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Bridging History of the Concept of Function with Learning of Mathematics: Students' Meta-Discursive Rules, Concept Formation and Historical Awareness

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Abstract In this paper we present a matrix-organised implementation of an experimental course in the history of the concept of a function. The course was implemented in a Danish high school. One of the aims was to bridge history of mathematics with the teaching and learning of mathematics. The course was designed using the theoretical frameworks of a multiple perspective approach to history, Sfard's theory of thinking as communicating, and theories from mathematics education about concept image, concept definition and concept formation. It will be explained how history and extracts of original sources by Euler from 1748 and Dirichlet from 1837 were used to (1) reveal students' meta-discursive rules in mathematics and make them objects of students' reflections, (2) support students' learning of the concept of a function, and (3) develop students' historical awareness. The results show that it is possible to diagnose (some) of students' meta-discursive rules, that some of the students acted according to meta-discursive rules that coincide with Euler's from the 1700s, and that reading a part of a text by Dirichlet from 1837 created obstacles for the students that can be referenced to differences in meta-discursive rules. The experiment revealed that many of the students have a concept image that was in accordance with Euler's rather than with our modern concept definition and that they have process oriented thinking about functions. The students' historical awareness was developed through the course with respect to actors' influence on the formation of mathematical concepts and the notions of internal and external driving forces in the historical development of mathematics.

1 Introduction

The aim of this paper is to bridge history of mathematics with theories from mathematics education research to argue, and illustrate through a teaching experiment that history of

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mathematics can function at the core of what it means to learn mathematics in a way that develop students' historical awareness. The paper addresses two audiences: researchers in the mathematics education research community and mathematicians and historians of mathematics who teach mathematics. For the former group, the theoretical foundation might be of primary interest and the experimental teaching course of secondary interest serving as an "existence proof" for the theoretical claims brought forward. The latter group might be more interested in the experimental teaching course, and satisfied with shorter descriptions supplemented with references of the theoretical ideas and frameworks from mathematics education research literature. We had a choice to make and we chose to structure the paper in accordance with the research literature since our experimental course was designed and implemented to test the theoretical argument. This means that we begin by outlining and explaining the theoretical framework for our analysis and our theoretical argument for how history of mathematics can function at the core of what it means to learn mathematics. Readers who are mostly interested in the experimental teaching course and its implementation in the classroom may wish to jump directly to the section "An Experimental Course in the History of the Concept of a Function: Design" and then refer back to the first part of the paper for clarification and explanation of the theoretical constructs.

In (Kjeldsen 2011a) it was argued that if we adopt a multiple perspective approach to history from practices of mathematics and a competence-based understanding of mathematics education, it is possible to design teaching and learning situations where history benefits both students' learning of mathematics and provides students with historical awareness in a scholarly sense (see also Jensen 2010; Jankvist and Kjeldsen 2011; Kjeldsen 2012). Kjeldsen and Blomhøj (2012) argued further that learning the history of mathematics may profoundly affect the development of students' meta-discursive rules of mathematics. They based their argument on a multiple perspective approach to history of mathematics studied from its practices and on Sfard's (2008) theory of thinking as communicating.

This article discusses an experimental course in the history of the concept of a function implemented in a Danish high school mathematics classroom as presented in Petersen (2011). The course was designed using the above-mentioned theoretical frameworks, implemented with a matrix-organized structure as described in Kjeldsen (2011b). The theoretical frameworks were used in the design of the experimental course to examine whether and if so in what sense history of mathematics activities can be used to (1) reveal students' meta-discursive rules in mathematics and make them objects of students' reflection, (2) support students' learning of mathematical concepts: here specifically the concept of function, while (3) at the same time, enhance students' historical awareness.[1] Hence, the course was designed to test the theoretical argument.

The first section of this paper outlines the theoretical frameworks, the second presents the design of the course and data collection procedures, the third is an analysis of data with respect to (1), (2), and (3) above, and the fourth summarizes and discusses the results.

2 Historiographical Approach: Multiple Perspectives and Practices of Mathematics

Historians and users of history have different perspectives on history depending on their aims. Our aim for integrating history into mathematics education is to develop historical

[1] Parts of results presented here were first presented at a talk held at the ICME conference in Seoul, Korea, July 2012.

awareness in a way that benefits the learning of mathematics. This requires that we, on the one hand, adopt a genuine approach to history and on the other hand, address historiographical issues that are connected to relevant mathematics, dealt with in a way that benefits the students' learning of mathematics. To fulfill this we have implemented a multiple perspective approach to history of mathematics from its practice.

The term *multiple perspective approach to history* is borrowed from the Danish historian Eric Bernard Jensen (2003). It is an action-oriented approach where people are thought of as both being shaped by (a product of) history and as shapers (generators) of history. Investigations of people's projects and actions are taken into account to achieve a historical-social understanding of how people have thought and acted in different cultures and at different times in order to understand and explain historical processes. History is studied from the perspectives of the historical actors, their intentions and motivations, as well as intended and unintended consequences of their actions. Such an approach can be adapted to history of mathematics by studying mathematics as a product of human intellectual activities under specific social and historical circumstances. Studying history of mathematics from practices of mathematics brings history close to mathematical activities, to processes of knowledge production in mathematics, and hence to mathematics education.

A multiple perspective approach to history of mathematics from practices of mathematics is in accordance with recent trends in the academic discipline. It can be adapted to history of mathematics teaching modules or projects in mathematics education on a small scale by having students study concrete episodes of (past) mathematicians' activities in these mathematicians' "workshops", following the development of their ideas and techniques within their practice of mathematics and from their perspectives, and paying attention to their intentions and motivations.

3 Meta-discursive Rules in Mathematics and the Role of History

In Sfard's (2008) *Thinking as Communicating*, learning is viewed as "becoming a participant in a certain discourse", where discourse is defined as "the totality of communicative activities, as practiced by a given community" (Sfard 2000, p. 160). This interpretation of learning emphasises the social nature of mathematical intellectual activities, which is in accordance with the multiple perspective approach to history of mathematics studied from its practices that was introduced above. Hence, there is no conflict between our approach to history and a discursive view of mathematics.

To communicate is an activity that is regulated by rules, and discursive patterns are the results of such rule-governed processes. Sfard distinguished between rules at the object level and rules at the meta level. The object-level rules have the content of the discourse as object. They regard the properties of mathematical objects in the form of narratives on these objects, for example, "the line segment between two points in a convex set lies in the set itself". Meta-discursive rules have the discourse itself as object. They are rules, not about mathematical objects, but about how the discussants behave and how they act in trying to produce object-level narratives (Sfard 2008, p. 201). These rules are often tacit. They are implicitly present in discursive actions when we, for example, judge if a solution or proof of a mathematical problem or statement can count as a proper solution or proof (Sfard 2000, p. 167). They govern "when to do what and *how* to do it" (Sfard 2008, p. 201, emphasis in the original).

The meta-discursive rules are connected to the object-level of the discourse. They have an impact on how participants in the discourse interpret its content. As a consequence, developing proper meta-discursive rules are indispensable for the learning of mathematics. This means that an essential aspect of mathematics education is to create teaching and learning situations where meta-discursive rules are revealed, and here—as argued by Kjeldsen and Blomhøj (2012)—history of mathematics seems to present itself as an obvious strategy. Meta-level rules are contingent; they change over time. Hence, meta-discursive rules can be treated at the *object level* of history discourse and thereby be made into explicit objects of reflection for students.

In the experimental course we focused on two meta-rules that were part of Euler's mathematical discourse of the eighteenth century but which were not part of Dirichlet's mathematical discourse from the nineteenth century. The work with the historical sources not only made the students aware that there are meta-level rules in mathematics and that these rules change over time, it also revealed that some of the students were governed by meta-rules that do not coincide with the rules held by the mathematical community of today.

4 Learning of Mathematical Concepts

To analyze whether the historical activities benefited students' learning of mathematical concepts, we used the notions of concept image and concept definition as defined by David Tall and Shlomo Vinner (1981), and Sfard's (1991) model for concept formation.

Tall and Vinner define the term *concept image* as the total cognitive structure an individual associates with a concept (Tall and Vinner 1981, p. 152). A person's concept image may contain mental pictures, properties, and processes that are tied to the concept. It is a person's conception of the concept, and is subjective, changing and developing over time as the person meets new challenges and examples. A concept image can be a quite comprehensive structure, and when a person is brought in a situation where he or she needs to activate his or her understanding of a particular mathematical concept, often only a part of the person's concept image is brought into play. The part of the concept image that is activated in a given situation is called the *invoked concept image*.

Tall and Vinner define a *concept definition* to be a written description that explains the concept. A concept definition is the definition that is generally accepted in the mathematical community. A *personal concept definition* is a person's definition of a mathematical concept. It is the form of words a person uses for his or her own explanation (Tall and Vinner 1981, p. 152). A concept definition generates an image of the concept which is part of a person's concept image. Depending on whether a person has learnt the concept definition by heart or has created his or her own concept definition, the image generated by the concept definition might be nearly non-existent or more or less coherently related to parts of the person's concept image (Tall and Vinner 1981, p. 152).

A concept image might include parts that are in conflict with each other and/or the concept definition. If conflicting parts of a concept image are invoked simultaneously they are called *cognitive conflict factors*; if they are not invoked together Tall and Vinner called them *potential conflict factors*. Research has shown that students often refer to (parts of) their concept image instead of formal concept definitions when they deal with a mathematical concept. In cases where students had a limited or a wrong concept image, drawing on their concept image instead of the formal definition created learning difficulties. In order to remedy wrong concept images or extend limited ones teachers need to be aware of them in students. Students' concept images are formed during teaching and learning processes, and sometimes wrong or limited concept images arise as an unintended consequence of teaching. As we will see below in the analysis of student dialogues from the experimental course, many of the students seemed to have a concept image of a function as an expression given by a formula, which apparently can be linked to the fact that these were the only kind of functions they had worked with.

Sfard (1991) operated with a dual nature of mathematical conception. She distinguished between an operational and a structural understanding of a mathematical concept. To have a structural understanding of a concept means to conceive the concept as an abstract object in itself. If a student has a structural understanding of a concept, he or she is able to handle and think about the concept as a whole. To have an operational understanding of a concept means to conceive of it as a process or as a product of a process. An operational understanding of the concept of a function, for example, means to conceive of it not as an abstract object in itself that can be manipulated, but as an action—a process—that leads from one element to another. Operational and structural understandings of a concept are two sides of the same coin.

In her model for an individual's formation of a concept, Sfard (1991), p. 19 distinguished between three phases: interiorization, condensation and reification. The interiorization phase refers to the phase when a student is learning to perform the processes which later will be transformed into an independent abstract object. These processes are performed on objects the student already knows, that is, objects of which the student has a (beginning) structural understanding. The condensation phase is reached when the student is able to talk about and deal with processes behind the object without having to perform the actual operations, when the student is capable of switching between different forms of representations for example. The reification phase occurs when the student conceives the concept as an independent abstract object that he or she can talk about and perform operations on. A structural understanding of a concept does not occur until the reification phase. Sfard defined reification as an ontological shift.

5 An Experimental Course in the History of the Concept of Function: Design

The experimental course "The concept of a function viewed through historical and contemporary glasses" was designed as part of a study for a master's thesis in didactics of mathematics (Petersen 2011). One of the purposes of the course was to test the theoretical argument that history can function at the core of what it means to learn mathematics by revealing students' meta-discursive rules. The main purpose was not to teach the students the concept of a function.[2] The teacher had covered the curriculum on functions before the experimental course was implemented. History of mathematics is included in the Danish high school curriculum and the history of the concept of a function was chosen because the students were supposed to be familiar with the concept. The course focused on the development of Leonhard Euler's concept of a function in connection with the debate of the vibrating string in the eighteenth century and changes in the organization of mathematics education in connection with the French revolution and Dirichlet's concept of a function from the nineteenth century. The course was implemented in a Danish high school classroom of seventeen 11th graders (17–18 year-old students). The students worked with

[2] Much work has been done concerning student understanding of the concept of a function. We refer to Breidenbach et al. (1992), Dubinsky and Harel (1992).

history of mathematics literature and extracts of primary mathematical sources. They discussed the development of the concept of a function within the mathematical and educational culture of the time and compared it with our contemporary conception of a function.

As mentioned in the introduction, we will focus on the development of students' historical awareness and instances where meta-discursive rules were revealed and reflected upon by students. Two historical meta-rules, referred to as the *general validity of analysis* and the *generality of the variable* were chosen. The general validity of analysis refers to the norm that the results, rules, techniques and statements of analysis should be generally valid. The generality of the variable refers to the norm that a variable in a function could take on all values and could not be restricted to an interval, for example. These two rules were generally accepted in the mathematical discourse of Euler and his time, but were not part of Dirichlet's meta-discursive rules. By focusing on these rules in Euler's discourse and their absence in Dirichlet's, the intention was that the students become aware that there are meta-rules in mathematics and that these are historically given. They should reflect upon the role of proofs and the domain of a function in contemporary mathematics. Here we focus on the generality of the variable.

The experimental course consisted of 13 lessons of 50 min each and was organized in two steps according to the matrix structure described in Kjeldsen (2011b). In the first step, the students were divided into four basis groups. Each basis group worked on a particular aspect of the theme guided by a worksheet that was designed according to different learning goals. The students worked in the basis groups for five lessons on the following:

Basis group 1: Historical definitions of a function
Basis group 2: The debate of the vibrating string
Basis group 3: Euler, Dirichlet, and the society in which they lived
Basis group 4: The modern concept of a function

Each basis group wrote a report completing the tasks formulated in their respective worksheet. Group 1, 2, and 3 worked with the history of the concept of a function from different perspectives.

In the second step the groups were re-organized and four new "expert" groups were formed. Each expert group consisted of at least one member from each of the basis group.

In this way every expert group in principle contained all the knowledge produced in the basis groups. In each expert group, the expert on theme 1, 2, 3, 4, i.e., the student who came from basis group 1, 2, 3, 4, respectively, was responsible for sharing his or her knowledge with the other members of his or her expert group with the help of the report that had been produced in his or her basis group. Every expert group was given the same task, namely to write a paper in which they, based on the knowledge gained in the basis groups, could contribute to a fictitious debate among two groups of mathematicians in a scholarly journal about how a mathematical concept emerged and what were the driving forces behind the development of the concept. Four lessons were spent in class on the expert groups' work.

Two lessons were spent on in-class discussions with all students. The first took place between step 1 and step 2. It was used to introduce the task for the expert groups, i.e., the idea of and intentions with the writing of the journal paper. It was discussed how students from the different basis groups could contribute to the task of the expert groups. After three lessons of work in the expert group, one lesson was used to discuss how the work in the expert groups progressed. To this class discussion, every expert group was asked to prepare a short presentation of their paper emphasizing one or two issues they wanted to discuss.

A set of materials was designed for the course. It consisted of the worksheets for each basis group and for the expert groups; extracts of sources from works of Euler and Dirichlet translated into Danish; a list of different kinds of resources in history of mathematics, and some history of mathematics books (e.g., Katz's (2009) *A History of Mathematics*), and articles, such as Lützen (1978), that were at the students' disposal throughout the course. Marks were placed in the material to help guide the students' use of the materials. To give an idea of how the students were guided in their work with the sources and the socio-historical context, we give a short description of the worksheets for basis group 1 and basis group 3.

5.1 Basis group 1

Based on extracts from Euler's book *Introductio in Analysin Infinitorum* (1748) and Dirichlet's paper *Über die Darstellung ganz wilkürlicher Functionen durch Sinus- und Cosinusreihen* (1837) and on a short explanation of Euler's extension from 1748 of his original concept of a function, the students were asked to complete the following task:

> Explain what a function is according to Euler's original definition of a function in *Introductio in Analysin Infinitorum*, Euler's extended definition of a function and Dirichlet's definition of a function. Describe how these three definitions of a function are different from each other and in what ways they are similar. Explain what the principle of the generality of the variable is all about and the relationship between this principle and the principle of the generality of the validity of analysis.

The formulation of the task is followed by eight questions (Petersen 2011, Appendix B):

1. What are the central concepts in Euler's definition of a concept?
2. Which principle characterizes a variable according to Euler, and what is this principle called?
3. What is the principle of the generality of the variable all about?
4. What are the similarities between the principle of the generality of the variable and the principle of the generality of the validity of analysis? Consider why both principles have been given names that contain the word "general".
5. How does Euler's extended concept of a function differ from his original concept, and what are the similarities?
6. Find three ways in which Dirichlet's concept of a function differ from Euler's definition.
7. Explain from text 3, what Dirichlet must have thought about the generality of the variable.
8. On page 10 there are four pictures. Which of these pictures are graphs of functions according to Euler's definition in *Introductio in Analysin Infinitorum*, Euler's extended definition and Dirichlet's definition respectively?

5.2 Basis group 3

The task was to explain who Euler and Dirichlet were and to describe their respective mathematical communities. The students had to write a report where at least the following eight questions were dealt with (Petersen 2011, Appendix B):

1. Provide a short biography of Euler and Dirichlet. Who were they, when did they live?
2. In 1789 something happened that changed in particular the French but also other European societies, what was that?

3. Where did research and teaching in mathematics take place before 1789/1793? Was it at the same place?
4. Who financed the academies before 1789 and why?
5. What happened with the academies in 1793?
6. Where did research and teaching in mathematics take place after 1793?
7. How were new mathematical results disseminated before and after 1789/1793? What was the significance of exchange of letters between mathematicians and of official mathematical publications?
8. What was the significance for the practice of mathematics of the changing relationship between teaching and research in mathematics and the change in ways of disseminating new mathematical results after 1793?

5.3 The Task of the Expert Groups

In contrast to the basis groups, the expert groups all worked on the same assignment. The students were told that there is a heated debate among two groups of mathematicians. One group claims that mathematical concepts are static, timeless entities that are uninfluenced by humans and conditions of society. The other group thinks that mathematical concepts are the result of a process of development, during which the ways in which people talk about the concept and the central ideas of the concept have changed. The students receive a fictitious invitation by the journal *Nordisk Matematisk Tidskrift* to contribute to this debate by writing a paper in which they express their opinion. They are requested to argue for their opinions based on the collected work that has been done in the basis groups. As was the case in the basis groups, the students are helped by the formulation of four issues in the development of the concept of a function they had to address and discuss: Euler's, Dirichlet's and our current concept of a function; the two meta-rules, domain, range of image and proofs; sociological factors; and human factors. They were also helped by a guide for authors specifying the formalities, e.g., that the paper must be about six pages long, should have an introduction, a conclusion, and references to literature.

5.4 Data

There were several data sources: (1) The students' written essays from the work in the basis groups and the papers written in the expert groups, (2) a questionnaire answered by the students at the end of the course, (3) video recordings from the lessons in the classroom and from the students' independent work in the groups.

6 Analysis: Meta-Discursive Rules

In the following we discuss some findings related to the issue of meta-discursive rules. This aspect was addressed in different ways in the design of the course. The students in basis group 1 worked with different, partly inconsistent concepts of a function, and the two meta-rules mentioned above. The students of basis group 2 worked with the discussion of the vibrating string that was a driving factor in the development of the concept of a function. The students in basis group 4 worked with modern textbooks' presentation of the concept of a function. In the expert groups, basis group 1's work on the historical texts was situated in the discussion of the vibrating string and contrasted to modern textbooks' conception of a function. The students were guided by worksheets that were explicitly designed to lead them into discussions of meta-discursive rules of the past and to compare them with how these issues are conceived of today. Working with small extracts of original sources from writings of Euler and Dirichlet and from modern textbooks was an essential aspect of the design. The extracts were thought to play the role as interlocutors governed by different meta-rules. The idea was that the students should be confronted with different conceptions of a function (the historical ones, the modern one from different textbooks, and their own) that were in accordance with different meta-discursive rules.

The students were presented with Euler's definition of a variable and a function (Petersen 2011, Appendix B):

> A variable quantity is an indeterminate or universal quantity which comprises in itself all determinate values without exception. A variable quantity thus covers all possible numbers.

And

> A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities.

Hence, Euler defined a function as a formula—as given by *one* analytic expression. In the data, we have no evidence that the students entered into a discussion of how Euler interpreted this, for example, what did he mean by "in any matter"? The students had access to history of mathematics literature in which this was discussed. However, whether they were aware of such a discussion does not present itself in our data. Euler's functions were defined everywhere, they were not limited to a specific domain, which was in accordance with the principle of the generality of the variable, a meta-discursive rule that was generally acknowledged by Euler and his contemporaries.

The students were introduced to various texts from historians of mathematics explaining how Euler extended his concept of a function due to the discussion of the vibrating string. The French mathematician D'Alembert, in 1747, found a solution for the wave equation that governs the position of a vibrating string. Euler held the opinion that mathematics should be able to account for all situations in physics, and since he thought that the motion of a plucked string was excluded from D'Alembert's solution (Lützen 1983), he extended his concept of a function accordingly. He believed that in order to describe the plucked string, it was necessary to extend his concept of a function to functions that are piecewise linear. He called functions that were given by different analytical expressions in different intervals discontinuous functions. He called functions given by one analytic expression continuous functions.

Dirichlet and his contemporaries were not governed by the meta-rule of the generality of the variable, which can be seen from his definition of (continuous) functions from 1837 that was given to the students:

> Let us suppose that a and b are two definite values and x is a variable quantity which is to gradually assume all values located between a and b. Now, if to each x there corresponds a unique, finite y in such a way that, as x continuously passes through the interval from a to b, $y = f(x)$ varies likewise gradually, then y is called a continuous … function of x for this interval. It is, moreover, not at all necessary, that y depends on x according to the same law in this whole interval; indeed, it is not necessary to think of only relations that can be expressed by mathematical operations.

It is evident from the following dialogue that working with the historical sources elucidated meta-discursive rules in mathematics and made them objects of students' reflections. The dialogue took place in expert group 1. The group summoned the teacher while they were working. They were discussing, with reference to the debate of the vibrating string, how the rule of the generality of the variable could go together with Euler's discontinuous functions. The dialogue shows that the students found that Euler's idea that a function could be described by different analytic expressions in different intervals was inconsistent with the generality of the variable:

Student 1: How would he have been able to describe a plucked string? If he calculated for the first part of the string and then for the second part of the string […] Wouldn't they cross each other and just continue straight ahead?

Teacher: Yes […] that is probably one of the problems, right? To describe the crack.

Student 1: But, he calculated an expression for the string on one side of the crack and then an expression for the string on the other side of the crack. They will touch each other. […] But would he [Euler] continue the lines so they form a cross instead? […] He [Euler] disregards that

The dialogue shows that the student is aware that introducing Euler's discontinuous functions creates a problem with the principle of generality of the variable. In the class discussion with the expert groups, this group wanted to discuss this point. The student explained the point as follows:

Student 1: We want to discuss Euler's conception of his discontinuous functions. […] A discontinuous function could, e.g., look like this [the student drew a graph on the black board which was composed of a horizontal straight line, followed by a decreasing straight line that is followed by a curved line]. This is what he [Euler] called a discontinuous function. But in the way he calculated them, they would in principle continue like this [the student continues the three lines by continuing the three lines in accordance with the analytical expression that defines them in the respective intervals.] This looks messy. Actually it does not make sense.

The student is reflecting on the meta-rule of the generality of the variable. One intention for learning was that the students, through discussions of this rule, would come to reflect upon the role of the domain of a function in our modern conception of a function. That this did not happen is illustrated in the next dialogue, which indicates that student 1 was governed by a version of the meta-rule of the generality of the variable. The dialogue took place in expert group 1. They had just begun to discuss whether there are differences between Euler's, Dirichlet's, and our modern concept of a function.

Student 1: Today we talk about that a *f*—that *y* is an independent—or is a dependent variable, i.e., it depends on others, where *x* is the independent variable, *a* and *b* are constants. […] and hence they are unchangeable in a function, right? "…and Euler defined in his time a constant magnitude as a magnitude that had the same value no matter what" [the student is reading aloud from the report from basis group 1]. That is, his [Euler's] constants, i.e. *a* and *b*, must have been the same as ours now a day. […] [the student continue reading from the report] "In contrast he [Euler] defined a variable as a quantity that could take on any value what so ever. A value, where the set doesn't matter." This is also—as I see it—this is partly how it is today

Student 2: Yes it is

Student 1: because *y* depends on something. […] so it [*y*] cannot be anything. But it can change. In principle *x* doesn't matter. It doesn't matter what it [*x*] is

From the dialogue one gets the impression that the two students are of the opinion that the generality of the variable applies for the independent variable today. There were several dialogues that suggested that the students themselves were governed by a meta-rule that has more in common with Euler's than with the accepted mathematical discourse of today. The data showed that some of the students followed the rule of Euler's of analytic continuation. In the papers written in the expert groups there are many incidents that show that the students operate according to the generality of the variable:

> The concepts of domain and range are not without significance when we talk about functions, because it is fundamental that we need to know the x-values (the domain), in order to find the y-values (the range), which constitute the function. These are just concepts we use, to make it easier to separate the different values.

The work with the historical sources created further situations that revealed that some students were governed by meta-rules that do not coincide with meta-rules of the mathematical community of today. In the dialogue below we see how the students' work with the extract from Dirichlet's text revealed that they acted according to a meta-rule that states that symbols and the meaning one attach to them are inseparable. The dialogue took place in expert group 1. The students are trying to understand what basis group 1 has written about Dirichlet's text:

Student 1: [reading aloud from basis group 1's report] … and a and b were two numbers on the x-axis

Student 2: It should have said the y-axis, that's what is meant. Isn't it? x is this way and y is that way [draws a coordinate system in the air]. Yes. Then it is the y-axis. I go tell the others [leaves the room]

Student 1: Not in our concept of a function [thinking break of 9 s]. It is true that. b is a number on the y-axis. But I wouldn't say that a is

Student 3: a that is the one that goes out and up, right? [draws with a finger in the air] [It seems that for this student a is the slope of a linear function]

Student 1: It depends which function it is [Student 2 comes back into the room]

Student 2: Hey, friends, it is the x-axis, but it is because it is formulated wrongly, right? [Draws a coordinate system on the black board]. You know what this is, right? […] Well, he [Dirichlet] thinks that there is a point here, right? [draws a point on the abscissa] […] This is a. And then there is a point here [draws yet another point on the abscissa]. This is b. […] And then he just says that in principle it can move like this [draws a graph that goes from a to b] from a to b on the x-axis. […] Do you understand?

Student 1: But this is not how we see a and b today

Student 2: No, but this is what he meant

The dialogue shows that Dirichlet's use of the symbols a and b created cognitive obstacles for the students. In the students' mathematical discourse a and b are, as a rule, constants in a function, and for student 3, they are constants in a linear function. The dialogue shows that the students translate mechanical from symbol to meaning. For them, it seems that the meaning is inherent in the symbol regardless of the context. This is an example of a situation where some students are governed by a meta-rule that is not in accordance with meta-rules accepted by the mathematical community, and we see here very clearly, how students' meta-rules have an impact on how they understand object level rules. We see here an example of how important it is for students to develop proper meta-level rules in order to learn mathematics.

7 Analysis of Some Results: Concept Formation

Basis group 4 worked with the modern concept of a function. They were asked to give a description of our contemporary concept of a function, of a graph and of the notions of domain and image. They were also asked to explain why proofs are necessary when we explore functions. In their work, they were guided by seven questions they had to answer. The questions were designed to qualify the students' reflections on our concepts of function, graph, domain, image and proofs. The students were asked to identify the central notions in five definitions of a function taken from five different contemporary textbooks in analysis. They were asked to compare the five definitions of a function, a graph, domain and image—the similarities and how they differ from one another. They were asked to reflect upon what a proof is and why proofs are needed for gaining knowledge about functions.

The following dialogue took place in basis group 4. They are discussing what a function is today:

Student 1: So, you are saying that today, by a function we understand a relation between x and y. […] The dependent variables … No, wait … what is it now? I need to think. […] Aren't they dependent variables, because they depend on a and b?
Student 2: No, aren't they the independent variables—that is x and y?
Student 1: But it can also be because a and b … they vary depending on which points you take … but they depend on …
Student 3: They still depend on x and y
Student 2: a and b depend on x and y—don't they?
Student 1: x and y depend on a and b
Student 3: What?
Student 1: Yes, they do!
Student 2: Hmmmmmm …
Student 1: But, look, a and b will always … won't a and b always be the same?
Student 2: No, x and y will always be the same
Student 1: Not x!

It is quite clear from this discussion that the students do not have a clear idea of the notions of dependent variables, independent variables and constants. Student 1(second line) seems to be thinking of a linear function with a and b denoting the slope and the function's value for $x = 0$. The remark that a and b depends on the points one takes, can be interpreted as if Student 1 is thinking about fitting a linear function to a set of data.[3] This can explain the students' confusion about what depends on what—they think a and b depend on x and y, because that is what they have experienced in trying to fit linear functions to data.

Student 1 also seems to conceive of a function as given by an analytic expression containing the symbols a and b. There were many examples in the data that show that many of the students had a concept definition of functions as analytic expressions. One example is the following quote from expert group 3's article:

> Dirichlet believed that functions do not necessarily have to be given by a formula or an analytic expression. This does certainly not comply with today's view of a function. Today we see a function as a form of analytic expression.

[3] Modeling, and especially linear regression and fitting functions to data, is part of the mathematics curriculum in upper secondary school and is practiced in exercises.

Another example is the dialogue below which took place in expert group 1. The students discussed how to explain how Dirichlet's concept of a function differs from Euler's:

Student 1:	Dirichlet said, or he didn't think that a function needed to be expressed as a mathematical expression. Euler thought so. He had that as a requirement. But this is also a requirement today … The teacher enters the room
Student 1:	[Talks to the teacher] We agree that Euler said that his functions should be given by a mathematical concept, right?
Teacher:	Yes, like a formula?
Student 1	Yes. … Or an analytic expression?
Teacher:	Yes
Student 1:	And Dirichlet said, that this was not necessary?
Teacher:	Yes
Student 1:	But now a day we need to be able to describe a function by an analytic expression
Teacher:	[Shakes his head]
Student 1:	No …
Teacher:	No, not necessarily
Student 2:	This was how it was with our former teacher [the student had another teacher the year before]—wasn't it?
Teacher:	Because the functions you have worked with have always been functions that were given by a particular analytic expression. You have had linear functions …
Student 1:	and exponential functions and power functions

This dialogue shows that student 1 came to reflect upon his/her understanding of a function through the work with the historical sources, by comparing Euler's and Dirichlet's concept of a function. The task of writing the history of mathematics article created a learning situation that challenged the students to articulate and reflect upon their own conceptions of a function. At this point in the course, student 1, and most of the students in the class, knew that Dirichlet's and our modern concept of a function are similar to one other, and it then surprises him that Dirichlet does not require a function to be given by an analytic expression. In Tall's and Vinner's terminology, we are here witnessing and incidence where the concept image of student 1 is in conflict with his (Dirichlet's) concept definition (namely that our concept of a function agrees with Dirichlet's). We have here a situation where the students' work with historical sources supported their conception of a function by forcing them to articulate and reflect upon their own conception, which then disclosed misconceptions.

From the work of basis group 4 and from the questionnaires it is evident that the students had an operational understanding of the concept of a function. Basis group 4 wrote the following in their report:

> A function is to find out which correlated values that satisfy x and y that is, it is not only about finding x as in ordinary equations

The students obviously have problems articulating what a function is, e.g., what does it mean to find values that satisfy x and y? From their explanation it is clear, however, that a function is about "finding something out", finding values—and not only x values as they do when they solve "ordinary equations". In one of the questions in the questionnaire the students are asked to explain what a function is today. A student answered:

> A function is a procedure for calculating. There are variables and constants. y depend on a, b and x. (a and b are constants and x and y are variables.)

Here again, we see a process oriented thinking about functions—an operational conception of functions. From both quotes it is evident that the students have a formula (an analytic expression) in mind when they describe their understanding of a function.

With respect to Sfard's model for individuals' formation of a concept, the data material show that most of the students are in the phase of condensation. The two quotes below from basis group 1's and basis group 4's reports show that the students are able to switch between different forms of representation and that they talk about and deal with processes behind the object without necessarily having to perform the operations:

> First we will try to define what we ourselves think a function is. […] A function is a relation between variables, where x and y usually are defined as the variables.

And

> The relation between a function and a graph is that the graph is a visual picture of a function. A function is a relation between x and y, which among others can be represented through a graph, just as it can be represented in the form of a table or a formula.

From these two quotes we can see that the students, through the work with the history of the concept of a function, have been challenged to begin to reflect upon the concept of a function on a structural level.

8 Analysis: Historical Awareness

The development of the students' historical awareness was assessed through the students' reports in the basis groups, the articles written in the expert groups and the questionnaires. The data shows that many of the students realized that the concept of a function had undergone a historical development, e.g., as expressed below in the article written by expert group 1:

> Since the beginning of the concept of a function there has been a debate about its usability. This has led to several changes and new interpretations of how to describe a relationship of the kind a function does.

The report from basis group 2 indicates that not only did the students realize that the concept of a function had undergone a historical development they had also gained more specific insights into this development, as they wrote in their report:

> Euler developed his first concept of a function in 1748. One of his claims about the notion of a function was that a function of a variable is an expression of a calculation composed in any way. However, the expression must include a variable, and one or more numbers or one or more constants. This meant that functions would be continuous (Note! The understanding of continuous functions then is different from how we understand it today!). Later that year Euler extended his concept of a function. He added the possibility that a function can be discontinuous (Note! The understanding of discontinuous functions then is different from how we understand it today!).

These students understood some of the important elements of Euler's first conception of a function and that the concept of continuity has been understood differently than it is understood today.

With respect to the question of driving forces in the development of mathematics the experimental course focused on the debate of the vibrating string which caused Euler to extend his original concept of a function. The quote below from the report written by basis group 2 illustrates that even though these students did not fully understand the debate, they did realize that the debate functioned as a driving force for the development of the concept of a function:

> The reason why Euler began to work with discontinuous functions was because of a debate between contemporary mathematicians. The debate concerned the fact that the functions the mathematicians worked with could not describe a vibrating string.

The debate of the vibrating string was discussed in basis group 2 as an example of a scientifically internal driving force in the historical development mathematics. The French Revolution and the subsequent changes in the organization of science and education were discussed by basis group 3 as a sociological (indirect) external driving force. Their report shows that these students understood in which kind of institutions in society the development of mathematics (the concept of a function) took place.

The question of actors' influence on the formation of mathematical concepts was discussed in the articles written by the expert groups. The students discussed in different ways the issue of human factors in the development of mathematics. The quote below is from expert group 1's article in which they point towards the role played by Euler's mathematics colleagues for the reception of Euler's extended concept of a function:

> Therefore, the development of the concept of a function was among other things due to human attitudes and interpretations, which were important factors. For example, some of Euler's contemporary mathematics colleagues were of the opinion that Euler's extended function concept should not be used because it went against the principle of mathematics. They thought it was cheating. This meant that Euler's extended function concept never came to be used as intended, and a new function concept was developed by Dirichlet.

9 Discussion and Conclusion

In the experimental course, history and extracts of original sources were used to have students reflect upon meta-discursive rules in mathematics, to support their learning of the concept of a function, and to develop their historical awareness.

The results show that it is possible to diagnose (some) of (some) students' meta-discursive rules, that some of the students acted according to meta-discursive rules that coincide with Euler's from the 1700s, and that reading a part of a text by Dirichlet from 1837 created obstacles for the students, that can be referenced to differences in meta-discursive rules. The data from the experiment showed that some of the students were governed by meta-rules that are not in accordance with meta-rules held by present day's mathematicians. The work with the historical sources placed the students in unfamiliar mathematical situations in which they could not rely upon standard solution methods for problem solving. The historical questions the students worked with in the two phases of the matrix-organized group work challenged them to discuss the notion of a function and related notions in a non-operational way. Pedagogical observations of the students' discussions of the extract from Dirichlet's text revealed that they acted according to a meta-rule that states that symbols and the meaning one attaches to them are inseparable. Similar kinds of pedagogical observations of dialogues between the students and their teacher exposed that, like Euler, many of the students were governed by the meta-rule of the generality of the variable.

The experimental course with history made it possible to diagnose some of students' meta-discursive rules that are in conflict with rules shared by the mathematical community today. It is not possible from this course alone to provide evidence that these students' meta-rules were changed. According to Sfard, for this to happen, the students need to experience what she has called a *commognitive* conflict, which happens when students are confronted with meta-rules that are different from those they are governed by. However, the discussions in expert group 1 showed that these students became aware that introducing

Euler's discontinuous functions caused a problem with the principle of the generality of the variable and analytic continuation: "it does not make sense", as one of the students concluded. This created an episode in the classroom discussion where it became clear to at least one student, that the meta-rule of Euler was in conflict with the example the student gave, of a function that was given by different analytic expression in different intervals—a function, which is included in our current concept of a function. This situation appeared in the classroom due to the students' work with historical sources governed by past meta-rules, and it shows that history of mathematics has the potential. To create commognitive conflicts and thereby a potential to cause a change in students' meta-level rules.

Many of the students had problems understanding the concept of a function, and some of these problems were revealed through the teaching experiment with history. The data revealed that the students did not have a clear idea of the notions of dependent variables, independent variables and constants. Analysis of dialogues indicates that some of the confusion was caused by the students' experiences with fitting linear functions to a set of data. There were several examples in the data that show many of the students had a concept image of functions as analytic expressions, and that this was caused by the teaching, due to the fact that up to the experimental course, the students had only worked with functions given by analytic expressions. The task of writing the history of mathematics article in the expert groups created a learning situation that challenged the students to articulate and reflect upon their own conceptions of a function. In this phase of the matrix-organized group work most of the students knew that Dirichlet's and our modern concept of a function are similar to one other. However, because the students' concept image is not in accordance with Dirichlet's concept definition, one of the students articulated that he did not understand why Dirichlet did not require a function to be given by an analytic expression. Due to the work with the historical sources an incidence in the learning situation occurred that corresponds to what Tall and Vinner called the activation of two cognitive conflict factors: the concept image of the student is in conflict with his [Dirichlet's] concept definition that our concept of a function agrees with Dirichlet's. The students' work with historical sources supported their conception of a function by forcing them to articulate and reflect upon their own conception, causing them to begin to reflect upon the concept of a function on a structural level. The experimental course on the history of the concept of function supported the students' learning towards a structural understanding of the concept of a function.

One of the aims of the experimental course was to bridge history of mathematics with the learning of mathematics. This is a prerequisite for using history to reveal to students that there are meta-level rules of mathematics discourse and make them objects of students' reflection by having them investigate meta-level rules at the object level of history discourse. Hence, this aim can only be fulfilled if the students become engaged in history of mathematics in a scholarly sense. That is, if they investigate how past mathematicians have developed, understood and worked with mathematics in general and the concept of a function in particular. In the experimental course this was accommodated by adopting a multiple perspective approach to history of mathematics studied from practices of mathematics. Through the matrix organization the students worked with the history of the concept of a function from different perspectives in the different basis groups. These perspectives were then brought together in the work in the expert groups.

The data from the work in the expert groups shows that many of the students realized that the concept of a function had undergone a historical development, and that they gained more specific insights into this development. They realized that some of the important elements of Euler's first conception of a function and that the concept of continuity has been understood differently than it is understood today. The students became aware that mathematics develops according to both internal scientific driving forces as well as to external sociological circumstances. They became aware that the formation of mathematical concepts is due to historical actors. Due to the writing assignment in the expert groups, the students came to discuss in different ways the issue of human factors in the development of mathematics, e.g., the role played by Euler's mathematics colleagues for the reception of Euler's extended concept of a function.

The results from the experimental course show that this way of integrating and using history to bridge history of mathematics with the learning of mathematics, can be used to pinpoint students' meta-discursive rules and to disclose misconceptions. This knowledge can then be used to target further teaching towards learning goals that can focus students' attention towards developing proper meta-rules and to support students' reification of mathematical objects. In this sense, history of mathematics can function at the core of what it means to learn mathematics.

References

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*, 247–285.

Dubinsky, E., & Harel, G. (1992). The process conception of function. In: Harel, G., & Dubinsky, E., (Eds.), The Concept of function: Aspects of epistemology and pedagogy, MAA Notes, 28, 85–106.

Jensen, B. E. (2003). *Historie—livsverden og fag*. Copenhagen: Gyldendal.

Jankvist, U. T., & Kjeldsen, T. H. (2011). New avenues for history in mathematics education: Mathematical competencies and anchoring. *Science & Education, 20*, 831–862.

Jensen, B. E. (2010). *Hvad er historie*. Copenhagen: Akademisk Forlag.

Katz, V. J. (2009). *A history of mathematics. An introduction*. Boston: Addison-Wesley.

Kjeldsen, T. H. (2011a). History in a competency based mathematics education: A means for the learning of differential equations. In V. Katz & C. Tzanakis (Eds.), *Recent developments on introducing a historical dimension in mathematics education* (pp. 165–173). Washington, DC: Mathematical Association of America.

Kjeldsen, T. H. (2011b). Does History have a significant role to play for the learning of mathematics? Multiple perspective approach to history, and the learning of meta level rules of mathematical discourse. In E. Barbin, M. Kronfellner, & C. Tzanakis (Eds.), *History and epistemology in mathematics education proceedings of the sixth European summer university ESU 6* (pp. 51–62). Vienna: Verlag Holzhausen GmbH.

Kjeldsen, T.H. (2012). Uses of history for the learning of and about mathematics: Towards a theoretical framework for integrating history of mathematics in mathematics education. Plenary address in The proceedings for the HPM 2012 satellite meeting of ICME-12, Daejeon, Korea, 2012, 1–21.

Kjeldsen, T. H., & Blomhøj, M. (2012). Beyond motivation: History as a method for the learning of meta-discursive rules in mathematics. *Educational Studies in Mathematics, 80*, 327–349.

Lützen, J. (1978). Funktionsbegrebets udvikling fra Euler til Dirichlet. *Nordisk Matematisk Tidsskrift, 25*(26), 5–32.

Lützen, J. (1983). Euler's vision of a general partial differential calculus for a generalized kind of function. *Mathematics Magazine, 56*(5), 299–306.

Petersen, P.H. (2011). *Potentielle vindinger ved inddragelse af matematikhistorie i matematikundervisningen.* Master Thesis in mathematics, Roskilde University, Denmark. http://milne.ruc.dk/ImfufaTekster/pdf/483web.pdf.

Sfard, A. (1991). On the dual nature of mathematical conception: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1–36.

Sfard, A. (2000). On reform movement and the limits of mathematical discourse. *Mathematical Thinking and Learning, 2*(3), 157–189.

Sfard, A. (2008). *Thinking as communicating*. Cambridge: Cambridge University Press.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151–169.