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# History, Applications, and Philosophy in Mathematics Education: HAPh—A Use of Primary Sources

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**Abstract** The article first investigates the basis for designing teaching activities dealing with aspects of history, applications, and philosophy of mathematics in unison by discussing and analyzing the different 'whys' and 'hows' of including these three dimensions in mathematics education. Based on the observation that a use of history, applications, and philosophy as a 'goal' is best realized through a modules approach, the article goes on to discuss how to actually design such teaching modules. It is argued that a use of primary original sources through a so-called guided reading along with a use of student essay assignments, which are suitable for bringing out relevant meta-issues of mathematics, is a sensible way of realizing a design encompassing the three dimensions. Two concrete teaching modules on aspects of the history, applications, and philosophy of mathematics—HAPh-modules—are outlined and the mathematical cases of these, graph theory and Boolean algebra, are described. Excerpts of student groups' essays from actual implementations of these modules are displayed as illustrative examples of the possible effect such HAPh-modules may have on students' development of an awareness regarding history, applications, and philosophy in relation to mathematics as a (scientific) discipline.

## 1 Introduction

In recent years, articles on the inclusion of history of mathematics in mathematics education have become more common in the research literature, for example in *Science & Education* where three were published in the past year alone (Heeffer 2011; Jankvist and Kjeldsen 2011; Panagiotou 2011). Also quite common are studies on applications of mathematics in mathematics education, in particular if one includes mathematical

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modeling within these.[1] Studies, however, on the inclusion of philosophy of mathematics in mathematics education, that is in the teaching and learning of mathematics,[2] are as of yet only very few in number.[3] And studies on attempts to include all three dimensions of history, applications, and philosophy of mathematics in mathematics education at the same time are practically non-existing. This article reports on such a study and in particular the design aspects of trying to align the dimensions of *H*istory, *A*pplications, and *Ph*ilosophy in teaching modules, so-called *HAPh-modules*, for upper secondary level. But before we get to the actual design considerations and technicalities, we must address the question of why one would go to the lengths of trying to accomplish this in the first place. The answer to this question is twofold. First, it is of course a matter of the national setting of mathematics education in Denmark, where the study stems from. But second, it is also a matter of the relatedness of the arguments for including these three dimensions in mathematics education, the so-called 'whys'.

## 2 The 'Whys' of Using History, Applications, and Philosophy in Mathematics Education

In Jankvist (2009a), it is argued that the arguments for using history in mathematics education, the 'whys', may be categorized into two different kinds, namely *history as a tool* and *history as a goal*.[4] First, history may be used as a *tool* for teaching and learning mathematical ideas, concepts, theories, methods, algorithms, ways of argumentation and proof, etc. The tool-arguments may further be subcategorized into being concerned with history as a motivational and/or affective tool; history as a cognitive tool (for example the idea of epistemological obstacles); and the role of history in what may be referred to as the evolutionary arguments (the recapitulation argument or historical parallelism). Second, history may be used as a *goal*, meaning that it is considered a goal to teach students something about how mathematics has come into being, the historical development of it as well as both human and cultural aspects of this development, etc. (Note that this is not the same as teaching the history of mathematics per se.)

In a rather similar manner, the arguments for mathematical applications (and modeling) in mathematics education may also be ordered into two categories of tool and goal, respectively. Niss (2009) argues that, on the one hand, applications (and modeling) may serve as a *means* (tool) to support the learning of mathematics, either by providing interpretation and meaning to mathematical ideas, constructs, argumentation, and proof, or by motivating students to study mathematics. On the other hand, it may be seen as an *end* in itself that students become acquainted with the use of mathematics in extra-mathematical contexts (and in relation to modeling, also become able, themselves, to actively put mathematics to use in such contexts; for a more elaborated discussion of this, see Blum and Niss 1991).

To my knowledge, no categorizations of the arguments for using philosophy in the teaching and learning of mathematics have yet been published, which of course is a

[1] Examples of recent collective volumes are: Blum et al. (2007), Kaiser et al. (2011).

[2] Philosophy in general, on the other hand, is often used within theoretical constructs and frameworks in mathematics education.

[3] The few I am familiar with are: Carson and Rowlands (2007), Chassapis (2010), Daniel et al. (2000), De La Garza et al. (2000), Iversen (2009), Kennedy (2007), Kjeldsen and Blomhøj (2009), Prediger (2010), Rowlands and Davies (2006).

[4] For further discussion of this framework, see also Tzanakis and Thomaidis (2012).

consequence of the limited number of papers available on the topic. Nevertheless, it seems to me that the arguments of using philosophy in mathematics education would fall into the same two main categories as the arguments for including history and mathematical applications, i.e. tool and goal. The category of *philosophy as a tool* would embrace arguments stating that philosophy may assist students in their sense-making of internal mathematical issues, for example mathematical argumentation and the notion of mathematical proof, including also how and why we argue and prove, mathematical ideas and constructs, etc.; arguments regarding motivation and/or affection; and arguments stating that philosophy may act as a vehicle for developing students' creative thinking. Arguments belonging to the category of *philosophy as a goal*, on the other hand, would include those stating that it serves a purpose in its own right for students to know something about for example the epistemology and/or ontology of mathematics and its concepts and constructs, the philosophical foundations of mathematics as a discipline as well as questions of why mathematics is constructed the way it is, its science-philosophical status, or even why we may need (a) philosophy of mathematics, etc.

Another way to think of the above is through a distinction of *in-issues*, which concern the internal issues of mathematics such as mathematical concepts, definitions, theorems, proofs, etc., and *meta-issues*, which concern more metaperspective aspects of mathematics as a scientific discipline, as for example the historical evolution of mathematics, its interplay with society through applications, or issues of epistemology, ontology, etc. A use of history, application, or philosophy as a tool is concerned with students' learning of mathematical in-issues, whereas a use of these dimensions as a goal focuses on students' learning of various meta-issues regarding these. Of course, this is not to say meta-issues may not act as a means (tool), for example by motivating students, to reach and end (goal) of learning specific in-issues. In a similar manner, certain meta-issues can clearly not be understood without an understanding of some mathematical in-issues (see Jankvist 2011a for examples). The distinction is merely a useful way to talk about the primary foci of the whys of using history, application and philosophy in mathematics education, as we shall see in the following sections.

Now, the fact that the arguments for history, applications, and philosophy in mathematics education may be categorized in a similar fashion provides an opportunity in terms of design, since similar purposes, 'whys', may be realized through similar approaches, 'hows'. I shall return to this later, but first let's have a look at the national setting of mathematics education in Denmark.

## 3 The Danish Upper Secondary School Mathematics Program and the KOM-Project

In 2005 a new reform of the Danish upper secondary school was initiated. For the mathematics program this meant an increased focus on both the historical development of mathematics as well as actual mathematical applications (see e.g. Jankvist 2009d). By 2008 the ministerial regulation for mathematics stated the following:

> When mathematics is part of a study program, an academic working relationship must be established, which involves a more comprehensive application of mathematics. By this, the student is to acquire a deeper insight into the descriptive power of mathematics as well as the importance in considering and discussing the premises for a mathematical description and the reliability of the results obtained by the description. A teaching module must be designed with the purpose of developing the students' knowledge about mathematics' interplay with culture, science, and technology. (UVM 2008, appendix 35, article 3.4, my translation from Danish)

Such teaching modules may well be included as part of the so-called "supplementary curriculum":

> The students will not be able to fulfill the academic goals on the basis of the core curriculum alone. The supplemental curriculum in the subject of mathematics, including the interplay with other subjects, must deepen and put the core curriculum into perspective as well as expand the academic horizon and leave room for local wishes and considerations for the individual school.
> In order for the students to fulfill all the academic goals, the supplemental curriculum, which takes up 1/3 of the total teaching time, must involve, among other things, […] modules in the history of mathematics. (UVM 2008, appendix 35, article 2.3, my translation from Danish)

The 'academic goals', which are referred to here, are in particular the following two:

> The students must be able to:
>
> – demonstrate knowledge about the development of mathematics and its interplay with the historical, scientific, and cultural development
> – demonstrate knowledge about application of mathematics within selected areas, including knowledge about application in the treatment of a more complex problem.
>
> (UVM 2008, appendix 35, article 2.1, my translation from Danish)

The ministerial regulation, and its rhetoric, is inspired by the outcome of the so-called KOM-project, a Danish abbreviation of "competencies and mathematical learning", which was published in Danish in 2002 and in an English version in 2011 (Niss and Højgaard 2011). Besides listing and discussing eight (1st order) mathematical competencies, for example mathematical thinking, reasoning, representation, modeling, problem tackling, etc.,[5] the KOM-project also lists three (2nd order) competencies, known as three types of *overview and judgment* (OJ). These are:

- OJ1: the actual application of mathematics in other subject and practice areas;
- OJ2: the historical evolvement of mathematics, both internally and from a social point of view; and
- OJ3: the nature of mathematics as a subject.

While mathematical (1st order) competencies comprise "having knowledge of, understanding, doing, using and having an opinion about mathematics and mathematical activity in a variety of contexts where mathematics plays or can play a role", or in other words a kind of "well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge", the three types of overview and judgment are "'active insights' into the nature and role of mathematics in the world" and Niss and Højgaard state that "these insights enable the person mastering them to have a set of views allowing him or her *overview and judgement of the relations between mathematics and in conditions and chances in nature, society and culture*" (Niss and Højgaard 2011, pp. 49, 73, italics in the original).

As may already be clear to the reader, these three types of overview and judgment to a large extent resemble the dimensions of history ($\sim$OJ2), applications ($\sim$OJ1), and philosophy ($\sim$OJ3) as *goals* in mathematics education, and for that reason I shall spent a little more space on the KOM-project's discussion of these.

OJ1 concerns actual application of mathematics to extra-mathematical purposes within areas of everyday life, society, or other scientific disciplines. The application is brought about through the creation and utilization of mathematical models. As examples of questions to be considered in relation to this, Niss and Højgaard (2011, p. 74) mention:

[5] For a brief discussion of these, see also Jankvist and Kjeldsen (2011).

- "Who, outside mathematics itself, actually uses it for anything?"
- "What for?"
- "Why?"
- "How?"
- "By what means?"
- "On what conditions?"
- "With what consequences?"
- "What is required to be able to use it?" etc.

OJ2 should not be confused with knowledge of the history of mathematics viewed as an independent topic (as taught per se). The focus is on the actual fact that mathematics has developed in culturally and socially determined environments, and is subject to the motivations and mechanisms which are responsible for this development. On the other hand, the KOM-project says, it is obvious that if overview and judgment regarding this development is to have any weight or *solidness*, it must rest on *concrete examples* from the history of mathematics.[6] As examples of OJ2 questions Niss and Højgaard (2011, p. 75) mention:

- "How has mathematics developed through the ages?"
- "What were the internal and external forces and motives for development?"
- "What types of actors were involved in the development?"
- "In which social situations did it take place?"
- "What has the interplay with other fields been like?" etc.

OJ3 concerns the fact that mathematics as a subject area has its own characteristics, as well as the characteristics themselves. Some of these, mathematics has in common with other subject areas, while others of them are unique. As the following examples of OJ3 questions (Niss and Højgaard 2011, pp. 75–76) show, this type of overview and judgment embraces quite a few elements of philosophy of mathematics:

- "What is characteristic of mathematical problem formulation, thought, and methods?"
- "What types of results are produced and what are they used for?"
- "What science-philosophical status does its concepts and results have?"
- "How is mathematics constructed?"
- "What is its connection to other disciplines?"
- "In what ways does it distinguish itself scientifically from other disciplines?" etc.

## 4 Problématique

As should be evident from the two previous sections, when using history, applications, and philosophy as *goals* in mathematics education, we are dealing with and trying to develop the students' *awareness* about these three dimensions of *mathematics as a discipline* (Jankvist forthcoming). In the context of using history, the dimension of the three that has been researched the most so far, aspects related to such awareness are discussed for example by Kjeldsen (2011), Kjeldsen and Blomhøj (in press), and not least Fried (2007, 2010). Fried argues that in order to truly embrace a use of history-as-a-goal, a rethinking of what mathematics education is about may be required in order to avoid the often occurring

[6] For a discussion of how such *solidness* may be brought about and evaluated through the construct of *anchoring*, see Jankvist (2009c, d, 2010, 2011a) and Jankvist and Kjeldsen (2011).

situation of the inclusion of history becoming shallow or perfunctory. Whereas Fried refers to mathematics education on the whole and from an international perspective, I shall take my departure point in the Danish situation of upper secondary level as outlined above. On the one hand, I do realize that this setting is quite special in terms of providing opportunities for an inclusion of the three discussed dimensions, and in particular those of application and history. But, on the other hand, it also offers a quite unique possibility for the conduction of educational research regarding these three dimensions in the teaching and learning of mathematics, since the Danish regulations for upper secondary level mathematics education seem to posses some of the elements which Fried is calling for.

Still, from a practice point of view, this leaves the question of how to actually, and meaningfully, put a use of these three dimensions as a goal into play in order to form a basis and create a setting for students' development of an awareness about these dimensions of mathematics as a discipline. That is, how may we design teaching activities that do this? It shall be my claim that we may attack the problem by trying to embrace aspects of all three dimensions, i.e. the history, applications, and philosophy of mathematics, in the same activity—thus, also allowing for possible interplay between the three dimensions to be part of the activity.

Before we go further, allow me to explain briefly how I intend to address the given problem and the posed question as well as what is to come in the remaining parts of the article. The first to be said is that my answer to the question will be by example(s), thus making it into a sort of 'constructive existence proof'. More precisely, I shall describe two actual teaching activities (modules) which have been designed for upper secondary level: one on the early history of graph theory (Euler), its use in shortest path and minimum spanning tree algorithms (Dijkstra), and a philosophical dimension related to a discussion of mathematical problems (Hilbert); and another on the early history of logic (Boole), its later use in electric circuit design (Shannon), and a philosophical dimension related to the 'unreasonable' effectiveness of mathematics (Hamming). In the design of these activities, I have relied on a special approach to the use of primary original sources referred to as a 'guided reading'. Having accounted for the content of the original sources, their interrelations, the design choices for the activities and aspects related to their actual implementation, I shall display a small selection of data from each activity supporting the 'existence proof' of the study, data which stem from implementations in a Danish upper secondary class in the years of 2010 and 2011. But first, the claim about why we should want to embrace all three dimensions, or types of OJ, in the same activity, which is related to the previously mentioned 'hows'.

## 5 The 'Hows' of Using History, Applications, and Philosophy in Mathematics Education

As discussed in Jankvist (2009a), strictly separating the purposes (the 'whys') of using history from the approaches to doing so (the 'hows'), is not a straightforward task since they often are intertwined. Nevertheless, attempting to do so anyway may provide us with new insights about the realizing of different 'whys' by means of different 'hows', not least from a design point of view.

In Jankvist (2009a), the approaches for using history are grouped into three different categories: the *illumination approaches*, where the ordinary teaching is somehow supplemented—or 'spiced up'—by historical information; the *modules approaches*, which are instructional units devoted to history, either small curriculum-tied modules or longer more free ones, but often on a specific case; and the *history-based approaches*, which are directly based on or inspired by the historical development of some mathematics or mathematical topic, the historical development setting the agenda for the presentation of this. Of course, each of these types of approaches may be scaled according to size and scope.

Blum and Niss (1991, p. 60) give a more fine-grain categorization of the "different types of basic approaches to including relations to applicational areas in mathematics programmes" into six different types: the *separation approach*, where applications and modeling are put in separate courses devoted to them; the *two-compartment approach*, where the mathematics program is parted into that concerned with 'applied' mathematics and that concerned with 'pure' mathematics; the *island approach*, where the ordinary 'pure' mathematics program is interrupted by 'islands'—or modules(!)—of applicational work; the *mixing approach*, where elements of application and modeling are invoked frequently in the ordinary teaching to assist—or illuminate(!)—the introduction of mathematical concepts, etc.; the *mathematics curriculum approach*, where mathematical or applicational problems define the actual mathematics that students work with; and finally, the *interdisciplinary integrated approach*, which differs from the previous one in that it operates with a full integration of mathematical and extra-mathematical activities in the sense that 'mathematics' is not organized as a separate subject. Thus, where the first three approaches to some extent resemble the idea of 'modules', the third more than the first and second, the fifth and sixth approaches may be seen as kinds of problem-oriented, problem-driven, or applicational approaches.

Again, to my knowledge, no categorization of the approaches for using philosophy in mathematics education exists. So I shall give a tentative one encompassing at least the four following approaches. As for history and applications, it seems clear to me that philosophy may play the role of a supplement to the ordinary mathematics instruction, i.e. through an *illumination approach*. In a similar fashion, we may also include philosophy through a *modules approach*. As history had its distinctive history-based approach, we may talk of a *philosophy-based approach*, where the instruction follows strictly and is dictated by a particular philosophical school of thought, as for example that of intuitionism. Further, it seems to me that philosophy offers us yet an approach, a *philosophical discussion-approach*, where (groups of) students representing different philosophical viewpoints or directions enter into discussion about philosophical questions, for example Platonism versus one of the non-Platonist directions discussing questions such as mathematics being discovered or invented (e.g. on the basis of Hersh 1997).[7]

## 6 Design Considerations Part 1: Interplay of the 'Whys' and 'Hows'

Despite the discrepancies between the 'hows' when using history, applications, or philosophy in mathematics education, it should be clear from the above that one factor which they do have in common are the modules approaches. As accounted for in Jankvist (2009a), a purpose of history as a goal is often best realized through exactly a modules approach:

[7] For an example of a research study, where students were asked to make up their minds about the question of discovery or invention of mathematics, see Jankvist (2009d, forthcoming). Yet an example may be found in Rowlands and Davies (2006).

> The modules approaches are […] a much more suitable place to try and bring out the different meta-issues of the development of mathematics, […]. In the more extensive modules […] an opportunity opens up for introducing new in-issues of mathematics, thus fertilizing the soil for a discussion of related meta-issues. Such modules further provide the opportunity to study branches of mathematics that are not normally part of the curriculum at a given school level. This again opens up for a larger variety of meta-issues to be discussed (Jankvist 2009a, pp. 251–252).

Since both using applications as a goal and using philosophy as a goal are concerned with the meta-issues of mathematics in a similar way as that of using history as a goal, the above situation must apply to these purposes as well, i.e. that a modules approach is a suitable way of trying to realize these purposes in a teaching situation. Furthermore, by addressing all three dimensions of history, applications, and philosophy in one and same activity, an opportunity is created to illustrate the interplay between these dimensions in mathematics and its development.

As shown in Jankvist (2009b), students often prefer history of mathematics which, on the one hand, is not too distant in time from their own and, on the other hand, includes elements of applications of the mathematical in-issues involved, preferably with some relation to their own everyday life.[8] With this in mind, I decided on the historical cases mentioned earlier: the use of graph theory in shortest path algorithms, something used in practically any Internet application telling us how to get from $A$ to $B$; and the use of Boolean algebra in electric circuit design, part of the basis for all the computer hardware surrounding us.

The chosen cases, partly due to being more contemporarily, do not necessarily lend themselves to draw in a more traditional use of philosophy of mathematics. However, as stated in the quote above, the non core-curricular topics of mathematics open up for a discussion of other types of philosophical meta-issues, as for example those addressed by Hilbert (1900) and Hamming (1980) on the role of mathematical problems in relation to the development of mathematical topics and the 'unreasonable' effectiveness of mathematics in other practice areas, respectively.

## 7 Design Considerations Part 2: A Use of Primary Original Sources

The next design aspect to be considered is that of how to actually present the chosen mathematical topics to the students. As also mentioned in Jankvist (2009a), primary original sources may play a central role in the modules approaches as a way of realizing the goal-arguments. This has to do with what Barbin (1997) and Jahnke et al. (2000)[9] refer to as the three general ideas best suited for describing the special effects of using original sources in mathematics education: *replacement* (or in the original French wording; *vicariante*), which refers to the replacement of the usual with something different, for example by allowing mathematics to be seen as more than just a corpus of knowledge and techniques; *reorientation* (*dépaysement*), which challenges one's perception by making the familiar unfamiliar, thus also causing a reorientation of our views, and; *cultural understanding* (*culturel*), which allows us to place the development of mathematics in a scientific, technological, or societal context of a given time and place. As may be observed, these ideas are not so very different from the purposes (or 'whys') of using history,

[8] These observations are based on studies reported in Jankvist (2009c, d, 2010, 2011a).

[9] Chapter 9 in the ICMI Study on *History in Mathematics Education*; the chapter is written by Jahnke, Arcavi, Barbin, Bekken, Furinghetti, El Idrissi, da Silva and Weeks.

applications, and philosophy as a goal in mathematics education as well as the various elements of the KOM-project's three types of overview and judgment regarding mathematics as a discipline. According to Jahnke et al. (2000, p. 291), "… the study of an original source is the most demanding and the most time consuming", at the same time such a study is "rewarding and substantially deepening the mathematical understanding."

One other reason for resorting to primary original sources, instead of just retelling the contents of these to the students, is that original sources are open to interpretation—something that secondary sources are not—which provides the students with a very different basis for making their own reflections as well as developing their overview and judgment. Yet a reason has to do with the problem of introducing a *Whig* interpretation of history (Butterfield 1931/1951), a term referring to a reading of the past with the eyes of the present. As also discussed by Fried (2001), this is much less likely to happen when reading and studying the original texts. However, because original sources generally *are* difficult to access, the presentation of these in the modules were supplied with explanatory comments and illustrative tasks along the way—something which actually also provides the opportunity of bringing the students up to date with modern notation, definitions, and terminology. This form of presentation and way of working is referred to as a *guided reading* of primary original sources, a format developed by Pengelley, Lodder, Barnett and others related to the group at NMSU who use original texts in such manner in their university undergraduate courses (see Pengelley 2011; Barnett et al. 2011).[10]

## 8 Design Considerations Part 3: Teaching Material and Student Essay Assignments

A decision was made to prepare a teaching material for each module. Besides a few introductory notes, a material mainly consisted of three primary original sources, in Danish translation, one for each of the three dimensions. Thus, for the first module the three original texts to be studied through 'guided readings' were:

- Leonhard Euler, 1736: *Solutio problematis ad geometriam situs pertinentis* (In: Fleischner 1990).
- Edsger W. Dijkstra, 1959: *A Note on Two Problems in Connexion with Graphs* (Dijkstra, 1959).
- David Hilbert, 1900: *Mathematische Probleme—Vortrag, gehalten auf dem internationalen Mathematiker-Kongreß zu Paris 1900* (the introduction only) (Hilbert 1900, 1902).

For the second module, the three original texts (also in Danish translation) to be studied were:

- George Boole, 1854: *An Investigation of the Laws of Thought on which are founded the Mathematical Theories of Logic and Probabilities* (Boole 1854).
- Claude E. Shannon, 1938: *A Symbolic Analysis of Relay and Switching Circuits* (Shannon 1938b).
- Richard W. Hamming, 1980: *The Unreasonable Effectiveness of Mathematics* (Hamming 1980).

[10] Links are: http://www.math.nmsu.edu/hist_projects/ and http://www.cs.nmsu.edu/historical-projects/ (retrieved on April 15, 2012). In particular the projects by Janet Heine Barnett (n.d., 2011a, b) have served as a source of inspiration for the work discussed in this paper.

Common features for these two selections of texts are that the first text, representing the historical dimension, is one which more or less constitutes the origin of a specific mathematical sub-discipline, here graph theory and Boolean algebra; the second text is a modern day use of this 'old' mathematics with some relevance for students' everyday life; and the final text is that which sets the philosophical frame for the module. Whereas the two first texts in a set are more directly connected, for example Shannon makes a direct reference to Boole, the third text is more 'independent'. However, the crucial point is that the mathematical cases of the first two texts constitute an illustrative example of the philosophical discussion or topic in the third text. I shall explain this more thoroughly in the sections to come.

Another important feature of the design is that of having groups of students prepare small essays. In a previous study I have found that this is a good way of bringing students to work with aspects of history as a goal (Jankvist 2011a). For this reason, the same approach was taken to bring in the two dimensions of applications and philosophy. The particular setting creates a scene, where students at the end of the implementation of the teaching module, after having read and worked with the mathematical case in question, are to discuss among themselves meta-issues regarding the case. These meta-issues are chosen beforehand and included in the description of the essay-assignment at the end of the teaching material; although in such a manner that the students can draw in additional meta-issues should they find it relevant. In the two modules under discussion here, the students first and foremost were to relate the two first texts on history and applications to the philosophical discussion in the third as part of their essay assignments—a task which to some extent also forces out some of the interplay between the three dimensions, although still exemplified by the concrete case.

I shall illustrate this as I provide the introduction to the texts of both modules in the following sections. Also, I shall provide a couple of illustrative examples of students' answers to the essay-assignments. But do bear in mind though that these examples are not meant to provide sufficient empirical evidence in any way, they are merely given to illustrate the kind of outcome that the HAPh-module design may result in.[11]

## 9 HAPh-Module 1: Historical Emergence and Modern Application of Graph Theory

As mentioned, the overall theme for the first module was 'mathematical problems', which is what Hilbert addresses in general terms in the introduction of his lecture from 1900. Besides posting a series of criteria for a good mathematical problem (for example that it must be explainable to laymen and that it must be challenging but not inaccessible, etc.), he also discusses the origin of good mathematical problems and the relationship between mathematics and the real world through these[12]:

> Surely the first and oldest problems in every branch of mathematics spring from experience and are suggested by the world of external phenomena. […]
> But, in the further development of a branch of mathematics, the human mind, encouraged by the success of its solutions, becomes conscious of its independence. It evolves from itself alone, often without appreciable influence from without, by means of logical combination, generalization,

[11] The illustrative examples stem from implementations of the two HAPh-modules in a Danish upper secondary class. Danish upper secondary school is 3 years, where students in first year are of age 15–16 years. The class under consideration followed the mathematics-science direction, meaning that they study mathematics through all 3 years of upper secondary school.

[12] Although not included in this HAPh-module, a relevant source on Hilbert's views is that of Corry (2004).

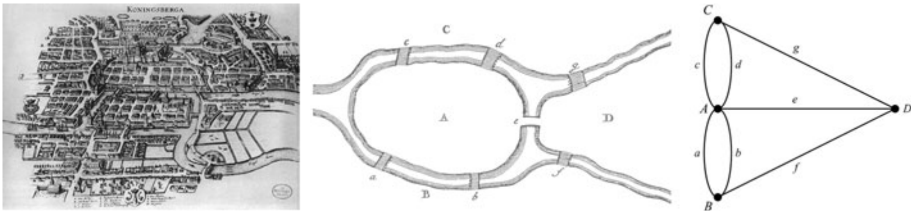


**Fig. 1** *Left* A picture of Königsberg with its 7 bridges from 1652. *Middle* Euler's (1736) simplification of Königsberg's bridges. *Right* A modern graph representation

> specialization, by separating and collecting ideas in fortunate ways, new and fruitful problems, and appears then itself as the real questioner. […]
> In the meantime, while the creative power of pure reason is at work, the outer world again comes into play, forces upon us new questions from actual experience, opens up new branches of mathematics, and while we seek to conquer these new fields of knowledge for the realm of pure thought, we often find the answers to old unsolved problems and thus at the same time advance most successfully the old theories. (Hilbert 1902, quoted from the 2000-reprint, p. 409)

In a certain sense, parts of the history of graph theory illustrate Hilbert's observations. Usually, Euler's paper from 1736 on the Königsberg bridge problem—how to take a stroll through Königsberg crossing each of its seven bridges once and only once—is considered the beginning of mathematical graph theory (Fleischner 1990). Euler's approach to the problem is first to model the real world situation (Fig. 1, left) into a more mathematical one represented by his Fig. 1 (middle) and then generalize the problem to: "whatever the shape of the river and its distribution of arms may be, and whatever the number of bridges crossing them may be, to discover whether one can cross the individual bridges once and only once" (Euler 1736, p. 129 in Fleischner 1990, p. II.12). By considering the number of bridges leading to a given region of land and whether this number is even or uneven (I shall not go into all the mathematical details here, for a discussion of those see e.g. Biggs et al. 1976), Euler concludes[13]:

> Therefore, in every possible case, one can immediately and very easily decide, with the help of the following rule, whether or not a walk without repetition can be taken across all bridges:
> If there are more than two regions with an odd number of bridges leading to them, it can be declared with certainty that such a walk is impossible.
> If, however, there are only two regions with an odd number of bridges leading to them, a walk is possible provided the walk starts in one of these two regions.
> If, finally, there is no region at all with an odd number of bridges leading to it, a walk in the desired manner is possible and can begin in any region.
> Consequently by this given rule the posed problem can be completely solved. (Euler 1736, p. 139 in Fleischner 1990, p. II.19)

As seen from the above, already from first instance Euler begins generalizing the problem by bringing it into mathematics and giving it a life of its own which results in a much more general solution than the original problem actually requires—the Königsberg bridge problem falls in Euler's first category, since every region (or node) has more than two bridges (or edges) leading to it (see Fig. 1, middle and right), meaning that the requested stroll is not possible.

Two centuries later, with the dawn of the computer era, graph theory in its fine-tuned mathematical form (and discrete mathematics in general) found new applications.

[13] The proof for the third part of Euler's result is ascribed to Carl Hierholzer (published posthumous in 1873). For a discussion of the students' work with these proofs, see Jankvist (2011b).

Dijkstra's algorithm from 1959 solves the problem of finding shortest path in a connected and weighted graph (Dijkstra 1959),[14] and as previously mentioned, today Dijkstra's algorithm finds its use in almost every Internet application that has to do with shortest distance, fastest distance or lowest cost. Furthermore, Dijkstra also discusses and provides a solution for the problem for finding minimum spanning trees:

> We consider $n$ points (nodes), some or all pairs of which are connected by a branch; the length of each branch is given. We restrict ourselves to the case where at least one path exists between any two nodes. We now consider two problems.
> **Problem 1.** Construct the tree of minimum total length between the $n$ nodes. (A tree is a graph with one and only one path between every two nodes.) […]
> **Problem 2.** Find the path of minimum total length between two given nodes $P$ and $Q$. (Dijkstra 1959, pp. 269–270)

The problem of finding minimum spanning trees was considered in order to minimize the amount of expensive copper wire to be used in the building of computers at the time,[15] this illustrating Hilbert's point of the outer world forcing new questions upon mathematics which in this case are solved by means of a well-developed graph theory—as evident also from the terminology used by Dijkstra above.

## 10 Illustrative Example 1: Essay Assignment and an Example Student Group Answer

To illustrate how the students were brought to work actively with the three dimensions of history, application, and philosophy as well as their interplay in the teaching module, I display an illustrative example of both the formulation of an essay assignment given to the students and a student group answer of this[16]:

(a) Scrutinize Hilbert's lecture and identify the criteria he posts for what a good mathematical problem is.
(b) To what degree (and how) do the Königsberg bridge problem, the shortest path problem, and the problem of finding minimum spanning tree fulfill the criteria of Hilbert?
(c) What does Hilbert say in his lecture about the general development of mathematics?
(d) When you view the beginning of graph theory (in the form of Euler's 1736-paper) and its modern application in finding shortest path and minimum spanning trees, is this then consistent with Hilbert's description of the development of mathematics?

The following translated excerpt stems from the final hand-in of an essay assignment from one out of seven student groups in a first-year upper secondary class (students age 16–17):

> [According to Hilbert] The solution to a problem must provide new methods, new outlook, and achieve a broader and freer horizon; [It] Must be easy to comprehend; [It] Must be difficult, but not impossible; You must be able to establish the correctness of the solution.
> *The Königsberg bridge problem*: The problem is easy to understand; by using every bridge once and only once can you construct a path where all bridges are used? Finding the solution was not easy, but in no way impossible. The given solution can be understood by all with some variation in the level of

[14] For a modern textbook presentation of Dijkstra's algorithms, see e.g. Rosen (2003) or Jankvist (2011c).

[15] The problem of finding minimum spanning trees had on several occasions been solved before though: by Prim in 1957; by Kruskal in 1956; by Jarník in 1930; and by Borůvka in 1926.

[16] For further examples of essay-assignments from this HAPh-module as well as examples of mathematical tasks included in the material, see Jankvist (2011b, 2012a).

> understanding, although the solution showed that it was impossible. From the solution to this problem, new sides of mathematics were opened and provided us with a broader horizon. The problem contains all of Hilbert's criteria and for that reason is a good mathematical problem.
> *The shortest path problem*: Again the problem is easy to understand, all we have to do is to find the shortest path. Finding the solution was again difficult but not impossible, and it is possible to establish its correctness. The solution to this problem is something that can be used in many different places today and which gives us a broader horizon. […] It fulfills all of Hilbert's criteria for a good mathematical problem. (Excerpt from the hand-in by Group 3, translated from Danish)

Although Dijkstra did not argue for the correctness of his algorithm, it was shown to the students in a commentary section after the presentation of the original source how to do so.

## 11 HAPh-Module 2: Historical Emergence and Modern Application of Boolean Algebra

The philosophical theme for the second module was Hamming's (1980) comment to a paper by the physicist Eugene Wigner from 1960, where he discusses the unreasonable effectiveness of mathematics in the natural sciences (Wigner 1960). While Wigner's examples stem from the physical sciences, Hamming sets out to illustrate this unreasonable effectiveness of mathematics drawing on his own experiences from engineering—and aspects of what we today would consider to be computer science:

> During my thirty years of practicing mathematics in industry, I often worried about the predictions I made. From the mathematics that I did in my office I confidently (at least to others) predicted some future events—if you do so and so, you see such and such—and it usually turned out that I was right. How could the phenomena know what I had predicted (based on human-made mathematics) so that it could support my predictions? It is ridiculous to think that is the way things go. No, it is that mathematics provides, somehow, a reliable model for much of what happens in the universe. And since I am able to do only comparatively simple mathematics, how can it be that simple mathematics suffices to predict so much? (Hamming 1980, p. 83)

As may be seen from the above quote, Hamming approaches his question from a rather constructivist point of view, which of course rules out some of the more Platonic explanations for the effectiveness of mathematics. This can also be seen from his statement that "Indeed it seems to me: The Postulates of Mathematics Were Not on the Stone Tablets that Moses Brought Down from Mt. Sinai" (Hamming 1980, p. 86). Nevertheless, Hamming does point out that even though the standards of rigor in mathematics may change over time, and with that definitions and proofs, the mathematical results often stay intact. After having discussed the effectiveness of mathematics and what mathematics is, Hamming (1980, pp. 88–89) goes on to provide some tentative (and partial) explanations for the unreasonable effectiveness of mathematics arranged under four headings: (1) "*We see what we look for*", meaning that we approach situations with an intellectual apparatus so that in many cases we can only find what we do—Hamming provides a parable by the physicist Arthur Eddington, saying "Some men went fishing in the sea with a net, and upon examining what they caught they concluded that there was a minimum size to the fish in the sea". (2) "*We select the kind of mathematics to use*", meaning that we select the mathematics to fit the situation, and that the same mathematics does not work in every place. (3) "*Science in fact answers comparatively few problems*", referring to far from all problems in our world falling under the domains of science or mathematics. Hamming mentions the questions of Truth, Beauty, and Justice as examples to which science essentially has contributed nothing. And finally, (4) "*The evolution of man provided the model*", which refers to that even though there in the earliest forms of life must have been the seeds of our current ability to form and follow chains of close reasoning, we are still experiencing problems when having to deal with the very small or the very large in our surrounding world. Just as there are odors we cannot smell, sounds we cannot hear, and wavelengths of light we cannot see is it then not possible that there are thoughts we cannot think, Hamming asks. Based on the above, Hamming concludes that indeed mathematics is unreasonably effective, but that his explanation for this still is far from satisfactory in its account: "The logical side of the nature of the universe requires further exploration" (Hamming 1980, p. 90).

Boole's *The Laws of Thought* … from 1854 is an example of Hamming's explanation 2, that we select what mathematics fits the situation, since Boole selects the elements from standard (arithmetic) algebra that applies to his logic system, the purpose of which he describes as follows:

> The design of the following treatise is to investigate the fundamental laws of those operations of the mind by which reasoning is performed; to give expression to them in the symbolical language of a Calculus, and upon this foundation to establish the science of Logic and construct its method; to make that method itself the basis of a general method for the application of the mathematical doctrine of Probabilities; and, finally, to collect from the various elements of truth brought to view in the course of these inquiries some probable intimations concerning the nature and constitution of the human mind. (Boole 1854, p. 1)

In chapters II and III of his treatise, Boole considers the role of language in relation to the above and introduces a number of signs and laws to do so. More precisely, he introduces literal symbols $x$, $y$, etc. representing classes, and signs of operation $+$, $-$, $\times$ and the sign of identity $=$ to be used on these classes. For example, if $x$ stands for 'white things' and $y$ for 'sheep', then the class $xy$ stands for 'white sheep', similarly if $z$ stands for 'horned things', then $zyx$ stands for 'horned white things'. After associating the sign $+$ with the words 'and' and 'or', Boole deduces a number of laws which have their equivalent counterparts in standard arithmetic, for example the commutative law $x + y = y + x$, the distributive law $z(x + y) = zx + zy$, and the associative law (although this is not done as explicitly as for the others), and he deduces laws for the operation $\times$ as well. The more interesting thing, however, is Boole's observation that in the context of his investigation we have that $xx = x$ (or $x^2 = x$). If for example $x$ stands for 'good', then saying 'good, good men' is the same as saying 'good men'. Boole then draws the consequence of comparing this to standard algebra:

> Now, of the symbols of Number there are but two, viz. 0 and 1, which are subject to the same formal law. We know that $0^2 = 0$, and that $1^2 = 1$; and the equation $x^2 = x$, considered as algebraic, has no other roots than 0 and 1. Hence, instead of determining the measure of formal agreement of the symbols of Logic with those of Number generally, it is more immediately suggested to us to compare them with symbols of quantity admitting only of the values 0 and 1. Let us conceive, then, of an Algebra in which the symbols $x$, $y$, $z$, etc. admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them. (Boole 1854, pp. 26–27)

Boole's ideas went on to be adapted within mathematical logic and set theory, and the notion Boolean algebra was conceived. Some eighty years later, however, the ideas showed valuable in a very different setting than that of language and thought, namely design of electric circuits.

Shannon was a student at MIT when he got the idea for describing electric circuits by use of logic. With a set of postulates from Boolean algebra ($0 \cdot 0 = 0$; $1 + 1 = 1$; $1 + 0 = 0 + 1 = 1$; $0 \cdot 1 = 1 \cdot 0 = 0$; $0 + 0 = 0$; and $1 \cdot 1 = 1$) and their interpretations in terms of circuits ($0 \cdot 0 = 0$ meaning that a closed circuit in parallel with a closed

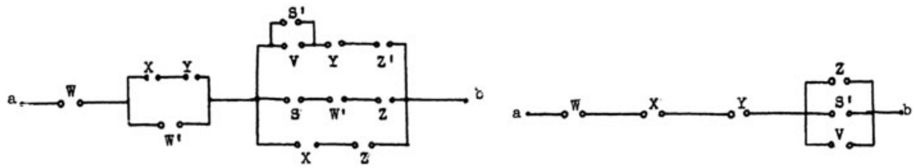


**Fig. 2** *Left* The circuit to be simplified. *Right* The simplified circuit after reductions on the hindrance function (Shannon 1938a)

circuit is a closed circuit; $1 + 1 = 1$ meaning that an open circuit in series with an open circuit is an open circuit), he was able to deduce a number of theorems which could be used to simplify electric circuits (see below). In an interview from 1987 in the magazine *Omni*, Shannon explained his use of Boolean algebra:

> It's not so much that a thing is 'open' or 'closed,' the 'yes' or 'no' that you mentioned. The real point is that two things in series are described by the word 'and' in logic, so you would say this 'and' this, while two things in parallel are described by the word 'or.' The word 'not' connects with the back contact of a relay rather than the front contact. There are contacts which close when you operate the relay, and there are other contacts which open, so the word 'not' is related to that aspect of relays. All of these things together form a more complex connection between Boolean algebra, if you like, or symbolic logic, and relay circuits.
> The people who had worked with relay circuits were, of course, aware of how to make these things. But they didn't have the mathematical apparatus of the Boolean algebra to work with them, and to do them efficiently. […] They all knew the simple fact that if you had two contacts in series both had to be closed to make a connection through. Or if they are in parallel, if either one is closed the connection is made. They knew it in that sense, but they didn't write down equations with plus and times, where plus is like a parallel connection and times is like a series connection. (Shannon 1987 in Sloane and Wyner 1993, p. xxxvi)

For a given electric circuit *a*-*b*, Shannon defined the hindrance function $X_{ab}$ to be 1 if *a*-*b* is open and 0 if closed. For example, Fig. 2 (left) has the hindrance function $X_{ab} = W + W'(X + Y) + (X + Z) \cdot (S + W' + Z) \cdot (Z' + Y{+}S'V)$, where + indicates series, $\cdot$ parallel and $W'$ is the negation of $W$. Now, by means of manipulations according to his theorems of the expression for $X_{ab}$, Shannon is able to reduce this to $X_{ab} = W + X + Y + ZS'V$, the circuit of which is illustrated on Fig. 2 (right). (For exact reductions and theorems used, see Shannon 1938b, p. 715 or Jankvist 2011d, pp. 62–63.)

## 12 Illustrative Example 2: Essay-Assignment and an Example Student Group Answer

Having been introduced to the mathematical theorems and their proofs behind the reductions of the hindrance function above as well as the mathematics of Boole, the students were given the following essay assignment[17]:

(a) According to Hamming, what does it mean that a piece of mathematics is effective?
(b) Do a comparison of the relative effectiveness of Boole's and Shannon's works (systems) distinguishing between effectiveness in terms of philosophy and effectiveness in terms of applications.

[17] For examples of students' reactions to this HAPh-module, see Jankvist (2012b).

(c) Based on your answers to the above questions, discuss different types of 'effectiveness of mathematics'. Recapitulate what Hamming means by the title of his paper 'The Unreasonable Effectiveness of Mathematics', and why it may be seen as 'unreasonable'.
(d) Do you consider Boole's introduction and Shannon's application of the idea of an algebra operating only on the elements 0 and 1 along with the mathematical interpretation of 'and' and 'or' as an example of Hamming's 'unreasonable effectiveness of mathematics'?

As an illustrative example of the students' work with this essay-assignment, I provide the following translated excerpt from one out of seven groups in the, now, third-year upper secondary class:

> According to Hamming a piece of mathematics is effective when it can describe and predict natural phenomena. He finds mathematics puzzling in the sense that it can describe Nature using relatively simple formulas and expressions, practically without doing any experiments. […]
> The philosophical effectiveness we see with Boole is connected to thoughts and the philosophy behind mathematics. It can say quite a bit about how we understand and confine mathematics with axioms [laws], for example when Boole makes it an axiom that $x$ must be either 1 or 0. On the basis of philosophy he concludes something about mathematics' ways of thinking and methods. Shannon, however, on the basis of Boole's philosophical effectiveness, uses a method leading to his own effectiveness regarding application when transferring the [mathematical] theories to real life, where he at the same time tests them and thereby obtains an evaluation of the effectiveness. This is seen from the system [of electric circuits] on which his theorems are used.
> What Hamming believes is that mathematics is unreasonable because you are able to describe real events by simple mathematics and that we as human beings are finding it difficult to comprehend that apparently there are no limits to the range of mathematics regarding use in everyday life. By 'unreasonable' he means that it seems illogical that nature can be described by such simple operations, since nature for us seems complex, incomprehensible, and unpredictable—i.e. that we 'just' cannot plot them on a graph and get a result.
> Yes, we do consider Boole's introduction and Shannon's application as being examples of the unreasonable effectiveness of mathematics, since Boole shows that mathematics can [help] explain composition of language and that you can translate language directly into mathematics. In addition to this, Shannon shows, by means of Boole's introduction, that [the idea of using] the elements 0 and 1 can be applied on a circuit and hereby find the most simple [circuit]. These two examples fall outside what is usually considered the main field of mathematics and should, following Hamming, be described by more complex systems—but since this is not the case, it is natural, according to Hamming's line of thought, to call this mathematics unreasonable. […] (Excerpt of the hand-in by Group 2, translated from Danish)

## 13 Discussion of Design and Use of Primary Original Sources

Although not correct in every possible aspect, it does appear, from the two illustrative examples of excerpts from the student groups' hand-in essay answers, that the students develop their awareness regarding the history, applications, and philosophy of mathematics to some degree through the setting of the modules. In this section, I shall discuss the design of the modules a bit further through the KOM-project's framework of overview and judgment (OJ) as well as Barbin's (1997) three general ideas describing the special effects of using original sources and along the way draw in the illustrative examples when possible. Also, I shall elaborate a bit more on the setting of the implementation of the modules in order to support the discussion of the design problématique in general. Finally, I shall discuss possible 'results' for the questions of design raised earlier in the article.

Regarding the KOM-project's OJ1—"the actual application of mathematics in other subject and practice areas"—the students are provided with actual examples of applications of mathematics: Dijkstra's algorithms in computer science, which are used in a variety of Internet applications today; and Shannon's logical description of electrical circuits in engineering, which finds it use, for example, in computer hardware.

The second kind, OJ2—"the historical evolvement of mathematics, internally as well as in a social context"—is dealt with specifically in both HAPh-modules. HAPh-module 1 illustrates how a puzzling question from everyday life (in Königsberg) spurs on the development of an entirely new branch of mathematics (graph theory), as well as how mathematicians (in this case, Euler) are able to transform such everyday life questions into purely mathematical problems and deal with these inside mathematics itself. HAPh-module 2, on the other hand, illustrates the case of applying mathematics within a different discipline, language, and how considerations about language leads to an entirely new mathematical construct, namely that of Boolean algebra. Furthermore, also the application-oriented aspects of the modules are embedded in their historical settings due to the use of original sources.

As for OJ3—"the nature of mathematics as a subject"—this comes into play on several levels. Firstly, the original texts by Euler, Dijkstra, Boole, and Shannon illustrate aspects of the characteristics of mathematical problem formulation, thought, and methods as well as provide the students with examples of how mathematics is constructed, the kind of results it produces, and not least possible applications of these. Secondly, the original texts by Hilbert and Hamming, respectively, introduce the students to quite different lines of thought than they are usually exposed to in their mathematics classes. Nevertheless, the students generally approached these texts with an open mind, taking in many of the messages, as seen from the essay excerpts above, and showing a preparedness to discuss the issues being raised.[18] In particular the second excerpt bears witness to this, for example when the students mention Shannon's opportunity to immediately evaluate the effectiveness of his system as well as their final remarks about complex situations being described with simple mathematical ideas, aspects neither of which were discussed in the teaching material of the module. Based on the above samples of essay answers alone, however, it could be argued that the formulation of the essay questions along with the tight structure of the modules may endanger the students' reflections to be overly determined. But interviews with the students after the implementations of the HAPh-modules provide evidence that genuine philosophical reflections—and 'philosophizing'—on the students' behalf did take place in one form or another. For further discussion and display of data regarding this aspect of the study, see Jankvist (2012a, b).

That the second student group excerpt appears more elaborating in nature than the first may have to do with the fact that the students were one and a half year older by the implementation of HAPh-module 2. The empirical setting consisted of the implementation of the two HAPh-modules in the same upper secondary class, the first in the spring of 2010 and the second the fall of 2011. The class consisted of 27 students who by the time of the second module were 18–19 years of age.[19] The modules were designed so that the students read the original texts as homework and then worked on the guiding tasks and the essay assignments in groups during classes, having the possibility of consulting their own teacher

[18] As part of the data collection, I video recorded one particular group of students (Group 7 out of seven) while they worked on the mathematical tasks and essay-assignments of the modules, allowing me an insight into this group's discussions prior to putting their answers down on paper.

[19] In fact, the research study involved a following of this class of students for a two-year period, during which they were given three questionnaires, 1 year apart, and interviewed afterwards in order to evaluate possible developments of their awareness, including beliefs/views/images (Jankvist 2009d), in relation to the KOM-project's three types of overview and judgment. For a preliminary analysis, see Jankvist (2012a).

if needed. Since upper secondary mathematics teachers are not always equally familiar with aspects of history, applications, and philosophy of mathematics, some coaching of the teacher(s) were done before and during the implementations.[20] Besides the hand-in student group essay assignments, the data collection also consisted of a collection of students' hand-in individual mathematical tasks and a test/questionnaire related to both the mathematics and aspects of history, application, and philosophy after the implementation.[21]

Of course, not every one of the example questions of overview and judgment from the KOM-project (cf. earlier) can be dealt with in every HAPh-module. The KOM-projects's description of the three types of overview and judgment is a normative one, meaning also that when put into practice not all of the ends may be reached, in particular since the topics to be discussed are dependant of the actual mathematical case(s) chosen. In fact, as pointed out by one of the reviewers of this manuscript, the KOM-project provides very ambitious standards, which, if taken absolutely literary, may be close to unreachable in practice. Therefore, in order to realize the idea of the HAPh-modules, including also that these can take up only a certain amount of teaching time, some compromises had to be made. For example, as the reader will have noticed, there is a large time span between the historical sources of Euler and Boole and those on application by Dijkstra and Shannon, respectively. In the material this was dealt with by giving commentary prologues and epilogues to the sources, explaining here the historical setting of the source as well as providing a brief account of the intermediate historical evolution of relevant mathematics having taken place.

As mentioned, with reference to Jahnke et al. (2000), original sources are usually considered to be some of the most inaccessible to students, but also some of the most rewarding when they actually succeed. Besides the method of 'guided reading', as previously described, another didactical method of trying to make the students' reading of the sources more accessible was to choose cases involving novel mathematical ideas. By this I mean that the historical cases of the HAPh-modules have been selected in such a manner that they represent the very beginning of a mathematical sub-discipline, i.e. the beginning of graph theory and Boolean algebra by Euler and Boole, respectively. Because the original texts by Euler and Boole each represent novel mathematical ideas, at the time, the authors are often more careful in explaining their line of thought and reasoning, which makes the arguments easier to follow.[22] And, quite importantly, practically no mathematical requirements are needed beforehand on the students' behalf to study these texts, because the authors more or less develop the conceptual apparatus as they go along.

The illustration of the development of these conceptual apparatus may also assist in 'reorienting' the students' views regarding how mathematical concepts and constructs come into being. According to Jahnke et al. (2000, p. 292), what often happens in teaching is that "… concepts appear as already existing" and "… are manipulated with no thought for their construction." For the cases dealt with in the two HAPh-modules, students saw how various constructs now belonging to graph theory and Boolean algebra were brought to life in much contextualized manners, i.e. those of solving the Königsberg bridge

[20] For a discussion of Danish upper secondary school teachers' attitudes toward history, application, and philosophy (or the three types of OJ) in teaching, see Jankvist (2009d).

[21] In Danish upper secondary school students have a national final written exam and a local final oral exam. Activities such as the HAPh-modules have the possibility of being part of the oral exam, if the teacher decides so.

[22] From a history point of view, such selection of sources of course also significantly reduces the problems associated with Whiggism.

problem and formulating a logic system capable of describing the “laws of thought”. There is in this setting also the possibility of a ‘replacement’, since the cases described “… allows mathematics to be seen as an intellectual activity, rather than just a corpus of knowledge or a set of techniques” (Jahnke et al. 2000, p. 292). All four mathematical texts (Euler, Boole, Dijkstra, and Shannon) illustrate how we go from an extra-mathematical problem (taking a stroll in a certain way, describing language with logic, finding shortest path, and simplifying electric circuits) to first formulating this in mathematical terms to then solving it by means of mathematics—what Freudenthal (1991) refers to as horizontal and vertical mathematizing, respectively—whether this mathematics is created in that process or is already available. Barbin’s third effect of using original sources, ‘cultural understanding’, concerns the invitation that original sources give us to “… place the development of mathematics in the scientific and technological context of a particular time, and in the history of ideas and societies” (Jahnke et al. 2000, p. 292). The texts by Dijkstra and Shannon, in particular, place mathematics in technological contexts i.e. that of computer related mathematics in the 1930s and late 1950s, respectively. Further, the very topics addressed in the texts by Hilbert and Hamming are concerned with history of ideas, the relationship between mathematics and the extra-mathematical world, including societal needs, technological breakthroughs, etc.

## 14 Conclusions

Let us return to the initially outlined ‘problématique’ and discuss possible conclusions and ‘results’. First of all, the above discussion illustrates that a well chosen selection of primary original sources may assist us in reaching the ‘whys’ related to using history, applications, and philosophy as *goals* in mathematics education. Next, the two HAPh-modules and their design show that it can make sense to address all three dimensions of history, applications, and philosophy in one and the same teaching module. As argued, this of course is related to the observation that because the purposes, or ‘whys’, of using history, applications, and philosophy as goals are similar in nature, then these goals may be reached through the same ‘how’, that is to say a modules approach.

In a sense, the very design of the HAPh-modules is a ‘result’ in itself, that is the idea of addressing all three dimensions of history, applications, and philosophy of mathematics in the same activity, realizing the goal arguments of including these three dimensions through a modules approach, having one original text for each dimension, relying on the method of guided readings of these, choosing texts which represent cases involving novel mathematical ideas, and using essay assignments to have student groups work on the involved meta-issues of these cases. Furthermore, the idea of the HAPh-modules may, on the one hand, illustrate how (parts of) the normative goals associated with the KOM-project’s overview and judgment (OJ) may be realized and operationalized in practice, and, on the other hand, it may provide input to the discussion of how to possibly rethink mathematics education so that the inclusion of such dimensions as history, applications, and philosophy does not become shallow and perfunctory (cf. Fried 2010).

Finally, the two illustrative examples taken from student groups’ hand-ins do render it probable that such HAPh-modules, with their emphasis on meta-issues, may assist in the development of students’ awareness of these three dimensions (history, applications, and philosophy as well as their possible interplay) of mathematics as a discipline. Of course, this topic should be researched further in order to describe the nature of the development of the students’ awareness as well as the nature of the awareness itself—which I also plan to at a later point in time, using the data material collected from the two implementations (but for now, see Jankvist 2012a). Nevertheless, it is clear that students' awareness regarding such matters does not develop on their own, in a sense it must be 'fertilized', which includes providing the students with concrete examples of mathematics in support of a development of awareness. And not least, a setting for this to take place in must be given. The HAPh-modules, and the idea behind the design of these, is but one way of providing such a setting—although it appears a promising one at such.

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