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How Archimedes Helped Students to Unravel the Mystery of the Magical Number Pi

Ioannis Papadopoulos

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Abstract This paper describes a classroom experiment where students use techniques found in the history of mathematics to learn about an important mathematical idea. More precisely, sixth graders in a primary school follow Archimedes's method of exhaustion in order to compute the number π. Working in a computer environment, students inscribe and circumscribe regular polygons inside and around a circle in order to find the approximate area of the circle. They then compute the ratio of that approximation to the area of a square with side-length equal to the radius of the circle. This ratio indicates how many times larger the area of the circle is than the area of the square. Mirroring Archimedes's findings, students discover that as they increase the number of sides in their polygons, the numerical results they obtain convince them that this number is almost equal to 3.14.

1 Prologue

In the last few years the research community has been increasingly interested in the inclusion of the history of mathematics in mathematics education. However, according to Siu and Tzanakis (2004), most of the contributions to the research literature remain at a theoretical level. In the same spirit Jankvist (2011) in his review of *The First Century of the International Commission on Mathematical Instruction* (*1908–2008*)*: Reflecting and Shaping the World of Mathematics Education*, edited by Menghini and associates points out that "while plenty of the volume deals with the history of mathematics education, there is little or nothing on the role of history in mathematics education." Teachers need examples showing: (i) how the history of mathematics leads them to develop a broader repertoire of mathematical proficiencies, and (ii) how history in teaching mathematics enables understanding and enables recognizing the role of history in the development of mathematical understanding (Clark 2012). Clark goes on to add that few if any such efforts have investigated how the history of mathematics contributes to the teaching of mathematical knowledge. On the other hand, there exist a plethora of anecdotal reports by primary teachers, who have found success in the

I. Papadopoulos (✉)
Aristotle University of Thessaloniki, Thessaloníki, Greece
e-mail: ypapadop@otenet.gr

practice of connecting the history of mathematics to the study of primary arithmetic (Michalowicz 2000).These anecdotes and the few papers that have been written about implementing such ideas in the education, including Smestad (2012), and Kjeldsen and Blomhøj (2012), make it clear that the approach can be productive. There is a need for more empirical investigation to better understand the most effective approaches and the range of mathematical ideas that can be effectively taught using the history of mathematics as a springboard in learning and teaching.

Responding to this need, this paper presents an empirical investigation of a teaching experiment based on the method of exhaustion of Archimedes for computing π. This experiment was motivated by a question posed by a student: "Why is [π] almost 3.14 and not almost something else, such as 3.13 or 3.15?"

If the question had remained unanswered, this would have left students with the impression that the discovery of important mathematical ideas such as this one are the products of the instant insights of an extraordinary mind, rather than the product of systematic investigation open to anyone willing to acquire necessary skills and do the systematic work of mathematical exploration. Moreover, if the concept is presented as a fact isolated from its original context, students are deprived of the important insight into how a person can figure out that mysterious number.

Using the terminology of Jankvist (2009) the aim of this teaching experiment was twofold: to use history-as-a-tool and, at the same time, to use history-as-a-goal. For the former, history becomes a motivating factor for students in their learning and study of mathematics; for the latter, the goal is to show the students that this mysterious number existed and evolved in time and space and has undergone an evolution, and that it is not something that emerged instantly. Because of the age of the students (primary school students), this experiment let them work with the topic at an informal level, previewing formalization that is more appropriately done at an older age (Furinghetti 2002). The teacher-researcher decided to follow this approach based on his conviction that it is important for young students to witness the development of concepts and to experience the interplay between rigor and imagination in the construction of mathematical ideas (Gulikers and Blom 2001). Helping young students understand the doing of mathematics is essential to the development of mathematical learning.

2 Introduction

The number π occupies a unique place both in the history of mathematics and in mathematics education.

In the history of mathematics, π has been defined as both the ratio of the circumference of a circle to its diameter and the ratio of the area of a circle to the area of the square with the side length equal to the radius of the circle. For both cases, π has been the epicenter of a search by many mathematicians from all nations since ancient times.

In mathematics education, the deeper one goes into π, the more mysterious it becomes (Galton 2009). Initially, it is met during primary school as one of the first formulas students work with; that is to find the perimeter or the area of a circle by substituting the approximation 3.14 for π. Later, students are told that it is not possible to compute an exact decimal equivalent for π though it is never made clear why this fact is worth knowing nor can the students understand any method by which it could be discovered. Even though it is a seemingly simple ratio ($C/2r$ and A/r^2), the decimal places continue forever. More precisely, π cannot be expressed as a definite fraction. Years later, usually during college, students learn that there exists a relationship of π to Euler's exponential *e* and the imaginary number $\sqrt{-1}$.

The presence of π in mathematics education does not end here, but it is beyond the aim of this work to describe all the interesting properties of π in mathematics. This paper, emphasizes the use of π in primary school in relation to the area of a circle and describes a classroom experiment that demonstrates and convinces students that π is approximately equal to 3.14. The design of this experiment was inspired by the original work of Archimedes based on polygons inscribed in a circle and circumscribed around the circle. His method was adapted to the level of the students' ability to understand certain mathematical concepts as well as to the tools that were available.

The next section describes how Archimedes computed π based on the above-mentioned method. Then, another section follows that presents similar experiments found in the research literature. The fifth section describes the context of the experiment and gives a step-by-step presentation of how this experiment was performed. The paper ends with a summary of the work done, highlighting the potential contribution of the history of the concept and its evolution to the students' conceptual understanding.

3 A Brief History of Archimedes's Method

In 1706, William Jones was the first to use the Greek letter π to represent this important ratio, but the earliest approximation of its value goes much farther back: the number 3 was used by the Hebrews, Egyptians and Babylonians (Harris 1959) and was accepted for many centuries. Later, in the second half of the fourth century BC, Euclid (∼325–265 BC) proved that "circles are to each other as the squares of their diameters" (Book XII, Proposition 2) making a contribution to the history of π which eventually led to the formula for the area of a circle. However, it was not until the third century BC that Archimedes of Syracuse (287–212 BC) undertook the task of computing π using the scientific method. In his famous book *Measurement of a Circle*, Archimedes proposed three approaches. The first rests on the area of a circle being equal to that of a right triangle where the legs of the triangle are respectively equal to the radius and the circumference of the circle. In that sense $A = 1/2 \times C \times r$, where A, C and r denote respectively the area, the circumference, and the radius of the circle. The second rests on the ratio of the area of a circle to that of a square, with sides equal to the circle's diameter, as being 11:14 (an estimated value). The third method consists of inscribing polygons in a circle and circumscribing polygons around the circle. The areas of the inner and outer polygon were used to bracket the value of π. After the first measure, the value of π was continually refined by successively doubling the number of sides of the polygons used. The process can continue until the polygons become virtually indistinguishable from the circle.

Trying to obtain an approximation for π, Archimedes recorded an upper and lower limit concluding that π was between these limitations. He began his measure of the upper limit by using a regular hexagon circumscribed about a circle. He then continued by circumscribing a succession of regular polygons whose number of sides were twice the number of the previous polygon. He determined the perimeter of each polygon (up to ninety-six sides) and this allowed him to resolve that π was less than $3\frac{1}{7}$ (upper limit). Next, he determined the lower limit using inscribed regular polygons in a succession similar to the one of the circumscribed polygons (i.e., six, twelve, twenty-four, forty-eight, ninety-six sides). Again, each time, he measured the lower limit perimeter. By that method, he found a lower limit for π of $3\frac{10}{71}$. Thus, he obtained the bounds for π: $3\frac{10}{71} < \pi < 3\frac{1}{7}$ or in decimal notation 3.140845… < π < 3.142857….

The bound $3\frac{1}{7} = \frac{22}{7}$ is often referred to, erroneously, as the Archimedean value: in fact $\frac{22}{7}$ was the upper bound that Archimedes calculated (Castellanos 1988). The process of circumscribing and inscribing polygons to determine a more accurate value for π was continued for a number of years by many mathematicians. Mathematicians in China and in the Islamic world made similar attempts that contribute in the approximation of π (Burton 2011). In the third century AD, Wan Fan in China obtained the estimate 3.1555. Two hundred years later, the distinguished mathematician-astronomer, Tsu Chungchih, with the help of his son, arrived at a value of π expressed by the bounds 3.1415926 < π < 3.1415927. The Chinese degree of accuracy was not reached in the West until the end of the sixteenth century. In the Islamic world, Ghiyath al-Din al-Kashi used a representation to correctly calculate π to 16 decimal places, greatly exceeding all previous calculations until the sixteenth century when the Dutchman Van Ceulen correctly calculated π to 35 decimal places.[1]

Even though more precise approximations were obtained by taking larger and larger doublings, Archimedes's upper-bound calculation of π is still in common use today as a quick approximation of π's value.

We know that π is the ratio of the circumference to the diameter of the circle (as well as the ratio of the area of the circle to the area of the square that has side length equal to the radius of the circle). When we use that definition to calculate π, we need to come up with a measure of the circumference. Archimedes's method of exhaustion is based on the idea that the circumference of a circle can be approximated by the perimeter of a circumscribed or inscribed polygon. Although it remains an approximation -error can never be completely eliminated- increasing the number of sides of these polygons increases the accuracy of the approximation.

Table 1 records the values obtained for π, when a unit circle (diameter = 1) is chosen and the series of regular polygons starts with hexagon (see Fig. 1).

The perimeter of a regular hexagon inscribed in a unit circle equals 3, whereas the perimeter of the circumscribed hexagon equals $2\sqrt{3} = 3.4641016$. For the inscribed and circumscribed dodecagon the corresponding numbers are 3.10582885 and 3.2153903. Continuing this process, one can obtain results for the perimeter as they are presented below in Table 1.

Archimedes concluded that the computed value for π would get closer and closer to the true value for π as the number of sides increased. So, the value for π would be the limit of the perimeter of the polygon in this unit circle as the number of sides goes to infinity.

4 Literature Overview: Classroom Experience

Archimedes's idea for calculating π has been proposed at various levels especially in secondary education.

Mason and Roth (2005) worked with high school (grade 9) students with the aim of encouraging academically oriented students to see mathematical formulas as descriptions

[1] In December 2002, computer scientists Kanada, Ushio and Kuroda, University of Tokyo, Japan, computed π to a world record 1.2411 × 10^{12} (more than one trillion) decimal digits. The computation consumed more than 600 h of time of a Hitachi SR8000 super computer (www.super-computing.org/pi_current.html. Accessed 23 October 2012).

Table 1 Perimeters of inscribed and circumscribed regular polygons

Number of sides	Perimeter of inscribed polygons	Perimeter of circumscribed polygons
6	3.0000000	3.4641016
12	3.1058285	3.2153903
24	3.1326286	3.1596599
48	3.1393502	3.1460862
96	3.1410319	3.1427146
192	3.1414524	3.14188730

All the arithmetical data of Table were retrieved from Harris (1959)

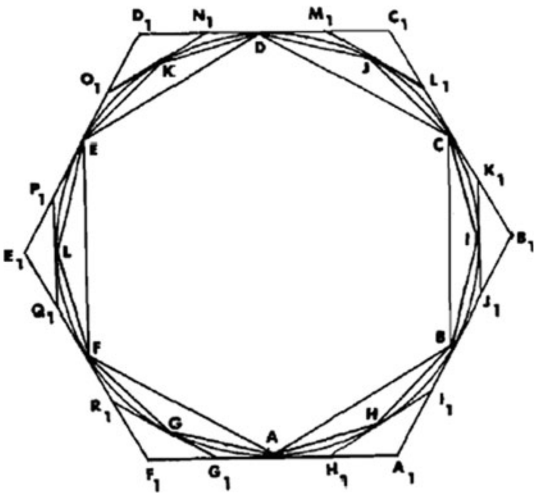


Fig. 1 Inscribed and circumscribed polygons in and around the circle. This figure is from Harris (1959)

of functional relationships as well as instructions for arithmetic in problem solving situations. This was done through activities developed from the historical record of Archimedes as a mathematical thinker. More specifically, in relation to π they wanted to engage students in the kinds of thinking that would lead the students to Archimedes's understanding. The students began to inscribe and circumscribe regular polygons for a given circle using straightedge and compass. They established upper and lower bounds, based on their measurements that were quite close to the actual value of π. Since it was difficult to obtain polygons with a large number of sides, the students were shown the results when this method was continued until a 96-gon was produced.

Another approach (although not so traditional) uses a computer programming environment. Costabile and Serpe (2010), working with secondary school students, proposed some problems, including Archimedes's idea, to be approached by formulating an algorithm and testing that algorithm in a programming environment. The students were asked to start the method of Archimedes by circumscribing a square around a circle using the programming environment MatCos. The authors found that their students learned to: (a) explore an interdisciplinary approach between mathematics and information technology, and (b) connect past and present.

But, the main inspiration for the teaching experiment that will be described in the next sections is a talk given by Serge Lang, who is considered by many of his Yale students to be one of the greatest teachers of mathematics (Kahn et al. 2011). Lang describes this talk in his book *MATH! Encounters with High School Students* (Lang 1985).

Lang starts not by seeking an approximation for the *value* of π, but by establishing that it has just one value, i.e., that it is constant. The problem of estimating π is directly motivated once the fundamental fact obtained already by Euclid is established, namely that C and A are proportional to r and r^2 respectively. Clearly, there would have been no reason to look for the ratios $C/2r$ and A/r^2, if it were not known already that they are constant for all circles. In fact Euclid's result was exactly this. Even though from a logical point of view the equations $C/2r = \pi$ and $A/r^2 = \pi$ are trivially equivalent to the equations $C = 2\pi r$ and $A = \pi r^2$, Lang points out that they are not the same from the didactical point of view. Starting with the notion that C and A are proportional to r and r^2 respectively leads to a natural path to plausibility by using the nontrivial concept of similarity. Lang took a simple example first—the dilation of similar rectangles—as a way to establish how the reasoning might then proceed.

In the first chapter, *What is pi,* Lang showed his students that when one dilates a rectangle by a factor r, its area changes by a factor r^2. He then repeated the process, but this time with a random curved figure. He continued his talk by negotiating the fact that a circle of radius r is the dilation of a circle of radius 1 by a factor r. In fact, this statement is obviously equivalent to Euclid's proposition mentioned above that, for a circle, the area is proportional to r^2. Lang and his students concluded that the area of a disc of radius r is r^2A, where A is the area of a disc of radius 1. They defined π to be the area of that disc (i.e., A). Thus, the area of the disc of radius r is now πr^2.

Later, Lang approximated the circle by regular polygons. The calculation of the area of the polygons combined the area of the triangles that had a vertex at the centre of the circle and the remaining two at the edges of each side of the polygon. The circumference was calculated by measuring the perimeter of the polygon (Fig. 2).

The triangle T_n is repeated n times. Let's call its base b_n and its height h_n. Then its area is $\frac{1}{2}b_nh_n$. Since there are n triangles the area of the n-gon will be $A_n = n\frac{1}{2}b_nh_n = \frac{1}{2}nb_nh_n$. The product nb_n is the perimeter L_n of the polygon. So, $A_n = \frac{1}{2}L_nh_n$. Then they let the number n of the sides of the polygon tend to infinity to get the approximation of the length of the circumference and the area inside the circle. As n approaches ∞, $nb_n = L_n$ approaches C (i.e., length of circumference), h_n approaches r (i.e., length of the radius), and A_n approaches πr^2. Therefore, $A = \frac{1}{2}Cr^2$. This crucial formula explicates the deep relationship between the two notions of π, namely, if the universal constant $\frac{A}{r^2}$ is denoted by π then $\frac{C}{2r} = \pi$ as well.

Adopting the notation shown in Fig. 2 above, if b_n, h_n, r, A_n, L_n and b_n', h_n', r', A_n', L_n', are the quantities corresponding to two different n-gons, then, their similarity follows from the fact that they have equal angles. Therefore we have similarity in the triangles (see

Fig. 2 Approximating the area of the circle. This picture is from Lang (1985)

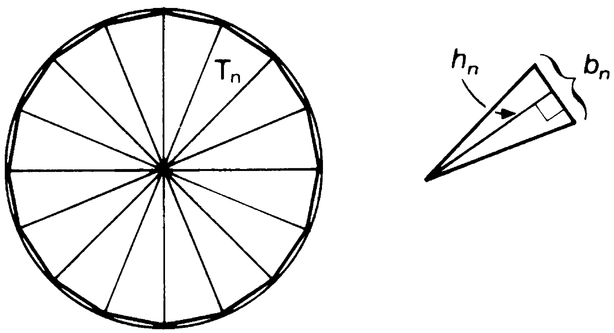


Fig. 2) that form these polygons, which implies that $\frac{b_n}{b'_n} = \frac{h_n}{h'_n} = \frac{L_n}{L'_n}$, hence that $\frac{h_n b_n}{h'_n b'_n} = \frac{A_n}{A'_n} = \frac{r^2}{r'^2}$. And then, because it is plausible to assume that as $n \to \infty$, L_n and A_n tend to C and A respectively (and the same for the other polygon as well), the proportionality between C:r and A:r^2 follows immediately. Moreover, from the equality of the above mentioned n triangles for each n-gon, $A_n = \frac{1}{2} h_n L_n$. So, it is plausible to assume that for $n \to \infty, A = \frac{1}{2} rC$.

5 The Classroom Experiment

5.1 The Experiment Setting

The classroom experiment took place in a public primary school in Thessaloniki, Greece. Twenty-seven 6th graders participated. During their regular classes in mathematics, the students had been taught the formulas for the calculation of the area of certain shapes, mostly limited to triangles, squares, and rectangles. During the 5th grade, they had been taught the formula for calculating the circumference of a circle. They also had experience estimating (or exactly calculating) the area of irregular plane figures: Students dissected unfamiliar shapes into shapes that were familiar in order to calculate the area of the unfamiliar shape. None of the students knew how to measure the area of a circle before the experiment took place.

For the current study, we chose to use area instead of perimeter for two reasons: (i) the students had already been taught the formula for the circumference during the 5th grade, and (ii) the question that motivated this study concerned the claim that it was the area of the circle that was almost 3.14 times the area of the square of its radius and we wished to remain focused on that perspective. Despite this focus on area, the *method* we chose for this study's classroom experiment was based on the method Archimedes used in his third proposition where he investigated perimeter, rather than the method he used in his first proposition where he investigated area. Though Archimedes's first proposition employs area (starting with a square and doubling the number of sides each time), the method of proof—double reduction *ad absurdum* (proof by contradiction) via computation of area—seemed less well suited for adaptation to the primary classroom. The methods of the third proposition—analyzing the perimeter by starting with a regular hexagon and doubling the number of sides each time—were more easily adapted.

The experiment was designed for a computational environment. Four sessions were needed to complete this intervention. The first three took place in the same week and were dedicated to the teaching part of the experiment. The fourth session took place during the next week (4 days later). It was dedicated to collecting evidence on how this experience aided the students in learning the concept of π. The students worked in groups of three, and the Cabri Geometry II Plus software was chosen to support the aim of the experiment. This choice was based on the availability of various tools in that software (measurement tools, calculator, macros, etc.), and on the fact that the students were familiar with the interface and basic functions of Cabri. This familiarity was restricted to the buttons and functions that were considered proper for the specific age and the corresponding mathematical topics in their mathematics courses. For example, they knew how to draw basic shapes (tools: *Triangle, Polygon, Circle, Line, Segment*), to drag (tools: *Pointer, Rotate, Dilate*), to measure lengths (tool: *Distance or Length*), areas (tool: *Area*), angles (tool: *Angle*), to construct perpendicular or parallel lines (tools: *Perpendicular Line, Parallel Line*), and to use the Cabri calculator (tool: *Calculate*…).

Using the computer environment allowed us to overcome a serious obstacle to using the method of exhaustion of Archimedes, which is based on constructing regular inscribed and circumscribed polygons. It is not reasonable to expect young students to be able to construct regular polygons by using straightedge and compass, nor to follow geometrical methods and advanced properties of the shapes to obtain lengths and/or areas. The relevant tools that are available in Cabri allow one to make automatic measurements of perimeter or area, but Cabri does not have tools for the construction of regular polygons. For the purpose of the experiment, the teacher defined macros (intermediate constructions that Cabri enables to be memorized and reproduced) extending thus the program's functionality and permitting students to generate inscribed and circumscribed regular polygons. Each macro was given a recognizable name (e.g., inscribed_12gon.mac, circumscribed_24gon_mac, etc.) to help students select the correct one.

Four days after the third session, another worksheet including four tasks was given to the students in order to gather evidence concerning how much this experiment helped them to grasp the meaning of the mysterious number π.

5.2 How the Experiment was Initiated

The students' mathematical textbook introduced the number π as the area of the circle (A) with radius α, and said that A is less than the area of four squares with side of length equal to α (i.e., $A < 4\alpha^2$, see Fig. 3). Then the textbook refined that statement by describing the area of the circle as almost 3.14 times the area of the square, that is $A = 3.14\alpha^2$. It was then that a student asked the question, which inspired this experiment:

S5: OK, and why is [π] almost 3.14 and not almost something else such as 3.13 or 3.15?

To respond, the teacher then designed an experiment based on the method of Archimedes, with the aim of leading students to compute π themselves.

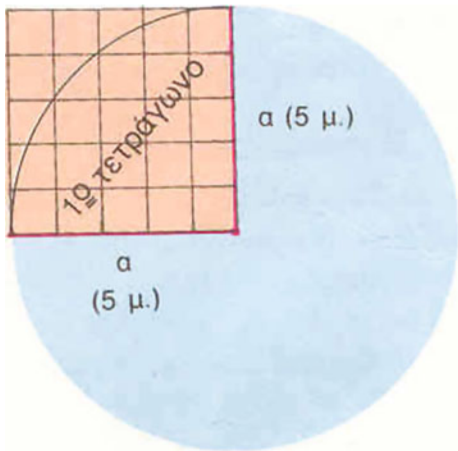


Fig. 3 Circle of radius α, square with side length equal to α (from students' textbook)

5.3 Archimedes in Action

The classroom discussion was recorded, transcribed, and translated. This subsection follows the dialogue between teacher and students.

First question posed to students:

T: What would you need in order to find out how many times bigger the area of the circle in your textbook is than the area of the square?
S5: The numbers for the area of each shape!
S3: Yes, but we cannot calculate both of them.
T: Why is that?
S3: OK, we know the formula for the square; it is 'base times itself'. But how will it be possible to find the area of the circle? If we knew it…
S6: There is not such a formula.

At that point the students were asked to open a Cabri file pre-made by the teacher that included a circle of radius 7, an inscribed equilateral triangle and a circumscribed one (see Fig. 4).

T: Now, do you think it is a good idea to accept the area of the inscribed triangle as almost equal to the area of the circle?
S: (many voices) No!!!
T: Why?
S7: It's too small compared to the circle!
S10: A lot of parts of the circle are left over.
S1: Its sides are far away from the line of the circle.
T: OK. Then, what do you think about the circumscribed one? Is it a good alternative?

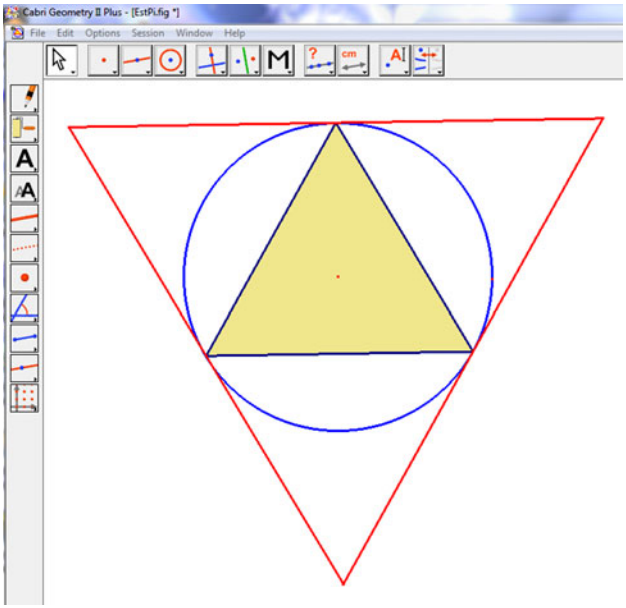


Fig. 4 Inscribed and circumscribed equilateral triangles

Again they refused to accept it.
S7: It is bigger than the circle.
S13: Many parts of the triangle are out of the circle.
This was a critical point.
T: Now, if the area of the first triangle is less than the area of the circle, and the area of the second triangle is bigger than the one of the circle, then, what do you think about the area of the circle?
S6: Obviously, it will be a number less than the one for the big triangle and more than the one of the small.
S16: Somewhere between the two.
S5: Maybe in the middle.

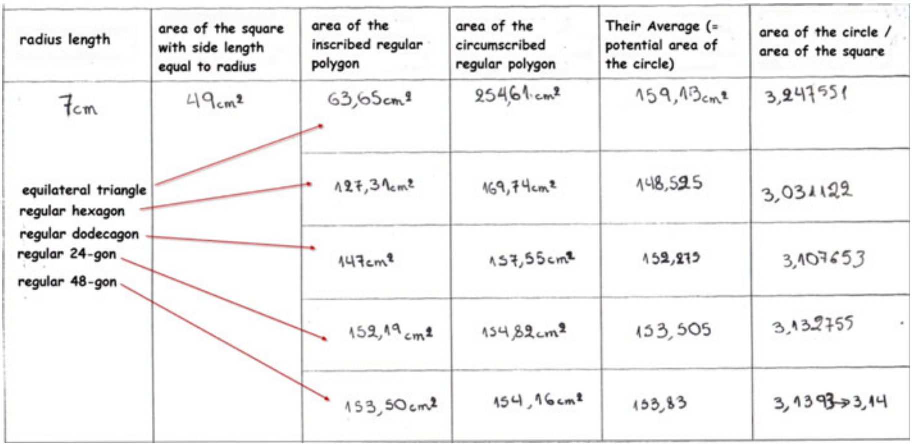
The students had already been taught the concept of the average and they did not have a problem interpreting this "number in the middle" as the average of the areas of the inscribed and circumscribed regular polygons. This average represented the area of the circle. Thus, the last thing they had to do was to compare this average with the area of the square; that is to find the ratio of the approximate area of the circle to the area of the square that had side length equal to the radius. In this way they would get an approximate value for π. Therefore, the students were able to write down their records in a worksheet as can be seen in Fig. 5 below.

It was now time to start progressively increasing the number of the sides of the polygon according to Archimedes's method.

T: It is obvious that you are not satisfied with the triangle I proposed. Is there any suggestion about how to proceed?
S: We need a bigger shape. *(Many students)*
S3: You mean a shape with more sides.

The students agreed that this was a good idea and the teacher asked them to use a pre-constructed macro that allowed them to draw a regular inscribed and circumscribed hexagon in the same circle (see Fig. 6). This new shape co-existed with the previous ones (i.e., the triangles) and thus it was easy for the students to realize how much better an approximation this new pair of hexagons was likely to provide.

For the inscribed regular hexagon their comments were:



radius length	area of the square with side length equal to radius	area of the inscribed regular polygon	area of the circumscribed regular polygon	Their Average (= potential area of the circle)	area of the circle / area of the square
7cm	49cm²	63,65cm²	254,61 cm²	159,13cm²	3,247551
		127,31cm²	169,74cm²	148,525	3,031122
		147cm²	157,55cm²	152,275	3,107653
		152,19cm²	154,82cm²	153,505	3,132755
		153,50cm²	154,16cm²	153,83	3,1393→3,14

Fig. 5 Measurements and computations made by the students

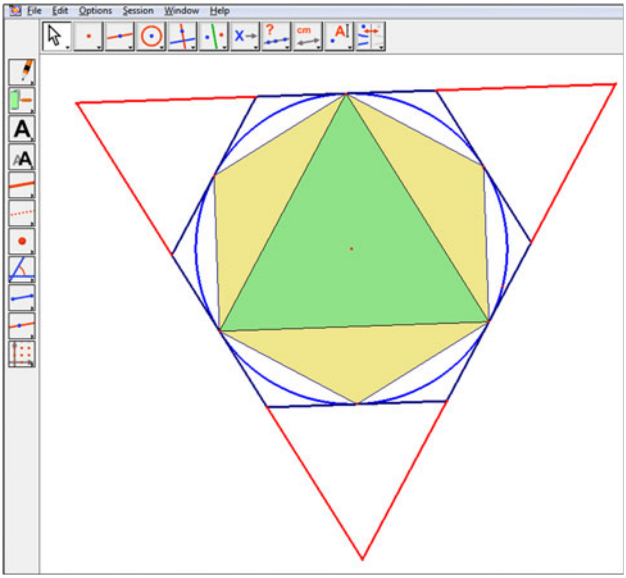


Fig. 6 Inscribed and circumscribed equilateral triangles and regular hexagons

S3: It is much closer to the circle.
S27: It covers a bigger area inside the circle.
For the circumscribed one their comments were:
S10: It's smaller than the triangle.
S18: It covers a smaller region.
S1: It's much closer to the circle.
S8: The outer hexagon covers the inner.

The students accepted the new average as a better approximation of the real area of the circle, but the ratio of the area of the circle to the area of the square was not yet "almost 3.14" (see Fig. 5), and consequently not a satisfying one.

They were ready now to follow the thinking of Archimedes.

S4: We need more sides to the polygon.
S8: It is better to have inside the circle a polygon with bigger number of sides since it will cover a bigger part of the circle.
S5: As we increase the number of sides, the polygon gradually reaches the line of the circle.

So, it was time for the next Cabri-macro to be used, creating regular inscribed and circumscribed dodecagons (see Fig. 7).

The students immediately used the proper tools to obtain automatic measurement for the area of the polygons and to calculate their average (as an approximation of the area of the circle) as well as the ratio (area of the circle)/(area of the square) as their current approximation of π. The value for π was still not a satisfying one and in the next minutes a new macro was used and the students found themselves working with 24-gons and then with 48-gons. This allowed them to obtain better approximations for π (3.132755 and 3.1393 respectively, see Fig. 5). By using Cabri, all students got the same answer for each regular polygon and therefore the numbers in Fig. 5 were the same for the whole class.

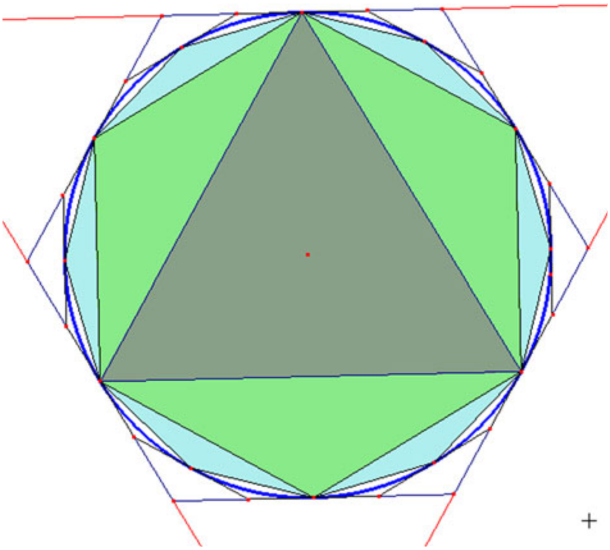


Fig. 7 Inscribed triangle, hexagon and dodecagon

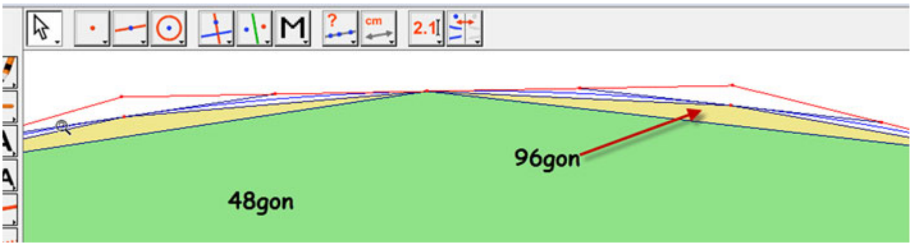


Fig. 8 Zooming into see the 96-gon

They seemed now to be satisfied. They found a value for π (i.e., 3.1393) that was almost equal to 3.14. However, the teacher challenged them to go one step further and use another Cabri macro for constructing an inscribed and a circumscribed 96-gon. Their reaction was that there was no noticeable change in the appearance of the sketch. This was because, for the given circle, it was hard to distinguish the sides of the polygons from the line of the circle. So the teacher suggested that the students use the "zooming in" function to get a closer view (see Fig. 8).

This helped them see the slight differences between the 48-gon and the 96-gon. Using the relevant measurements, they immediately assigned to π the value 3.1414.

Given that students felt that this was the end of the process, they were surprised when the teacher made the following comment:

T: I am glad you are now satisfied and convinced that the area of the circle is almost 3.14 times the area of the square with side length equal to the radius of the circle. But don't you think it's a good idea to continue with polygons that have more and more sides until they cover the circle completely?

S4: I think that some parts of the circle will always be leftover.

S5: I don't think that the circle will ever be completely covered. The polygon is comprised of line segments and angles whereas the line of the circle is a curve.
However, the most impressive comment at this point was made by another student.
S6: As we progress, the area of the inscribed and the circumscribed polygons tend to be very close.

Though, naturally, in a somewhat naïve way, the student actually reached the notion of the area of the circle as the limit of the area of the polygons when the number of their sides goes to infinity.

After being convinced that the ratio of the area of the circle (A) to the area of the square (r^2) is almost equal to π it was easy to conclude that $A = \pi r^2$.

They were excited when they were informed that Archimedes shared their interest in the number π, and that he followed more or less the same path they did in order to compute that number.

Obviously, from the mathematical point of view, because these students could not use geometric similarity arguments to generalize, this conclusion of the students is valid only for the specific circle. On the other hand, it is equally obvious that such young students could not be expected to work in that formalistic direction, i.e., to show that the value they computed for π is always the same no matter the size of the circle (and subsequently the size of the polygon). Even though these students could not finish the proof, it is important that they learned how this proof begins, and something about how proof works. The dynamic geometry environment is suitable for obtaining many instances of the same idea by dragging. Thus, in a following session the students had the chance to experiment with trying to figure out how the size of the circle affects the value of π. They were amazed by realizing the "universality" of their previous computing of the value of π.

The students were convinced. This is a critical point since it is possible to confuse being convinced as being identical to proof. Indeed, as Harel and Sowder (2007) explain, the definition of "proof scheme" requires, among other things, the notion that an assertion can be conceived of either as a conjecture or as a fact. It ceases to be a conjecture and becomes a fact once the person becomes certain of its truth. The teacher is responsible at this point to make clear this distinction. The students were invited to be involved in an age-appropriate discussion aimed at making clear that in mathematics it is not enough to be convinced that one's findings seem to work, but it is necessary to be able to explain "why" these findings always work (i.e., to prove it). As Mason et al. (1982) very suitably describe:

> …Often it is easy to conjecture WHAT, but not so easy to see WHY. To answer why satisfactorily means to provide a justification for all the statements, which will convince the most critical reader. To achieve this usually requires a strong sense of some underlying structure, the link between KNOW and WANT… (p. 115).

Thus, the teacher told the students that if they remain equally interested after some years to continue their research on π they will have the chance to complete the whole story of π.

6 After the Teaching Sessions

Four days after completing the teaching experiment, a worksheet was distributed to the students that included four items. The aim was to examine whether this teaching intervention influenced students' conceptual understanding in relation to the area of the circle and the number π. The students were given 20 min so as to respond to these four items.

Item 1 examined the choice of the regular polygon that would offer a better approximation of the area of the circle (Fig. 9).

Twenty-five out of twenty-seven students answered correctly that it was the dodecagon that had to be selected. They gave a lot of explanations to justify their decision.

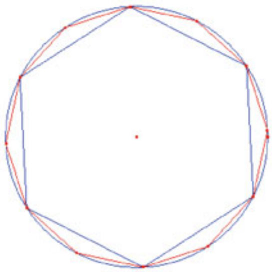
(i) It is much closer to the line of the circle. *(S1, S4, S6, S7, S8, S10, S14, S15, S16, S18, S22, S23, S24, S26)*
(ii) It covers a bigger region in the interior of the circle.*(S27)*
(iii) It covers almost the whole circle. *(S9)*
(iv) It fits better. *(S20, S21)*
(v) It has more angles and thus it is able to cover bigger part of the circle. *(S3, S19)*
(vi) The area between the circle and the dodecagon that is left over is now less than the area between the circle and the hexagon. *(S2)*
(vii) It obtains a better approximation of the real area of the circle. *(S5, S11, S12, S17)*

It seems that the case of the inscribed polygons was well understood since (as already mentioned) there were 25 correct answers distributed among these various explanations.

The next item (Fig. 10) concerned the case of circumscribed polygons and it appeared that these shapes were more difficult for the students.

The difference between this item and the previous one was that there was no picture to help them to see the two shapes at the same time. The statement of the task did not include any visual representation of the abstract objects (i.e., the polygons). The aim was to evaluate the students' answers on the basis of their understanding of the situation and not on the basis of the support given by the visual impression of the task. Besides, the lack of the visual information would let the students express their confusion with connecting the number of the sides of the polygon to its area (inscribed vs circumscribed polygons). Finally, keep in mind that the worksheet was administered to the students 4 days after the last session of the teaching experiment. So, these objects could only be visualized in the minds of the students. Being capable of somehow "seeing" these objects is an essential mathematical ability and it actually constitutes what Sfard (1991) calls *structural conception of a notion*. Thus, the students had to base their decision upon their understanding of the situation. Only eight students (out of twenty-seven) answered correctly that the area of the dodecagon will be less than 120 cm^2. The explanation given by the students was:

> Since we are out of the circle, the more we approach the line of the circle the less becomes the area of the polygon. (S4, S5, S6, S7, S9, S15, S16, S20).



Item 1

Two regular polygons, a hexagon and a dodecagon, are inscribed inside the circle. Which one could be considered most proper for approximating the actual area of the circle? Why?

Fig. 9 Worksheet: Item 1

> *Item 2*
>
> *The area of a circumscribed hexagon around a circle is 120 cm². There is also a circumscribed dodecagon around the same circle. Do you think the area of the dodecagon will be less or more than 120 cm²?*

Fig. 10 Worksheet: Item 2

It can be said that the presence of structural conception allowed them to know both "what" and "why," in other words to have *both rules and reasons* (Sfard 1991).

Eleven students claimed that the area of the dodecagon will be more than 120 cm^2 and their argument was:

> The dodecagon has more sides than the hexagon. So, its area will be bigger than the area of the hexagon. (S1, S2, S3, S10, S11, S12, S17, S22, S23, S24, S25).

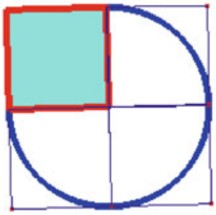
The rest of the students either answered that the area will remain the same or avoided answering that item.

At least three possible explanations for the mistaken answers can be given: (i) students did not have a visual representation describing the situation, (ii) there was a confusion connecting the number of the sides of the polygon to its area, and (iii) they were picturing the wrong construction (inscribed instead of circumscribed polygons). Although this is a valid property for the inscribed polygons, exactly the opposite happens with the circumscribed ones: The greater the number of the sides of a circumscribed polygon the smaller its area.

It is interesting to present the answers of the two students (S13, S25) who failed to answer Item 1 correctly. The first one did solve Item 1 but then worried that his answer was wrong, and crossed it out. Moreover, he avoided answering Item 2. The second student did not answer either item.

Items 3 and 4 were almost identical and were designed to reveal whether the understanding of the formula for the area of the circle was independent of the context, and whether the students were able to make the connection between specific situations and the relevant formula that describes the situation. For Item 3 the context is purely mathematical, while Item 4 hides its mathematical background behind a somewhat naïve description (Fig. 11). In Item 4 it is of course possible to consider the number of marbles that completely cover the square as expressing the area of the square. To the students this likely did not seem to have a direct connection with the understanding of the notion of π. Because of this, it stands as an indication of whether the acquired knowledge (i.e., the knowledge of π as it is certified from Item 3) can be transferred to another context. The teacher-researcher was interested to discover whether the students were able to recognize the applicability of their learning in this novel context that does not seem to be a purely geometrical one.

For Item 3 there were fifteen (out of 27) correct answers (S1, S2, S3, S4, S5, S6, S7, S8, S9, S10, S11, S12, S13, S14, and S15). These students multiplied the area of the square by 3.14 to find the area of the circle exploiting their previous knowledge that the area of the circle is almost 3.14 times bigger than the area of the square that has side length equal to the radius of the circle. However, not all of them were able to recognize that the same knowledge could be transferred and applied to the environment of the marbles. Thus, only eight of the fifteen students (S1, S2, S3, S4, S5, S6, S7, and S8) correctly answered that the



Item 3

The area of the square is 16 cm^2. What is the area of the circle?

Item 4

I need 300 marbles to completely cover the square. How many marbles do I need to cover the circle?

Fig. 11 Worksheet: Items 3 and 4

total number of marbles needed to cover the circle will (approximately) be $3.14 \times 300 = 942$ marbles. The remaining students either multiplied 300×4 (S10, S11, S12, S13, S15, S18, S22, S23) or did not answer that item. A potential explanation for the product 300×4 might be that it was an influence from the existence of the four squares which means that the students did not notice that the total number of marbles they calculated covered the big square (i.e., four small ones) rather than the circle.

7 Some Final Remarks

In this paper an empirical study is presented aiming to demonstrate how the students can use techniques found in the history of mathematics to learn about a certain mathematical idea. This study is an existence proof (Schoenfeld 2007). It demonstrates that it is possible to realize an intervention based on history that can bring students to (i) understand the number π better (tool), and (ii) appreciate how mathematics comes into being by the hands and minds of human beings (goal).

Even though all students shared the experience of this journey through the thinking of Archimedes, not all of them were successful in their answers concerning the four items. This was expected since a lot of parameters could influence the performance of the students: tiredness, inadequate mastery of skills relevant to mathematics, lack of available time for completing a task or for thinking about and incorporating the learning, lack of motivation, state of health and so on, just to mention some of them. But, the important thing was that the students had the chance to get involved in a process that demonstrated how a mathematical concept is born and how it evolved in history. Students had a chance to learn the "back-story" of a mathematical concept, not just the fact itself. In other words, students had the chance to experience the development of mathematical concepts as a human activity.

The students were excited by being informed that they shared much the same line of thought with Archimedes. But, even more important was the fact that they were convinced about the validity of the formula because of their own investigation rather than of the teacher's authority. Importantly, the vehicle to obtain that was the history of mathematics.

The positive feedback received from similar efforts highlights both the issue of the importance of the history of mathematics in primary school teachers' education, and the role that history might be given in mathematics curricula.

Obviously, this study took place under specific circumstances with a small number of students, so its results are not yet general, but there is potential for the findings to be generalized. The design of the study tried to show a high level of trustworthiness by responding to certain criteria such as the explanation of how and why π functions in the way it does, the precision in describing the various notions, and making it possible to replicate some key aspects of the study elsewhere (Schoenfeld 2007).Because of this, the study offers the possibility of creating a classroom mathematical community in which students engage meaningfully in some of the activities that mathematicians used in the past and still continue to engage in today.

References

Burton, D. M. (2011). *The history of mathematics. An introduction* (7th ed.). New york: McGraw-Hill.

Castellanos, D. (1988). The ubiquitous Pi. *Mathematics Magazine, 61*(2), 67–98.

Clark, K. M. (2012). History of Mathematics: illuminating understanding of school mathematics concepts for prospective mathematics teachers. *Educational Studies in Mathematics, 81*(1), 67–84.

Costabile, F., & Serpe, A. (2010). Archimedes in secondary schools: A teaching proposal for the math curriculum. In S. A. Paipetis & M. Ceccarelli (Eds.), *The genius of Archimedes: 23 centuries of influence on mathematics, science and engineering* (Vol. 11, pp. 479–491)., History of Mechanism and Machine Science Netherlands: Springer.

Furinghetti, F. (2002). On the role of the history of mathematics in mathematics education. In *Paper presented at 2nd international conference on the teaching of mathematics at the undergraduate level.* Crete, Greece.

Galton, D. (2009). The magic of Pi (π): Coda. *The Quarterly Journal of Medicine, 102*(6), 439–440.

Gulikers, I., & Blom, K. (2001). 'A historical angle'. A survey of recent literature on the use and value of history in geometrical education. *Educational Studies in Mathematics, 47*(2), 223–258.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspective on the learning and teaching of proof. In F. K. Lester Jr (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805–842). Charlotte: National Council of Teachers of Mathematics.

Harris, H. (1959). The history and calculation of Pi. *The Emporia State Research Studies, 8*(1), 5–39.

Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235–261.

Jankvist, U. T. (2011). Essay Review. A century of mathematics education: ICMI's first hundred years. *Historia Mathematics, 38*(2), 292–302.

Kahn, K., Sendova, E., Sacristan, A. I., & Noss, R. (2011). Young students exploring cardinality by constructing infinite processes. *Technology, Knowledge and Learning, 16*(1), 3–34.

Kjeldsen, T. H., & Blomhøj, M. (2012). Beyond motivation: History as a method for learning meta-discursive rules in mathematics. *Educational Studies in Mathematics, 80*(3), 327–349.

Lang, S. (1985). *Math!: Encounters with high school students*. New York: Springer.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. New York: Addison Wesley.

Mason, R., & Janzen Roth, E. (2005). Thinking like Archimedes: An instructional design experiment. In *Proceedings of the eight international congress of the history, plilosophy, sociology and science teaching,* Leeds, UK, www.ihpst2005.leeds.ac.uk/papers.htm. Accessed 10 October 2012.

Michalowicz, K. D. (2000). History in support of diverse educational requirements: Opportunities for change. In J. Fauvel & J. van Mannen (Eds.), *History in mathematics education: The ICMI study* (pp. 172–200). Dordrecht: Kluwer.

Schoenfeld, A. H. (2007). Method. In F. K. Lester Jr (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 69–107). Charlotte: Information Age Publishing.

Sfard, A. (1991). On the dual nature of mathematical conceptions. *Educational Studies in Mathematics, 22*(1), 1–36.

Siu, M. K., & Tzanakis, C. (2004). History of mathematics in classroom: Appetizer? Main course? or dessert? *Mediterranean Journal for Research in Mathematics Education, 3*(1-2), v–x.

Smestad, B. (2012). Not just "telling stories". History of mathematics for teacher students: What is it and how to teach it. *Paper presented in ICME-12, Seoul, Korea*, Retrieved from, www.icme12.org/upload/UpFile2/TSG/0484.pdf. Accessed 20 May 2013.