead of working directly with Planck’s equation for the full spectrum of blackbody radiation? All papers on Einstein’s view on the particle nature of light start, like him, with the Wien’s approximation without asking why he does that. That question has been raised by John Stachel but in relation to Einstein 1924–1925 papers on atomic gas and quantum statistics, whereas I here raise it in relation to his 1905 and 1909 papers, the second of which made visible for the first time what became known as “wave–particle duality” (Stachel (2002, 427–444).[4]

In addition to contributing to a better understanding of Einstein’s style of theoretical physics, our analysis wish to call attention more generally to the role of formal analogies in the practice of physicists. Many recent papers have been devoted to the role of models in

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[2] Prepared for the special issue of *Science and Education*, on the relations between physics and mathematics, this paper is a revised English translation of a previous paper published in French in Gingras (2011).

[3] See for example Klein (1980, 1982), Bergia and Navaro (1988), Soler (1999, 2001), Norton (2006), Brush (2007).

[4] On the history of wave-particle duality, see Hendry (1980) and Wheaton (1983).

scientific practice and distinctions between different kinds of analogies have been proposed but the actual use of formal analogies in physics and the cognitive work they do still await further analysis.[5] Given that models are often taken as closely related to analogies and even synonymous with them,[6] let us recall briefly some basic distinctions. Though many models are based on analogies (like the atomic model based on the solar system) not all models are analogical. Moreover, making analogies between two systems is not necessarily the same as constructing a model of one of the system. For instance, constructing a model of the moon as a three-dimensional structure with a complicated distribution of its mass is not based on an analogy with another system and is just a model of the actual moon. The other distinction we need is between material and formal analogies.[7] The former refers to material structures (the comparison between water wave and light wave or between a water drop and an uranium atom in the case of nuclear fission) while the latter only compares equations drawn from two different domains (electricity and mechanics for example, without inquiring about the kind of substance electricity could be). As we will see below, formal analogies compare the *syntactic* form of an equation from a known physical system to that of another equation in a (usually) less well-understood physical system in order to shed light on the *semantic* of the latter. Such comparisons thus go beyond the substance of the system (like matter vs. radiation) and look at their structure, thus making very different kinds of system (like mechanical ones vs. electric ones) look similar and even identical in their dynamic.

We are interested here in studying "analogies in action", and we will look at the manner in which they were used by Einstein in his search for understanding the nature of radiation and matter. In the following sections, we will recall the main arguments he developed in his papers devoted to the structure of radiation with an emphasis on the physical insight he derived from the mathematical form of the equations he obtained. We will also look at his 1925 paper on the quantum theory of monoatomic gas for it was by using the very same formal analogy on the fluctuations of energy in the reverse direction (*from* radiation *to* matter instead of *from* matter *to* radiation as he did in 1905) that he was able to generalize to material particles the wave–particle duality he had first attributed to radiation in 1909. In conclusion, we will briefly come back to the more general question of the limits of formal analogies as a tool for discoveries.

Probably since the end of the 1890s, Einstein was convinced of the particle structure of matter. His thesis on molecular weight as well as his first papers on statistical mechanics are based on that intuition and one can see his research on the nature of radiation as an extension of that research program. In 1905, as he was trying to get a hand on the structure of radiation, Einstein knew perfectly that the results he obtained using Wien's law for the distribution of radiation density ρ in terms of frequency v and temperature T:

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[5] For a detailed survey of the use of models in different disciplines see Morgan and Morrison (1999).

[6] See for example Hesse (1966, 8–10), where *models* are discussed in terms of positive and negative *analogies*. See also W. H. Leatherdale (1974, 1). The confusion between models and analogies is often due to the fact that most authors concentrate on the special cases of analogical models. For a recent analysis, see Bailer-Jones (2009).

[7] For a useful discussion of models and analogies, see Redhead (1980). Most authors discuss material analogies. See for example Holyoak and Thagard (1995, 186–188), where among their list of sixteen well-known scientific analogies of "enduring significance", none is a formal one.

$$\rho = \frac{8\pi h v^3}{c^3} e^{-hv/kT} \tag{1}$$

could only be an approximation and thus served only as a “heuristic point of view”, “not strictly valid” but “fully confirmed by experiment for large values of [v/T]” (Einstein 1989, 93).[8] But then, why not have worked directly with Planck’s law:

$$\rho = \frac{8\pi h v^3}{c^3} \frac{1}{e^{hv/kT} - 1} \tag{2}$$

and obtain more than a “heuristic” view of the structure of radiation? Martin Klein has noted that Einstein “deliberately did not use Planck form of the spectral distribution despite its greater power to account for experiments over the whole measured range of wavelengths” (Klein 1967, 512). Our question is why?

It has been suggested that in 1905, Einstein did not trust Planck’s equation because its foundations were not secure (Stachel 2002, 236–239). Though it is no doubt true that the foundations of this equation were not conceptually clear, we must distinguish between the empirical validity of the law, relatively well established in 1905, and the understanding of its theoretical foundation. Einstein’s problem with Planck’s law concerned its foundations, not its empirical validity.[9] As much as he knew that Wien’s law did not apply generally, there is no reason to think he could not accept the empirical validity of Planck’s law and use it to make calculations as he did with Wien’s law. Thus, the lack of trust in Planck’s equation is probably not the main reason that led Einstein to limit himself to Wien’s approximation in his 1905 paper.

I think the reason for choosing Wien’s law lies in the formal tools he was using at the time, namely entropy calculations based on Boltzmann’s equation, which were not well suited for “seeing” completely the hidden structure of Planck’s equation. Thus, one should search the answer to our question in the nature of the formal analogies with which Einstein was trying to “probe” the structure of blackbody radiation. As we will see, the two major tools he used were the equations for entropy and the equation for energy fluctuations, both stemming from his knowledge of thermodynamics and statistical mechanics, a branch of physics to which he contributed major papers between 1901 and 1904. As Martin Klein showed, all of Einstein’s most original ideas are “intimately related to his understanding of thermodynamics”.[10] While the use of entropy made it possible to create the concept of “light quantum” (photon) in 1905, only the use of energy fluctuation made visible the dual nature of light (in 1909) and, much later (in 1925), the similarly dual nature of material particles.

Einstein’s work thus provides a nice example of the fact that the very choice of mathematical formalisms and the formal analogies they may suggest can play a creative role in physical thinking. By defining an angle of vision, they make visible, otherwise invisible physical structures and different mathematical tools make visible different aspects of the object studied.

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[8] For convenience we use modern notations.

[9] Einstein’s reflections on those foundations led to his 1906 paper “On the Theory of Light Production and Light Absorption” in which he clarified Planck’s derivation, concluding that “Mr Planck introduced into physics a new hypothetical element: the hypothesis of light quanta”, thus confirming that his own views on the quantum of light and those of Planck were not incompatible as he first thought in 1905; see Einstein (1989, 192, 196).

[10] Klein (1967, 509). For more recent analysis see Baracca (1985), Abiko (2000), A. Kojevnikov (2002).

In his 1905 paper on the light quantum, Einstein used entropy calculations based on Boltzmann's equation:

$$S = k \ln W \tag{3}$$

in order to probe the nature of radiation. Although he left no scrap of papers on which he wrote the calculations that led to this famous paper in which he explained the photoelectric effect, it is likely that he first played with Planck's equation but got no insight into it using his entropy approach, while he found that for the Wien's approximation, the calculation of entropy led to a nice formula:

$$S - S_0 = k \ln (V/V_0)^{E/h\nu} \tag{4}$$

for the variation of entropy of radiation contained in a sub-volume V of the total volume V_0. Having obtained this form, he immediately remarked that "this equation shows that the entropy of a monochromatic radiation of sufficiently low density varies with the volume according to the same law as the entropy for an ideal gas or that of a dilute solution" (Einstein 1989, 94). He derived the equation for a collection of n independent gas particles submitted to the same change in volume and obtained:

$$S - S_0 = k \ \ln (V/V_0)^n. \tag{5}$$

It is the formal analogy that makes *visible* the *structure* of radiation that Einstein was looking for. Equations (4) and (5), combined with the fact that the argument of their logarithm is the expression of the probability of finding all radiation (or particles) in volume V, led him to the conclusion that "monochromatic radiation of low density (within the range of validity of the Wien's radiation formula) *behaves thermodynamically as if* it consisted of mutually independent energy quanta of magnitude [$h\nu$]" (Einstein 1989, 97, our emphasis). Based on this formal analogy, he thus made the bold step of equating the exponents of both equations to get the famous equation:

$$E = nh\nu. \tag{6}$$

Far from being a valid deduction, this equation could only be *suggested* by the *formal* identity of the entropy equations, hence Einstein's explicit use of the expression "behaves as if".[11]

The problem with Eq. (6) and the "particle" interpretation of radiation is of course that it does not apply generally since the true equation empirically tested was Planck's and not Wien's. Einstein was conscious of this limitation and, a few years later, explained his views in a letter to Lorentz written in May 1909:

> As far as the light quanta are concerned, it seems that I did not express myself clearly. For I am not at all of the opinion that light has to be thought of as being composed of mutually independent quanta localized in relatively small spaces. To be sure, this would be the most convenient way to explain the Wien end of the radiation formula. But the splitting of light rays on the surface of refracting media makes already this approach absolutely inadmissible. A light ray splits, but a quantum cannot split without a change of frequency (Einstein 1995, 123).

Interestingly, he wrote these lines just a few months after having published a paper "On the Present Status of the Radiation Problem" in which he had analyzed the complete Planck's formula looking at it this time through different "glasses", namely energy fluctuations instead of Boltzmann's equation.

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[11] We are not discussing here the *logical* necessity of the analogy used by Einstein but the *historical* fact of its particular use in his argumentation for the particle structure of radiation. For a logical analysis see Dorling (1971).

2 Changing Glasses: From Entropy to Energy Fluctuation

For some unknown reasons, Einstein finally came to approach the question of radiation from a different angle. Instead of thinking in terms of entropy, as he had done in his previous papers of 1905 and 1906, he tackled the problem in terms of energy fluctuations, thus providing, as he said, “a simple interpretation of the content of Planck’s radiation formula”, something he did not achieve in 1905 (Einstein 1989, 264).

But before analyzing further his new results, we must first recall that Einstein was at the time among the very few experts on statistical mechanics. He had, as was noted earlier, published fundamental papers on that subject in the period 1901–1904. In one of them (published in 1904), he derived the equation governing the mean square of the fluctuation ε of energy E at temperature T:

$$\overline{\varepsilon^2} = kT^2 \frac{\partial \overline{E}}{\partial T}. \tag{7}$$

Always sensitive to the formal structure of equations, he immediately remarked that “the relation just found is interesting because it no longer contains any quantity reminiscent of the assumptions on which the theory is based”, thus implicitly suggesting its *general* applicability outside the limits of mechanical systems of point particles endowed with mass from which it was derived (Einstein 1989, 75). Einstein was alone in thinking that statistical mechanics could be applied directly to free radiation (as opposed to the interaction of matter with radiation). To test his idea, he derived Wien’s displacement law (relating the maximum wavelength of radiation to the inverse of the temperature in the cavity) from the Stefan–Boltzmann law (stating that the average energy density of radiation contained in a volume V is proportional to T^4).[12] He concluded his paper by stating that the result obtained being of the right order of magnitude, “this agreement must not be ascribed to chance” and thus suggests the legitimacy of using Eq. (7) even for free radiation (Einstein 1989, 77).

In 1905, Einstein thus had already in his possession the basic equation for energy fluctuation but for some unknown reason he did not think of applying it directly to Planck’s law thus delaying by a few years his discovery of wave–particle duality. The fact that he had both tools in hand in 1905 gives even more weight to the idea that the choice of formalism plays an important role in *suggesting* physical interpretations of physical entities entering into mathematical equations. Though this cannot be proven, it is likely that being used to looking at radiation through the glasses of entropy, only the failure to understand Planck’s law in this way finally led him to change his lenses and try something else.

So, in his continuing quest to obtain information on the nature of thermal radiation, he came to calculate, in his 1909 paper, the fluctuations of energy in the case of radiation, an idea he had already hinted at in 1904. In essence, he used Eq. (7) instead of entropy (3) in relation to Planck’s Eq. (2) to obtain:

$$\overline{\varepsilon^2} = \left(h\nu\rho + \frac{c^3}{8\pi\nu^2}\rho^2 \right) V \mathrm{d}V. \tag{8}$$

As he was quick to point out, “the first term, if present alone, would yield a fluctuation of the radiation energy equal to that produced if the radiation consisted of point quanta of energy $h\nu$ moving independently of each other” (Einstein 1989, 366). It could be obtained

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[12] For details on this question, see Pais (1982, 68–70).

by applying (7) to the Wien's approximation (1). As for the second term, it has the form one would find when applying (7) to the Rayleigh–Jeans equation for waves:

$$\rho = \frac{8\pi v^2}{c^3} kT \tag{9}$$

In other terms, Einstein finally saw the structure of radiation by changing the "glasses" through which he looked at radiation and the wave–particle duality first appeared through a formal analysis based on a statistical mechanical treatment of Planck's equation. Einstein noted that the additivity of the two terms suggests that the effects of fluctuation are "arising from mutually independent causes", and thus he concluded that "the constitution of radiation must be different from what we currently believe", that is, different from a wave theory of light (Einstein 1989, 369).

Transposing the reasoning of his 1905 Brownian motion paper to the case of a small mirror immersed in a radiation field (instead of a gas of atoms), he also showed in the same paper that the existence of an equilibrium and the conservation of energy required that light also carried momentum according to an equation for the fluctuation of momentum similar to (8) but divided by c and where VdV is replaced by $S\tau dv$, the surface S of the mirror, the time interval τ, and frequency interval dv. He emphasized that the close relation between this relation and Eq. (8) "is immediately obvious and exactly analogous considerations can be applied to it" thus reinforcing the dual nature of light (Einstein 1989, 369).

From these considerations he concluded, during a talk at the Physics division of the meeting of the *Deutsche Physikalische Gesellschaft* given a few months after the publication of this paper, that in his opinion, "the next stage in the development of theoretical physics will bring us a theory of light that can be understood *as a kind of fusion* of the wave and emission [read: particle] theories of light" (Einstein 1989, 379, our emphasis). And indeed "fusion" rather than the now usual "duality" is probably a better term to characterize the peculiar structure of light.[13] In a brief paper published in 1910 under the title "On the Theory of Light Quanta and the Question of the Localization of Electromagnetic Energy", he repeated that he thought that we should assign to the "electromagnetic radiation itself, besides a wave structure, a second kind of structure such that the energy of the radiation itself is already divided into definite quanta" (Einstein 1993, 207–208). Citing his 1905 paper, he summarized again his argument based on Boltzmann's equation applied to Wien's law. It is worth stressing again that all these considerations were based solely on a study of the *formal* properties of the equations using the tools of statistical mechanics applied to radiation.

Einstein's strategy was to analyze the consequences of Planck's radiation law from as many different angles as possible and in 1911, as part of his presentation at the Solvay congress "On the Present State of the Problem of Specific Heats", he analyzed the dual nature of radiation from another point of view by looking at the vibrations of a solid. Calculating again the average fluctuation of energy in a solid vibrating at frequency v, he found:

$$\overline{\left[\frac{\varepsilon}{E}\right]^2} = \frac{1}{z_q} + \frac{1}{z_f} \tag{10}$$

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[13] Though it is not the place to develop that argument here, I think the acceptance of the idea of "duality" between particle and wave is closely linked to the Copenhagen interpretation of Quantum mechanics and its underlying operationalist philosophy in which the ontology is defined by measuring instrument. Though always marginal, some major physicists always opposed the idea of "wave-particle duality" as absurd; see for example Landé (1965).

where $Z_q = \frac{E}{h\nu}$ is the average number of light quanta in the solid, and $Z_f = 3nN$ is the total number of degrees of freedom of a solid with N (Avogadro's number) atoms. He immediately noted that "one sees from this equation that the system's relative energy fluctuations, which are produced by the irregular thermal motion, *result from two completely different causes*" (Einstein 1993, 414, our emphasis). While the second term "is the only fluctuation according to our mechanics [and] results from the fact that the number of degrees of freedom of the body is finite", the first term "shows an exact agreement with the quantum hypothesis, according to which energy consists of quanta of magnitude $h\nu$, which change their location independently of each other" (Einstein 1993, 415). Making the inverse calculation *from* the fluctuation equation *to* Planck's equation, he concluded that the fluctuation equation "exhaust[s] the thermodynamic content of Planck's radiation formula" (Einstein 1993, 415). After many years of intense work, Einstein had thus finally clarified the physical content of Planck's radiation law, at least as far as it concerned its *thermodynamic* properties.

During the discussion at the Solvay congress, Einstein had reaffirmed his 1905 idea on light quanta and added that by conceiving this idea he had "found an intuitive interpretation of the probability law for low-intensity radiation" (Einstein 1993, 431). As we saw, this "intuition" was strongly guided by his peculiar use of formal analogies between matter and radiation, a view Planck could not accept. In response to Einstein, he said that he was of the opinion that for thermal radiation in a vacuum, "entropy (or probability) cannot really be derived from the energy fluctuations of free radiation alone", the very hypothesis that was at the foundation of all the equations that led Einstein to his revolutionary ideas of photon and wave–particle duality (Einstein 1993, 432). And it is worth noting that in all these formal manipulations, Einstein did not aim to produce numbers for comparison with experiment but was in search of a *form* from which he could infer the physical *structure* of radiation.

3 Reversing the Analogy: Generalizing the Wave–Particle Duality to Matter

That analogical thinking was far from "uncharacteristic of Einstein" as Jon Dorling suggested,[14] but was on the contrary typical of his works on radiation, is further shown by his 1925 paper in which he predicted what have come to be known as "Bose–Einstein condensation". In this paper, he used an equation analogous to Eq. (10) in order to show that the wave–particle structure of *radiation* also applies to *material* particles. Thus, 20 years after having compared light to matter, he now looked at the problem the other way around and compared matter to light using the same formal tools provided by statistical mechanics for the calculation of energy fluctuations.

After having read and translated into German S. N. Bose paper on the new quantum statistics that the Indian physicist had sent him in 1924—in which he derived Plank's law in a way that is independent of classical electrodynamics—Einstein immediately applied the same method to material particles to develop a quantum theory of monoatomic ideal gas.[15] For Einstein, the interest of Bose's approach resided in the fact that "it is based on the hypothesis of a far-reaching *formal similarity* between the radiation and the gas"

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[14] Dorling (1971) writes that the use of arguments by analogy "seemed quite uncharacteristic of Einstein" and that it does not occur "in any of his other major writings". As this paper shows, this is far from being the case.

[15] For more details on the relation between Einstein and Bose, see Stachel (2002, 519–538).

(Einstein 1997a, 89, our emphasis). If Bose was right in suggesting that one can conceive radiation as a "gas of quanta", then "the analogy between the gas of quanta and the gas of molecules *must be a complete one*" (Einstein 1997a, b, 89, our emphasis). On this basis, Einstein used calculated energy fluctuations to probe the quantum behavior of molecules following exactly the same line of reasoning he had used to probe the quantum behavior of light. In a section devoted to "the fluctuations properties of the Ideal gas" of the second of his three parts paper, he wrote the equation for the mean square of the fluctuation Δ_ν of the average number of molecules n_ν in the energy range ΔE in the form:

$$\overline{\left[\frac{\Delta_\nu}{n_\nu}\right]^2} = \frac{1}{n_\nu} + \frac{1}{z_\nu} \tag{11}$$

where z_ν is the number of cells in phase space in the energy range ΔE. Having observed that this fluctuation law "is the perfect analog of the quasi-monochromatic Planck's radiation result, he added—in a kind of 'cut and paste' from his 1911 paper—that "the mean square relative fluctuation of the molecules is the sum of two terms. The first would occur alone if the molecules were independent of one another" (Einstein 1997a, 94). As for the second term, Einstein suggested that one could interpret it, by direct analogy with the case of radiation represented in Eq. (8), as the fluctuations arising from the interference of a radiation field suitably associated with the molecules of the gas. Convinced that this interpretation is "more than a mere analogy", he presented Louis de Broglie's recent and novel ideas of matter waves as a new approach confirming his own analysis that it is indeed possible to associate a scalar wave field to a gas. And he added that he had convinced himself through calculations (not given in the paper) that $1/z_\nu$ is effectively the mean value of the square of the fluctuations of this wave field (Einstein 1997a, 95).

What is striking here is the fact that Einstein was again using the very same tools of formal analogy he had used in 1909 and 1911, the sentences following Eq. (11) being nearly identical to those which followed Eq. (10) in his 1911 paper, and those he wrote following Eq. (8) in his 1909 paper. And here again, Planck remained skeptical for the very same reason that it "seems to be based on an analogy with the radiation quanta", thus considering, as Mehra and Rechenberg observed, that this analogy with radiation was "a weak point in Einstein's gas theory". By contrast, it was for Einstein more than a "mere" analogy and had in fact to be a *complete* one (Mehra and Rechenberg (1982, 617).

4 The Creative Power of Analogies and its Limits

In all the cases analyzed above, we see in action one aspect of Einstein's way of doing physics that is not often enough emphasized: his unique capacity to take seriously formal analogies between different systems. Our analysis also suggest that had Einstein not been the expert he was in statistical mechanics and thermodynamics geared as it were to see the world through these particular lenses, he would not have been in a position to propose the peculiar idea of light particle nor that of the dual nature of light and matter.

Philosophical reflections on the meaning of formal analogies in physics emerged only when mathematics came to play a central role in the construction of physical theories. James Clerk Maxwell seems to have been the first to raise the question of the legitimacy of the inference to a common semantic out of a common mathematical syntax. After having constructed mechanical analogies to derive equations relating electric currents and magnetic fields, he asked in the conclusion of his 1861 paper "On the Physical Lines of Force":

"how far ought we to regard a coincidence in the mathematical expression of two sets of phenomena as an indication that these phenomena are of the same kind?" (Maxwell 2003, 488). Though it is not entirely clear if "kind" refers to the substance or the dynamic of the two analogous systems, we have seen that Einstein's response would have been that these formal analogies do indeed go very far in suggesting the *physical* meaning of theoretical terms, although few of his contemporaries agreed with him. He seemed also conscious of their limits and the very fact that he never again, after his 1925 papers, came back to the question of light quanta and entropy or energy fluctuation, suggest that he thought he had exhausted the *heuristic value* of that particular formal analogy. Einstein also noted that the final argument for the legitimacy of their use ultimately lies in the success obtained, which cannot be guaranteed a priori. At the beginning of the third and last paper on the quantum theory of the ideal gas, he admitted that the postulate based on the analogy between matter and radiation, that a light quanta differed from a monoatomic molecule (apart from its polarization properties) only in that the rest mass of the quantum "is vanishingly small", was "by no mean approved by all". However, though "the statistical method used by Bose and by me is not free of doubt […] it does appear a posteriori, from its success in the case of radiation, to be correct" (Einstein 1997b, 100).

Obviously, one can never be sure in advance of the ultimate fertility of formal analogies, but it remains that they very often play an important role in physics as a heuristic device. After all, analogical reasoning is one of the few if not even the only way of moving from the unknown to the known.[16] Einstein is thus far from peculiar in his use of formal analogies as a guide to physical understanding and modern physicists use them frequently. A striking recent example is the use a formal analogy between electromagnetic waveguides and black holes physics. The wave equation for the vector potential in a waveguide having a special configuration is formally analogous to the one "that describes the propagation of photons in a curved space–time with a horizon" (Blau 2005, 19). This formal similarity provides a foundation to *transfer* the meaning of "Hawking radiation" (produced through black hole evaporation) to a waveguide configuration and thus *predict* the emission by the waveguide of the electromagnetic equivalent of Hawking radiations. The authors of that analogical analysis note that, "apart from experimentally testing this striking prediction", the comparison "would facilitate the investigation of the trans-planckian problem" (Schützhold and Unruh 2005, 31301-1). This approach based on formal analogies was also used in the case of supersonic fluid flow; the physicist W. G. Unruh having found that "the same arguments which leads to black hole evaporation also predict that a thermal spectrum of sound waves should be given out from the sonic horizon in transonic fluid flow" (Unruh 1981, 1351). The electromagnetic waveguide simply provides a more suitable arrangement to experimentally test a prediction that should apply to all equivalent configurations including, for instance, Bose–Einstein condensates (Garay et al. 2000). These examples, which could be easily multiplied,[17] clearly show that formal analogies are often used as a heuristic tool in the same manner that Einstein used them to shed light on the physical properties of a given system by interpreting the equations on the basis of those of a physically different but formally analogous system. Though this is usually done from the better known system to a lesser known one, the case of black holes and waveguides suggests that this particular direction, though the most frequent, can be reversed; in the work of Unruh and his collaborators, it is the black hole that sheds light on the waveguide!

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[16] For en entry into this literature see Gentner et al. (2001).

[17] For other recent examples, see Schliemann et al. (2005), Barcelo et al. (2005).

Beyond their heuristic and creative power, formal analogies are a also of pedagogical interest for they can serve as a mnemonic device to learn basic equations. Two simple examples will suffice to show that important aspect for the teaching of physics. Students often think they have to learn by heart Newton's second law of motion as well as the basic equation for the dynamics of rotation. In fact, both are formally analogous, so that once one knows $F =$ ma one can write directly the equivalent for rotation, once it is noted that for rotation, the force F is replaced by the torque Γ, the mass m by the moment of inertia I, and the acceleration a by the angular acceleration α. One thus simply have:

$$\Gamma = I\alpha$$

Similarly, starting from the equation for displacement

$$x = vt + 1/2at^2$$

one gets directly the equivalent equation for rotation with the transformation $x \rightarrow \theta$ (position → angular position) $v \rightarrow \omega$ (speed → angular speed) and $a \rightarrow \alpha$ (acceleration → angular acceleration):

$$\theta = \omega t + 1/2\alpha t^2$$

Other more complicated formal analogies exist between mechanical and electrical systems, and their comparisons also help to grasp the meaning of the analogous concept like inertia in mechanics and inductance in electrical systems.[18] Thus, understanding inertia for linear movement helps understanding the analogous notion for rotation, the moment of inertia, incarnated in the flywheels of rotating systems.

Approaching the history of physics from the angle of formal analogies helps understand the manner in which some invisible physical structures can become vivid using a particular mathematical representation of the phenomena investigated. This approach can also contribute to facilitate the understanding of concepts pertaining to different systems that are analogous to each other. And despite the fact that analogies are a mode of inference less certain than formal deduction, the cases analyzed here clearly show that they are part and parcel of the theoretical physicist tool-kit and that they can have a surprising creative power.

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[18] For an analysis of the analogy between mechanical and electrical systems, see Gaston Bachelard, *Le rationalisme appliqué*, Paris, PUF, 1949.

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