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Using History to Teach Mathematics: The Case of Logarithms

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Abstract Many authors have discussed the question *why* we should use the history of mathematics to mathematics education. For example, Fauvel (For Learn Math, 11(2): 3–6, 1991) mentions at least fifteen arguments for applying the history of mathematics in teaching and learning mathematics. Knowing *how* to introduce history into mathematics lessons is a more difficult step. We found, however, that only a limited number of articles contain instructions on how to use the material, as opposed to numerous general articles suggesting the use of the history of mathematics as a didactical tool. The present article focuses on converting the history of logarithms into material appropriate for teaching students of 11th grade, without any knowledge of calculus. History uncovers that logarithms were invented prior of the exponential function and shows that the logarithms are not an arbitrary product, as is the case when we leap straight in the definition given in all modern textbooks, but they are a response to a problem. We describe step by step the historical evolution of the concept, in a way appropriate for use in class, until the definition of the logarithm as area under the hyperbola. Next, we present the formal development of the theory and define the exponential function. The teaching sequence has been successfully undertaken in two high school classrooms.

1 Introduction

The initial reason for teaching of logarithms was their usefulness in doing complicated numerical calculations more quickly and easily. To this end many instructional hours were spent on teaching the use of logarithmic tables. This reason no longer applies, since calculators and computers have simplified the problem of calculations with large numbers. So, reality has imposed a change of the objectives and content of teaching the unit on logarithms. The computational part of the theory of logarithms was pushed aside. The logarithmic tables, that for centuries were the tool for each serious calculation, have passed

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into history. As Guedj (1998: 503), professor of history of sciences at Paris VIII University, says "*even the Mathematics are getting old*". The functional part of the concept has progressed into the limelight. The study of the properties of the logarithmic and exponential functions will always remains an important subject, because these functions describe many natural and social phenomena, such as radioactive decay, evolution of populations, and spread of contagious diseases.

Thus, textbooks, up to 25 years ago, began the introduction of logarithms with the definition given by Leonard Euler (1707–1783) in his famous book *Complete Introduction to Algebra* (St. Petersbourg 1770). Euler's definition, in modern form, goes as follows: "*If* $x > 0$*, the logarithm of* x *to base* a $(a > 0,\ a \neq 1)$*, is the real number* y *such that* $a^y = x$ *and is symbolized with* $y = \log_a x$" (Euler 1770/1984: 63–64).

In modern textbooks the above definition is usually repeated but they stress the functional form of logarithms: "*The base a logarithmic function* $(a > 0,\ a \neq 1)$ *is the inverse of the base a exponential function. If* $f(x) = a^x$ *then* $f^{-1}(x) = \log_a x$" (e.g., Brown 1992: 193).

However, both then and now, we encounter the following difficulties and questions as we teach this unit to 11th grade students:

- How can we explain the meaning of a^x when x is irrational, so that the definition of logarithm is understandable?
- How will we justify the exponential law $a^x \cdot a^y = a^{x+y}$, a law essential for the proof of the logarithmic properties?
- How will we demonstrate that for each $y > 0$ there is a number x such that $a^x = y$?
- Why and how we were led to the definition of logarithm?
- While the term exponential function for $f(x) = a^x$ does appear justified, what is the origin of the term logarithm?
- How were we led to the eminent number $e \cong 2.7182 \ldots$ as a limit of the sequence $(1 + 1/n)^n$.
- Why the logarithms to base e named *natural* and why they are particularly useful in comparison with others?
- How were logarithms calculated?
- Why should we continue teaching logarithms now that the initial reason for teaching them no longer applies?

There is no simple way to address the first three questions and for the other ones it is impossible to give an answer as long as we follow a treatment of logarithms in which the initial grappling with the material has been obliterated; this is the case in several of the schools textbooks I examined in preparing this paper. If we wish students to comprehend the importance of the theory of logarithms we have to follow the history of its creation. Katz (1995) and Fauvel (1995) have also recognized the importance of teaching logarithms by using history and proposed some classroom activities.

We describe briefly the organization of this paper. In Sect. 2 of the paper we follow the historical path to the creation of logarithms. The principal purpose in Sect. 2.1 is to highlight *the difficulty in doing the basic operations* in early numerical systems but even in the decimal system of today in the case of many-digits numbers. In Sect. 2.2 we establish a *correspondence between arithmetic and geometric progressions*, from which arose the concept of logarithm. We illustrate that computations with numbers appearing among the terms of the geometric progression are greatly simplified by this correspondence. The problem was that any two given numbers are unlikely to be found in any given geometric progression. In Sect. 2.3 we present how Napier built a suitably dense geometric progression in order to be useful for practical computations and employed the key idea of the above correspondence to construct his logarithmic tables. In Sect. 2.4 we discuss the surprising *connection between the logarithm and the hyperbola area* noticed by de Sarasa reading through a work of Gregory of St. Vincent. This recognition served to stimulate the study of hyperbolic areas and these investigations led to the densest geometric progression and turned logarithms from a practical to a theoretical tool.

In Sect. 3 we illustrate how to use the work of Gregory of St. Vincent on the hyperbola $y = 1/x$ to define natural logarithms and then derive their major algebraic properties. The discussion also shows how the mysterious number e is determined as the limit of the sequence $(1 + 1/n)^n$, $n \in N$ in a quite natural way. In the end we use logarithms to define the exponential function.

In the final section, we discuss some arguments and related goals in favour of teaching logarithms via their history.

2 The Historical Path to the Creation of Logarithms

2.1 First Attempts to Convert Multiplication into Addition

Students may not appreciate how difficult it was in the past the problem of performing the four arithmetic operations because we now have a suitable numerical system and there are algorithms to perform the calculations. They recognize, perhaps, how laborious and boring it is to perform multiplications and divisions with many-digits numbers and calculating square and cubic roots, since these operations require a lot of time and are prone to error. However, no operation poses a serious problem if one can use a calculator. This fact undermines the real understanding of the problem.

A good way for students to understand the problem and not downgrade the achievements of the past should be to engage them in some classroom activities on calculations in ancient Egypt, Babylon and Greece as well on calculations with large numbers in our decimal system before the invention of the known algorithms. The extent to which one would discuss these matters depends on the available time. To restrict the length of this article, we avoid giving ready made activities. To construct activities dealing with ancient systems we refer the interested reader to the material in Kline (1972), Struik (1948/1987), Van der Waerden (1961), and the most elaborate work of Resnikoff and Wells (1984).

The Indian-Arabic positional system of numeration certainly simplified calculations in the usual daily circumstances, but, however, did not suffice for scientific calculations. In astronomy, the demands for increasing precise predictions made the calculations difficult and time-consuming. By the end of the 16th and the beginning of the seventeenth century, the difficulty that everyone met in the multiplication of large numbers led to the invention of several techniques to perform this operation. The idea behind these techniques is to convert the multiplication problem into an addition problem. We suggest here an activity based on the '*jalousie method*' (e.g., Smith 1923/1958: 114–117) or on the '*Napier's method of rods*' (e.g., Coolidge 1990) and another activity on '*prosthaphaeresis*' (e.g., Pierce 1977).

By the end of these activities, we pose the questions: *By these methods can you calculate powers? Can you perform divisions? Can you calculate square roots*? The answers will show that our problem has not been solved.

2.2 Converting Multiplication into Addition by Comparing Arithmetic and Geometric Progressions

The invention of logarithms sprang up from the comparison between arithmetic and geometric progressions that had repeatedly attracted the attention of mathematicians. The explanations we find of the rule about the multiplication of the terms of a geometric progression by addition of their ranks have a long prehistory; Smith (1915) mentions a whole list of sixteenth century books of arithmetic where this ability is noted.

The relationship of the two progressions is exemplified by a table of numbers given by the French mathematician Nicolas Chuquet (c.1440–c.1488) in his *Triparty en la Science des nombres* (Lyon 1484) (e.g., Flegg et al. 1985: 247–249; Fauvel and Gray 1987: 248).

Chuquet notes that multiplication between two terms (numbers) of the geometric progression can be reduced to the addition of the respective denominations (numbers of their 'natural order') in the arithmetic progression. For example $2 \times 4 = 8$ corresponds to $1 + 2 = 3$. Chuquet describes the correspondence as follows: "…*Whoever multiplies* 2^1 *by* 4^2*, it comes to* 8^3*. For 2 multiplied by 4 and 1 added with 2 makes* 8^3*. And thus whoever multiplies first terms by second terms, it comes to third terms… And whoever multiplies 128 which is the 7th proportional by 512 which is the 9th, it should come to 65536 which is the 16th*" (Fauvel and Gray 1987: 248) (Table 1).

While this passage is useful to grasp the idea of simplifying multiplications, it is noticeable that Chuquet does not show any interest for this application. Actually, his aim was to define multiplication (and division) of monomials (of one variable). Note that in the previous passage the raised numbers, for example 3 in 8^3 is not a power of 8 but registers the denomination (the rank) of 8 in that place; we would write the expression as $8x^3$.

Table 1 The table of Chuquet (Fauvel and Gray 1987: 248

Denominations	Numbers
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024
11	2048
12	4096
13	8192
14	16384
15	32768
16	65536
17	131072
18	262144
19	524288
20	1048576

Table 2 The table of Stifel (Kouteynikoff 2006: 19)

Exponent	−4	−3	−2	−1	0	1	2	3	4	5	6
Number	1/16	1/8	1/4	1/2	1	2	4	8	16	32	64

The product of two monomials can be found by multiplying the ‘numbers’ (the coefficients of monomials) and adding their ranks (the denominations). In modern terms: $2x^1 \times 4x^2 = (2 \times 4)x^{(1+2)}$. He uses then these rules to solve some forms of equations (Cajori 1913: 13; Kouteynikoff 2006: 15–16).

An outstanding treatment of the conversion of the basic operations to simpler ones was given by the remarkable German mathematician Michael Stifel (1486–1567). In his *Arithmetica Integra* (Nuremberg 1544) employs the same progressions and states quite clearly that addition in arithmetical progression corresponds to multiplication in geometrical progression, that subtraction to division, multiplication to the finding of powers and division to the extracting of roots (Table 2) (e.g., Kouteynikoff 2006: 17; Smith 1915: 86).

It is important that Stifel goes onto examine the ‘formal’ way of forming differences and extends this correspondence ‘to the left’. He notes that in the same manner that we place whole numbers after the unit and fractions of unit before it, we place the unit and the whole numbers after 0 and the ‘fictitious’ unit and numbers before it. This time, as he remarks, it is the geometric progression that serves the arithmetic one; multiplications and divisions involving fractions help him to explain additions and subtractions of the fictitious numbers. For example, like 1/8 dividing 64 gives 512, thus −3 subtracted from 6 gives 9, namely $6 - (-3) = 9$; however, 9 is the exponent of 512 (e.g., Kouteynikoff 2006: 20).

However, Stifel, like Chuquet, used the correspondence between series to calculate with monomials and to solve algebraic equations. The originality of Stifel was the rule A.M.A.S.I.A.S., an algorithm for the solution of any second degree equation (Kouteynikoff 2006: 18).

Giving Table 2 (extended to the right) to students, we will ask them the question: *Is there any relation that would connect the two rows*? Most likely, in the first line, one recognizes a simple numeration rather than an arithmetic progression. And naturally, it is more difficult for the beginners to recognize that the list answers the question: *To which exponent the number 2 should be raised in order to obtain a given number*? In any case, students will be led to observe that the product of two terms in the geometric progression (i.e., $32 \times 256 = 8192$) is found precisely under the sum of the corresponding terms in the arithmetic ($5 + 8 = 13$). That is to say, multiplication is essentially reduced to addition.

$$\begin{tikzcd}
5 + 8 \arrow[r, dotted] & 13 \arrow[d, dotted] \\
32 \text{ x } 256 \arrow[u, dotted] \arrow[r] & 8192
\end{tikzcd}$$

They will realize that division is reduced to subtraction:

$$4096 : 128 = 32 \dashrightarrow 12 - 7 = 5,$$

raising to power is reduced to multiplication by the exponent:

$$16^3 = 4096 \dashrightarrow 4 \,.\, 3 = 12,$$

and the extraction of any root is reduced to division by the index $\sqrt[4]{4096} = 8 \dashrightarrow 12 : 4 = 3.$

Before continuing, we would ask the students to make some more computations by using Table 2.

One year later Girolamo Cardano (1501–1576) published the very important book *Ars Magna* (Nuremberg 1545) which contains the first signs of modern algebra. However, at that time, in fact up to the beginning of seventeenth century, there did not exist a broadly acceptable symbolism for powers and it is indeed one of the paradoxes in mathematics that logarithms were invented long before the prevalence of the powers. Stifel, despite lucking the modern symbolism to write the relation $2^{m} \cdot 2^{n} = 2^{m+n}$, named the terms of the arithmetic progression '*exponents*' of the corresponding terms of geometric progression. Later the terms of the arithmetic progression were named '*red numbers*', since they were printed in red ink in the tables. Even Napier, before the use of the term *logarithm*s, named the terms of the arithmetic progression '*artificial numbers*'. The term logarithm means precisely: the number that measures the ratios (the Greek *logos*). Indeed, number 6, in Table 2, that corresponds to 64 shows '*how many ratios* (*logos*)' are required in the *continuous proportion*

$$\frac{2}{1}=\frac{4}{2}=\frac{8}{4}=\frac{16}{8}=\frac{32}{16}=\frac{64}{32}=\dots$$

to reach the term 64 (at the time of Napier, *a geometric progression was defined as a sequence of numbers that are in continuous proportion*).

Today, it is obvious that the terms of the arithmetic progression are the base-2 logarithms of the corresponding terms of the geometric progression. At this stage we would introduce students to the modern symbolism, writing $\log_2 1 = 0$, $\log_2 2 = 1$, $\log_2 64 = 6$, $\log_2 256 = 8$, …, as well as the properties $\log_2 (a \cdot b) = \log_2 a + \log_2 b$, $\log_2 (a/b) = \log_2 a - \log_2 b$, $\log_2 (a^n) = n \log_2 a$.

We would now ask the question: *Given this table, can we multiply any numbers*? Almost certainly the answer will be negative; otherwise we might ask the students to compute, with the help of the table, the product 27 × 243. Consequently, the multiplication has been turned into addition only for certain privileged numbers. In order to increase our possibilities, *we can ask the students if something similar is valid for powers with base 3 and examine if this is generalized to every geometric progression.*

The answer to both questions is affirmative. It is simple to check it for powers of 3. This makes the calculations with these numbers easier. Certainly this will be true as well for calculations with integers that are powers of 4, powers of 5 and so on. So, our computational efficiency is improved. With regards to the generalization, consider the arithmetic progression of the non-negative integers and an arbitrary geometric progression beginning with 1 and having common ratio $r > 1$.

$$\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots \\ 1 & r^1 & r^2 & r^3 & r^4 & r^5 & r^6 & r^7 & \dots \end{array}$$

It is easy to recognize once again that multiplication in the geometric progression corresponds to addition in the arithmetic one. The terms of the arithmetic progression are the base-r logarithms of the corresponding terms of the geometric progression. Very well! But what does this mean in practice? *Can we multiply any two numbers*? Yes, provided that we have an infinite catalogue of geometric progressions and that we are lucky enough to see these two numbers appear among the terms of one of these progressions!!

These remarks would have practical value if we had a geometric progression sufficiently 'dense', so that we could find among its terms any two numbers or at least, the numbers that appear often in computations (at that time, the values of trigonometric functions). One way to increase the density of the geometric progression 1, r, r^2, … is to insert, for example 500 terms, between 1 and r, and at the same time find the corresponding 500 terms between 0 and 1 in the arithmetic progression. Then the logarithm of the nth inserted term in the geometric progression is equal to the nth term of the arithmetic. If instead of 500 terms we insert 5000, then we would have the possibility of finding the logarithms of many more numbers. The need for some denser geometric progression was met through the invention of logarithms by Napier, as we shall see in the next section.

2.3 A Denser Geometric Progression

The late sixteenth century was an age of numerical computation, as developments in astronomy and navigation called for complicated trigonometric computations. The prosthaphaeretic rules were the principal means used in the major astronomical laboratories of Europe to simplify multiplication. We have seen in Sect. 2.2 that writers like Stifel had called attention to certain relations between the terms of a geometric progression and the terms of an arithmetic one. But those writers did not realize the possibilities of this idea nor did they conceive and execute the plan of computing a pair of corresponding progressions sufficiently dense for practical use in computation. The clear understanding of this idea led John Napier and others to the logarithms about the turn of seventeenth century.

The Napier family belonged to the so called aristocracy of landowners. John Napier was born at Merchiston, near Edinburgh in 1550, when his father was only 16 years old. He died on 4 April 1617 in Edinburgh, three years after the announcement of his great invention. Little is known about the early years of his life. He entered in 1563 St. Salvator's college at St. Andrews University. Napier spent some time there but his name does not appear in the list of those graduating in the following years, so he must have left for continental Europe before completing his studies. It is highly likely that he studied at the University of Paris and also that he spent some time in Italy and in Holland, but there are no records to corroborate this.

In 1571 he returned to Scotland as a scholar competent in Greek, but without any degree in any science. The lack of information with regard to his studies does not allow any answer to the question: Why and when did he engage in mathematics? To simplify the computations he devised a mechanical means that became known as *rods* or *bones* of Napier (see Sect. 2.1). In 1614 Napier published in Edinburgh his work *Mirifici Logarithmorum Canonis Descriptio* that contained tables and instructions on how to use them but there were no proofs of the statements. In a second work, published posthumously in 1619 and titled *Mirifici Logarithmorum Canonis Contructio* Napier gives the proofs of the statements and also the steps for the constructions.

Napier wanted to simplify multiplication and division of sines. Perhaps for this reason he initially limited to logarithms of sines of angles. In his time, the sine of an angle t was not defined as a ratio but as half the length of the chord subtended by the angle $2t$. The sine of t was taken to be AD (Fig. 1). Of course, this length depends on the radius of the circle. The choice of the radius indicated the precision of the calculations, given that with a large radius of one's choice, the trigonometric values can be both integers and as accurate as one wants (or can calculate!). Napier took a radius equal to 10^7, because the table of sines that he had at his disposal gave the numbers up to seven digits. The sines he sought to provide simpler calculations for were, in our terms, integer approximations to the numbers $10^7 \sin t$, where t varies from 0° to 90° and is given in minutes since the instruments of the time did not allow better precision. Because $10^7 \sin 90° = 10^7$ and $10^7 \sin 1' = 2909$, we conclude that his sines were integers lying between 2909 and 10,000,000.

Fig. 1 $\sin t = \mathrm{AD}$

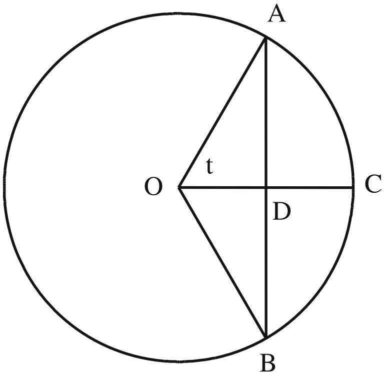

Table 3 The two progressions that Napier used originally

A.P. (b_n)	0	1	2	3	–	n	–
G.P. (a_n)	10^7	$10^7(1-10^{-7})$ $\|$ 9999999	$10^7(1-10^{-7})^2$ $\|$ 9999998	$10^7(1-10^{-7})^3$ $\|$ 9999997	–	$10^7(1-10^{-7})^n$	–

Napier began with a comparison of arithmetical and geometrical sequences. He considered a geometric progression with first term $a_0 = 10^7$ and common ratio $r = 1 - 10^{-7} = 0.9999999$. The choice of common ratio close to 1 was an intelligent idea because it made the progression sufficiently dense, so that the gaps between successive terms would remain small. Then, Napier set down, side by side, the terms of the geometric progression $a_n = 10^7(1 - 10^{-7})^n$, $n \in N$ and the terms of the arithmetic progression $b_n = n$, $n \in N$ (Table 3). Initially, he called the numbers b_n the logarithms of the numbers a_n, $n \in N$.

Napier should calculate over 81,000,000 terms of the sequence (a_n) to reach 2909 and then the logarithms of all the sines between them by interpolation. This meant that one had to perform an overwhelming volume of calculations. But Napier observed that it was enough to calculate the terms down to $10^7 \sin 30° = 5{,}000{,}000$. For if $b < 10^7/2$ we can find a natural number m such that $a = b \cdot 2^{\,m} \geq 10^7/2$. Then, one can apply the logarithmic properties to determine the logarithm of b from the logarithms of 2 and a. After this observation Napier had to calculate about 7,000,000 terms instead of 81,425,000!

Indeed, the work was decreased considerably but still remained a lot to do. However, the intelligence of Napier led him to a new invention. He calculated only 100 terms of the above geometric progression. Then, he continued making jumps of 100 terms. To this end, he used another geometric progression which starts from $a_0 = 10^7$, but has common ratio $r = 1 - 10^{-5}$. The second term of the new sequence is 9,999,900 and it is approximately equal with the 100th term of the previous one. Thus the table becomes less dense after the hundredth term. However, it would still require over 69,000 steps to reach 5,000,000. But, Napier had no any intention to continue this way. He calculated only 50 terms of the new progression. Then, he continues with bigger jumps doing the same with 69 geometric progressions of common ratio $1 - 5 \cdot 10^{-4}$ and 21 of common ratio $1–10^{-2}$, always being careful that his geometric progressions "tie up" one into the other (the last term of the first progression is the second term of the second one and so on). This tactic enables him to get to 5,000,000 in 1600 steps, a number far distant from 81,450,000!! (e.g., Ayoub 1993; Coolidge 1990; Edwards 1979: 144–146).

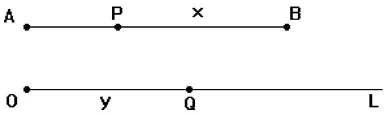

Fig. 2 Napier's definition of logarithm: N log $x = y$

Now Napier had to calculate the logarithms of these 1600 reference terms (apart from the first 100). Edwards (1979: 146–147), for purpose of simple illustration, has outlined a reconstruction of the table of logarithms of the 1600 reference terms using *linear* interpolation. However, Napier was perfectly aware of the non-linearity of the logarithmic function and therefore employed a shrewder method of interpolation. For the purpose of this non-linear interpolation, Edwards (*ibid.*: 148) writes: "… *Napier required a continuous definition of the logarithm function, rather than a discrete definition based on geometric progressions*". This may answer Coolidge's (1990: 72) wonder: "… *it is hard to see why he (Napier) set up his logarithms in the clumsy fashion which I shall describe presently*, …".

Napier's curious definition of logarithms was based on a geometric model in which he conceived two correlated points moving along two different lines. The first point P starts at the initial point A of a segment AB of fixed length 10^7 with initial speed 10^7, and move toward B, with its speed decreasing (from 10^7 at A to 0 at B) in such a way that it always equals the remaining distance PB from its ultimate goal. The second point Q starts at the initial point O of a ray OL, and moves to the right with constant speed 10^7 (Fig. 2).

At $t = 0$, let the two points start moving at A and O respectively. Napier proves that as P moves geometrically, Q moves arithmetically (e.g., Ayoub 1993: 355).

If at a time t, the point P is at a distance x from B and Q at a distance y from O, then Napier defines the segment $OQ = y$ to be the logarithm of the segment $PB = x$.

We shall write

$$y = \mathrm{N}\log x$$

to distinguish this function from the standard logarithm. Napier considers the same 1600 reference points, as above, calculates first their logarithms and then the logarithms of sines of angles between 0° and 90° at intervals of one minute by means of a subtle interpolation scheme. But now the logarithm of $a_n = 10^7\ (1 - 10^{-7})^n$ is approximately n and not exactly n as it was originally. For example, N log $(a_1) = 1.00000005$, Nlog $(a_{100}) = 100.000005$ instead of 1 and 100 correspodingly. For further details we refer the reader to the excellent article of Ayoub (1993).

In class, we should not enter in so many details, which would probably disorientate and confuse the students and make them forget that our objective is to place in correspondence the terms of an arithmetic progression with the terms of a dense geometric progression. It is preferable to pose the problem: *Given the geometric progression* a_0, $a_0\ r$, $a_0\ r^2$, *…, we require each of the terms to be close to its neighbours. How will we achieve it? Does it depend from* a_0? *Does it depend from* r? After a little experimentation, we inform the students that Napier considered the geometric progression with first term $a_0 = 10^7$ and common ratio $r = 1 - 10^{-7} = 0.9999999$. This is in reality the first of the geometric progressions he used. The choice of r is already justified but you should explain briefly the reason for the choice of the first term. Then, it is correct with great accuracy, in view of the previous paragraph, to say that Napier determined a correspondence between the terms of the geometric progression $a_n = 10^7\left(1 - \frac{1}{10^7}\right)^n$ and the terms of the arithmetic progression $b_n = n$, $n \in N$ (Table 3); that is,

$$n = \mathrm{N}\log a_n \Leftrightarrow a_n = 10^7\left(1 - 10^{-7}\right)^n$$

Here we can ask the question: *Is the standard logarithmic property log AB = log A + log B valid in Napier's system? Could you estimate Nlog1? Is it zero or not? Does this system have a base?* This is a chance for the students to work experimentally or, alternatively we can help them answer the questions using a work sheet with portions of this system.

Obviously, we have Nlog1 $\neq$ 0. As a consequence of this the basic logarithmic property Nlog $(AB) = \mathrm{N}\log A + \mathrm{N}\log B$ fails and instead of it we have (e.g., Ayoub 1993: 356; Burn 2001: 5):

$$\mathrm{N}\log(AB) = \mathrm{N}\log A + \mathrm{N}\log B - \mathrm{N}\log 1.$$

Napier had perceived this computational inconvenience and near the end of his life he suggested to the English mathematician Henry Briggs (1561–1631), professor at Gresham College in London, that the process could be simplified considerably if 1 was a term of the geometric progression and matched with 0 in the arithmetic one. This, as well as the adaptation to a decimal base, was subsequently completed by Briggs. He published his *Arithmetica Logarithmica* (London, 1624) which contained a usable table of logarithms to base 10, with 14 decimal digits, of the first 20 thousands natural numbers and the natural numbers between 90,000 and 100,000. This book was translated from Latin into French from the Dutch bookseller and editor Adriaen Vlacq (1600–1666), who filled the gap from 20,000 up to the 90,000 with a precision of 10 decimal digits and published it as *Arithme´tique logarithme´tique* (Gouda 1628). These last tables of logarithms were the basic method to simplify the calculations for the next three centuries.

Returning to Napier's system (Table 3) we note another crucial disadvantage. The notion of a base, as used in the modern definition, is inapplicable. The terms of a geometric progression were simply matched with the terms of an arithmetic progression and the terms of the arithmetic progression were named logarithms of the corresponding terms of the geometric progression. According to Napier (e.g., Cajori 1913: 7):

> *Logarithmi dici possunt numerorum proportionalium comites aequidifferentes* (Logarithms are numbers with constant differences matched with numbers in continued proportion).

This description characterizes all the logarithmic systems of seventeenth century, particularly those before 1649 (e.g., Burn 2001).

The system of Napier would have a base if one replaced Napier's progressions with those that result by dividing all its terms with 10^7 (Table 4). The new system has base $a = (1 - 10^{-7})^{10^7} = 0.367879422$, that coincides up to 8 digits with the value of $1/e = 0.367879441$.

In this modified system we can write in an approximate manner and in modern symbolism, that:

$$\log_{1/e}\left(1 - \frac{1}{10^7}\right)^n = 10^{-7}n \Leftrightarrow \log_{1/e}\frac{a_n}{10^7} = \frac{b_n}{10^7} \tag{1}$$

But in the system of Napier (Table 3) we have that Nlog $a_n = b_n$ and so Eq. 1 becomes:

$$\mathrm{N}\log a_n = b_n = 10^7\log_{1/e}\frac{a_n}{10^7} \Leftrightarrow \mathrm{N}\log a_n = b_n = -10^7\ln\frac{a_n}{10^7} \tag{2}$$

If we name $x = \frac{a_n}{10^7}$ and $y = \frac{b_n}{10^7}$ then Eq. 2 becomes:

Table 4 Modification of Napier's progressions to show their relation with the natural logarithms

A.P.	0	$10^{-7} \cdot 1$	$10^{-7} \cdot 2$	–	$10^{-7} \cdot n$	–
G.P.	$(1-10^{-7})^{0}$ $\parallel$ 1	$(1-10^{-7})^{1}$ $\parallel$ 0.9999999	$(1-10^{-7})^{2}$ $\parallel$ 0.9999998	–	$(1-10^{-7})^{n}$	–

$$y = -\ln x \tag{3}$$

Consequently, if we divide the terms of the two progressions by 10^7, then the logarithms of Napier are approximately the opposites of the natural logarithms. Thus the frequent designation of natural logarithms as "Naperian logarithms" is inaccurate. This could be a nice activity after the discussion about the change of base formula.

Lastly, I would like to stress once again that in class we should not go into the details of the definition and calculation of the logarithms of Napier. I believe it is better for the students to give the definition as in Table 3 or even Table 4. If you want to help your students get an idea of the difficulties Napier and his contemporaries encountered, you can prepare a project leading them to construct, for themselves, a miniature version of a logarithmic system (Burn 1998; Resnikoff and Wells 1984: 193). Basically, the exposition could be supplemented with stories from the rich biography of Napier (e.g., http://en.wikipedia.org/wiki/John_Napier, and the references cited there) and the reception of logarithms that spur on the interest and give cultural elements of the era and this fulfills one of the goals of using history of mathematics in teaching. For example, so new and original did logarithms seem to Briggs that he left London for Scotland, to visit Napier, the inventor. One can give students the description of the first meeting and ask questions of various levels about mathematics and the pedagogical reflections which it generates (Fauvel and van Maanen 2000: xi–xiii).

I will now try to explain briefly why we should avoid the perplexities of the course that Napier followed. The link between historical developments in mathematics and the students' learning of mathematics has often been done in terms of the '*historical-genetic principle*'. It came to education at the turn of the ninteenth into the twentieth century from biology as a result of Haeckel's biogenetic law: '*Ontogeny recapitulates Phylogeny*'. The historical-genetic principle is the adoption in education of a psychological version of the biological recapitulation and may be stated as follows: *effective learning requires that each learner retrace the steps in the historical evolution of the subject under study*. But a little knowledge of mathematical history shows the course of its development to be much less simple and linear than the historical-genetic principle would require. The relations between history of mathematics and learning and teaching of mathematics can be much more complex than was originally believed. What happened in the past and what will probably happen in the classroom are two different things because they are based in very different cultural, psychological and didactical environments (e.g., Furinghetti and Radford 2002).

The difficulties encompassing the simplistic version of psychological recapitulation encouraged new reflections on the historical-genetic principle. Pólya (1981: 132) formulated a temperate form of the historical-genetic principle and sounded a warning: "*The historical-genetic principle is a guide to, not a substitute for, judgement*". Freudenthal (1984) also provided a weaker interpretation of the historical-genetic principle. In the conclusion of his lecture at the 1983 International Congress of Mathematicians at Warsaw, said:

> If mathematics teaching proves to be a failure, the reason is often, if not always, that we do not realize that young people have to start *somewhere* in the past of mankind and *somehow* repeat the learning process of mankind. This is the lesson historians and educators can learn from each other (italics added).

That seems to be like the historical-genetic principle. But Freudenthal carefully states the principle in a weaker form with a 'somewhere' and a 'somehow'.

Waldeg (1997) has, also, discussed the theoretical underpinnings of the use of history, including the question of the sense in which it is true that ontogeny recapitulates phylogeny. She cites several studies which indicate how one can understand this idea; namely, works on epistemological obstacles (e.g., Bachelard 1938/1983; Brousseau 1983; Sierpinska 1994), on the mechanisms of passage from one stage of understanding a mathematical idea to a following stage (e.g., Sfard 1995), on the approach of *didactic transposition* by which modern teaching can in fact utilize old mathematics (e.g., Chevallard 1985), and on the status of mathematical objects (e.g., Sfard 1991). She has noted that the actual learning of mathematics has increasingly been described in constructivist terms, and in this context a similar approach to the use of the history of mathematics is warranted. In other words, attention should be focused on the process of reconstructing mathematics rather than rediscovering it. The studies seem to indicate specific ways one can use history to help students understand particular points and even how to use historical methods in teaching a modern course.

The two most commonly presented ways for the inclusion of a historical dimension in the teaching strategy are depending upon whether the presence of history is *explicit* or *implicit* in the teaching situation. In the case of an explicit use of history the educator presents in detail the evolution and the stages in progress of a concept. In this case, even with the necessary simplifications, the emphasis is on history. In a reconstruction in which history enters implicitly, there is no need to mention every historical detail. The historical development acts as a guideline; to take *ideas*. This means that history is not an aim for itself, but teaching a subject one may use concepts, methods and notations that appeared later than the subject under consideration. What counts is to keep in mind that the ultimate goal is to understand mathematics in its modern form. In such an approach, the constructive perspective would highlight the pedagogically more valuable order of presentation; one enters into the area of didactic transposition and looks at the historical evolution from the current stage of concept formation. Several researchers have developed historical studies driven by the previous didactical aims (e.g., Bartolini-Bussi and Sierpinska 2000; Friedelmeyer 1990; Kronfellner 1996; Radford and Guérette 1996). At this point we also want to refer to *The ICMI Study*, Fauvel/Van Maanen (2000). Chapter 7 from Tzanakis and Arcavi and chapter 8 from Siu give an analytic survey of how history of mathematics has been and can be integrated into the mathematics classroom.

However, the above two possible forms of reconstruction are not mutually exclusive but both may be used in teaching a concept in complementary ways. Vasco (1995, p.62) has proposed a methodology, called '*forward and backward heuristics*', to help us find the optimal teaching sequence of a mathematical subject. The forward heuristics propose efficient ways of reviewing the phylogenesis of the particular mathematical subject, in order to optimize the ontogenetic mastery of that conceptual field. The backward heuristics propose ways to trim, compress, and even alter the sequences found through the forward heuristics. Forward heuristics lay out the rough draft of the roads on the mathematical map; backward heuristics do the redesigning, the shortcutting, and the road signaling. To that end, Menghini (2000) pointed out that a didactical competence is needed more than a historic one. Note that Halmos, in his response to receiving the Steele Prize in 1983, said (Alexanderson 2003):

> I enjoy studying, learning, coming to understand, and the explaining, but it doesn't follow that communicating what I know is always easy; it can be devilishly hard. To explain something you must know not only what to put, but also what to leave out; you must know when to tell the whole truth and when to get the right idea across by telling a little white fib.

Historians of mathematics may adopt different views of what history means. Most historians of mathematics have not always found it easy to accommodate their sense of the complexity and subtlety of history to the somewhat distorted and adjusted interpretations given to a certain event for pedagogic purposes. Their work shows off a critical interest in and respect for the integrity of past events. Rowe (1996: 10) notes that this perspective was often absent in studies of the history of science prior to 1950. After the appearance of Herbert Butterfield's influential *The Origins of Modern Science* (New York, 1957), however, which decried the evils of "*Whing history*" this principle of approaching the past on its own terms has come to be regarded as an axiom within the field.

The analysis of how a dialogue might be promoted between historians and mathematicians is in need of investigation. Rowe (1996: 12) discusses two recent trends in philosophy of mathematics that suggest promising possibilities for a fruitful such dialogue. The first trend is the role of intuition in the process of mathematical discovery highlighted by Pólya (1954/1990) and the related philosophical ideas of Imre Lakatos (1976) regarding the nature of mathematical knowledge. Their work has broken down rigid stereotypes about mathematics and has challenged historians and philosophers of mathematics to re-examine more closely how mathematical knowledge grows. The second trend is that mathematics is an activity set firmly in a socio-cultural context. So, in a sense, historical research is moving away from a monolithic image of mathematics.

Bartolini-Bussi and Bazzini (2003) describe also the problematic relationship between historians and didacticians. In some cases mathematics educators consider history as a solution of the problem of involving students in mathematics, thus the transposition into classrooms was carried out mainly relying on the glamour and neglecting a careful set up of methodologies and objectives. On the other hand it is true that some times historians do not view their primary task as showing how past achievements were absorbed into a more familiar body of modern knowledge.

Many conferences have been organized in the past years to try to establish a connection between historians and didacticians. Jahnke et al. (1996: ix) organized a conference in Essen (1992), for historians and mathematics educators, with the next motivation:

> Questions and viewpoints resulting from teaching must become an integral part of the methodology of historical research. Although many historians may fear that this could impose limitations on their own work, the opposite is true. Education will raise a host of questions which will substantially enrich historical research.

Of course, this brief synopsis cannot possibly do justice to the works discussed above or to the larger directions in research associated with them. We conclude pointing out the need for a continuous collaboration between interested mathematics educators, historians of mathematics and research mathematicians.

2.4 The Densest "Geometric Progression"

The tables of Napier and Briggs and their followers revolutionalised the art of numerical computation. The repeated taking of square roots and insertion of geometric means could refine a geometric progression to any required degree of denseness. This was a basic property of progressions that had been used in the making of all known logarithmic tables. While the numerical aspect of logarithmic computation is not devoid of theoretical interest, the importance of logarithms in the historical development of calculus stems from a discovery published in 1647 by the Belgian Jesuit Gregory of St. Vincent, that implies a surprising connection between the natural logarithm function and the rectangular hyperbola $xy = 1$.

Gregory of St. Vincent was born in Bruges in 1584. He entered the Society of Jesus in Rome, in 1607. There he studied mathematics under Christophorus Clavius (1538–1612), a famous mathematician in the Roman College, who had appreciated the mathematical talent of Gregory. Gregory stayed in Rome until the death of Clavius in 1612. Then he returned to Belgium and after he was appointed to various positions and for short time intervals, he taught mathematics for three years in Antwerp (1617–1620) and four years in Louvain (1621–1625). This was the most creative period of his life and is characterized by important mathematical inventions. It seems that in this period he wrote the bigger part of his important work: *Opus Geometricum Quadraturae Circuli et Sectionum Coni* (Antwerp, 1647). He died in 1667 in Ghent after successive attacks of apoplexy (Van Looy 1984).

Up to the middle of the twentieth century Gregory of St. Vincent had not been appreciated as an important mathematician, despite the clear statement of Leibniz (Dhombres 1993: 403):

> During my own apprenticeship, more essential help came from the famous triumvirs: from Fermat by his invention of a method pro maximis et minimis, from Descartes by his showing how to describe curves of usual geometry by means of equations, and from Father Gregory of St-Vincent by his numerous bright inventions.

After his death, his manuscripts were bound in 17 volumes but unfortunately without any regard to content or chronology. The number of sheets per volume varies from 319 to 583. This enormous and disorganized mass of manuscripts had discouraged many investigators. However, during the last 60 years many studies about his life and work have attributed to him an important role. Gregory, along with Cavalieri, Fermat and Descartes is among those who paved the way for the invention of calculus from Newton and Leibniz.

The aim of Gregory's book is, as it is also written in the title, the squaring of circle. At that time almost no one still believed in the possibility of resolution of the famous old problem and thus no few people tried to locate an error in a book that had 1200 pages, perhaps the only book of mathematics with so many pages that was ever published. Four years later, in 1651, Christian Huygens finally found a serious error in the last chapter of the book, on page 1121.

Unfortunately the words *quadruturae circuli* attracted most of the attention of mathematicians, so that they ignored a lot of other important results in the work of Gregory. One of them appears in the unit where he investigates the problem of finding two geometric means x and y between two known line segments a and b, that the ancient Greeks had connected with the conic sections. Gregory, examining the properties of conic sections proves the next proposition, which gives, for first time in the history of mathematics, the logarithmic property of the hyperbola:

> "*Let Ox and Oy be the asymptotes of a hyperbola ABD. Divide ON so that OK, OL, OM, ON are in a continuous proportion. Then, the curvilinear figures ABLK, BCML, CDNM have equal areas*" (Fig. 3) (Proposition 673 in volume 13 (Van Looy 1984: 63); Proposition 130 in Book 6 (*de Hyperbola*) in Opus Geometricum (Burn 2001: 2)).

The term *continuous proportion* means that, as we have already noted, the segments *OK*, *OL*, *OM*, and *ON* form a geometric progression. Thus, we can state the above proposition as: *If the segments OK, OL, OM, and ON are consecutive terms of a geometric progression, then the areas (ABLK), (BCML), (CDNM) are equal*. Naturally one would expect at this point that the logarithms should make their appearance. But Gregory refers nowhere to logarithms even though Napier had used this term with this meaning much earlier, in 1614. It appears that Gregory had another aim; probably it was the first step of the Belgian priest towards the squaring of a hyperbolic segment.

Gregory limited himself to two successive bands and he proves that they have equal areas. He gives two proofs, one similar to Archimedes' squaring of a parabolic segment, and the other one similar to the current technique of calculating definite integrals by approximating the integral by increasing the number of inscribed rectangles in the hyperbolic segments (Burn 2000; Dhombres 1993). The first proof shows a good knowledge of the properties of conic sections that he obtained, as he says, from the study of *Conics* of Apollonius. He did not use coordinates because he gave the proof before 1625 and at that time Gregory did not have at his disposal the methods of analytic geometry. Using coordinates the proof of this important proposition can be shaped so as to be a very good project for the students.

In the second proof the Jesuit mathematician developes techniques that could lead to a theory of limits. He gives the next definition for the limit (*terminus*) of a progression (Van Looy 1984: 64):

> The limit of a progression is the end, which none progression can reach, even if she is continued in infinity, but which she can approach nearer than a given segment.

It seems to be the first to give a definition of limit. In his study of infinite series he uses Proposition 1 of Book *X* of the *Elements* of Euclid (Heath 1956: 14):

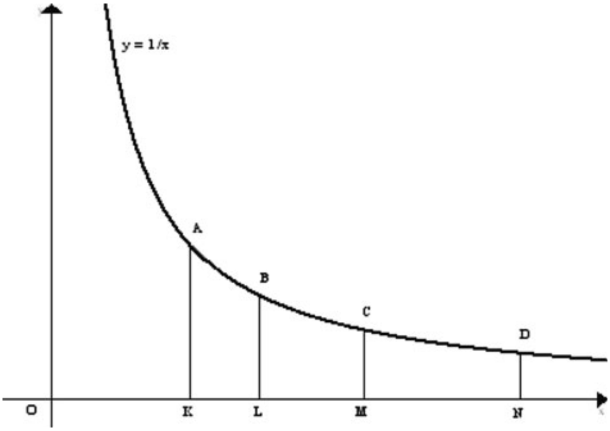

Fig. 3 The logarithmic property of the hyperbola

> If two unequal quantities are given and if one takes more than the half from the bigger quantity, and from what remains also more than the half, and if this process is continued, then will remain at last something smaller than the smallest quantity.

Gregory of St. Vincent advances little further than Euclid because he adds to the conclusion the phrase '*the quantity will be exhausted*'. It is to this phrase that the well-known method of exhaustion owes its name. The second proof, shaped suitably, could be used for the teaching in class and will be presented below in Sect. 3.1.

In 1648, one year after the publication of *Opus Geometricum*, Marin Mersenne (1588–1648) published a review of this quadrature, in which he did not locate any error but claimed that the problem was not yet solved and challenged Gregory with the next problem: *Given three rational or irrational magnitudes and given the logarithms of two, to find the logarithm of the third geometrically*. Mersenne posed the problem because he believed it was as difficult as the quadrature of the circle. Both circle-squaring and finding a third logarithm, given two, require the construction of transcendental numbers.

Father Alphonse Antonio de Sarasa answered, with the help of Gregory, the criticism of Mersenne with the publication of his book *Solutio problematis a R.P. Marino Mersenno propositi* (Antwerp, 1649) (e.g., Burn, 2001). De Sarasa was born in Nieuport, in Flanders, in 1618 and he died in Brussels in 1667. He became member of the Jesuit order in Ghent in 1632, on the year that Gregory of St-Vincent also went there. He became a student and later a colleague of Gregory in the local college for seven years. He also occupied academic positions in Antwerp and Brussels. When the correctness of the solution given by Gregory to the problem of squaring the circle was discussed, de Sarasa became one of his supporters.

Before looking at de Sarasa's response to this problem, we must acknowledge the difference between the term 'logarithm' in this context and our modern, narrower use. At least from the time of Euler, once the base had been chosen, the logarithms of all positive numbers were uniquely defined. But to say this we suppose that log 1 = 0 and logarithms are defined on the continuum. Before 1650, as already we have noticed in the discussion of Napier's system, there was neither the concept of base, nor any agreement about which number had logarithm zero. Logarithms were simply the terms of an arithmetic progression matched with the terms of a geometric progression:

$$\begin{array}{rccccc} \text{Logarithms in}\,A.P. & a & \alpha+d & \alpha+2d & \dots & \alpha+nd \\ \text{Numbers in}\,G.P. & c & cr & cr^2 & \dots & cr^n \end{array}$$

The correspondence $cr^n \leftrightarrow a + nd = \log\,(cr^n)$ shows that in the definition of a logarithmic system there are two degrees of freedom. If the logarithms of c and cr are given, then one may find the logarithms of all the terms of the geometric progression. So, de Sarasa considers that the solution of Mersenne's problem depends from the existence of a geometric progression that would contain the three arbitrary numbers. If this happens and $A = cr^k$, $B = cr^m$ and $C = cr^n$ are the three numbers and $\log A = K$, $\log \mathrm{B} = M$ the two known logarithms, then the logarithm of third number $\log C = N$ can be calculated. Indeed, from $K = a + k\omega$, $M = a + m\omega$ and $N = a + n\omega$ we take:

$$\frac{N-M}{M-K} = \frac{n-m}{m-k},$$

from which we may find N as a function of the known n, m, k, K, M.

But, *is it possible to find a geometric progression containing three given numbers among the host of geometric progressions containing any two of them*? Unfortunately, it is not certain that any given three numbers are terms of a geometric progression. Johannes Kepler (1571–1630) had already given in his *Chiliades Logarithmorum* (Marburg, 1624) an example of a triple (8, 13, 18), which does not belong to any geometric progression—another nice exercise for our students. Of course there is also the possibility of making denser a geometric progression containing two of the given numbers by repeated insertion of geometric means and by this limiting process to obtain the desired geometric progression. But such a solution would not be acceptable. The term *geometrically* in the statement of the problem, as de Sarasa also conceives it, has two senses: *geometric construction* and *geometric rigour*. In the last paragraph of the preamble of his book writes (Burn 2001: 9):

> In order to deal with the problem with geometric rigour, we will repeat here the most important teaching from Part 4 of Book [6] *de Hyperbola* from *Opus Geometricum* of Gregory of St-Vincent. The foundations of the teaching embracing logarithms are contained there.

To this end he restates as proposition 3 the basic Proposition of *Opus Geometricum*, that we have already mentioned, as follows: "*If Ox is divided so that the segments OK, OL, OM, ON are consecutive terms of a geometric progression, then the areas (ABLK), (ACMK), (ADNK) are consecutive terms of an arithmetic progression and conversely*". This is what de Sarasa derives from Gregory. Thus, he establishes a correspondence between the terms of the two progressions (Table 5).

This correspondence is the basic principle for each logarithmic system. But in this case logarithms have a **natural** meaning: they express the areas of certain geometric figures. This discovery has been variously attributed to Gregory and to de Sarasa (Burn 2001: 1, 15; Coolidge 1950).

In propositions 6–8, de Sarasa determines precisely when there exists a geometric progression containing among its terms three given line segments; this can be done if and only if the corresponding hyperbolic areas are commensurable. In other words: the segments *OK*, *OL* and *ON* (Fig. 3) belong to a geometric progression (without to be consecutive terms necessarily) if and only if the areas (*ABLK*) and (*BDNL*) are commensurable. After that de Sarasa adds (Burn 2001: 12):

> Consequently it is obvious that the problem of Mersenne isn't properly formulated; that which is sought is clearly contrary to the nature of logarithms, and cannot always be solved.

Clearly, for de Sarasa 'the nature of logarithms' was discrete and an answer to the problem depended on the existence of a geometric progression containing the three numbers. We naturally think that he should have adopted a continuous image of the logarithms, given that Napier had already considered a continuous model in his definition of logarithms. However, the recognition of hyperbola-area as a model of the logarithmic function was a richly suggestive idea, both in practice and in theory. In practice, the new model changed the structure on which the numerical calculations were made. In theory, it led to the definition of the logarithm of any positive number.

Table 5 The two natural progressions of Gregory

0	(*ABLK*)	(*ACMK*)	(*ADNK*)
OK	*OL*	*OM*	*ON*

At first sight this area redefinition seemed merely to convert a difficult analytical concept into an equally difficult one of hyperbola area. *Is the numerical computation of areas under a hyperbola any simpler than a Naperian calculation*? It is this question, defined and solved with increasing precision from the early 1650's on, which finally, with the aid of integration techniques, triggered the elementary infinite sum-series developments for the logarithm in the late 1660's.

Perhaps the first attempt to calculate hyperbola-areas systematically was formulated by William Brouncker (1620–1684) and Pietro Mengoli (1626–1686) some time in the mid-1650's (e.g., Whiteside 1960–1962: 222–225). Their methods are, in practice, laborious and unwieldy but an interesting conceptual development arises from Mengoli's method in his attempt to create an analytical theory of the logarithm, inspired by the model of hyperbola-area but independent of the geometrical form (*ibid*.: 224). Well into the 1660's it remained the ideal of many mathematicians to construct methods which, based on the model of hyperbola-area for their justification, would give a close approximation without undue computation. However, the problem of an adequate analytical definition of the logarithmic function was resolved with the aid of integration techniques by several sum-series expansions.

The first published account of the development was given by Nicholas Mercator (1620–1687) in his *Logarithmotechnica* (London 1668), though several people developed the method independently. Mercator studies the area under the curve $xy = 1$ over the interval $[1, 1 + x]$. The real significance of the investigation comes from the fact that he writes the equation of the hyperbola in the form $y = 1/(1 + x)$, which enables him to start from 0. He computes by long division the geometric series:

$$y = \frac{1}{1+x} = 1 - x + x^2 - \ldots \tag{4}$$

Mercator gives a crude explanation of the process which he followed to compute the area under this curve over the interval $[0, x]$. In fact, he used some results of John Wallis (1616–1703) for the area under the curves $y = x^m$ over the interval from 0 to x (m a non negative integer) but worked out only some examples for particular values of x ($x = 0.1$ and $x = 0.21$). He does not write down the general result of this integration. A much clearer exposition was published in the same year by Wallis (e.g., Coolidge 1950), who was the first to state Mercator's famous formula in general symbols (Cajori 1913); a result that today we express in the form:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad |x| < 1 \tag{5}$$

But should any area under the hyperbola and over the interval [0, x] be a logarithm? For Mercator the answer it seems to be affirmative. Probably using the results reached by Gregory and de Sarasa, Mercator connects his own results with logarithms and ends his *Logarithmotechnica* with the claim that: "*From the above, it is also obvious how Mersenne's problem could be solved, if not geometrically, at least numerically, to any desired degree of accuracy*" (cited in Boyé 2006: 226). In a note by Mercator in the *Philosophical Transactions* of 1668, the areas determined under a hyperbola are referred to as *natural* logarithms.

At the end of seventeenth century the computations of logarithms by infinite series had been accepted by the mathematicians. The reference to the hyperbola soon disappeared. Edmund Halley (1656–1743), in the *Philosophical Transactions* of 1695, gave the first derivation of infinite series for the computation of logarithms without any regard to hyperbola (e.g., Whiteside 1960–1962: 230–231). The significance of the infinite series became clear, adding both a further conceptual tool to mathematics and a means of calculating logarithmic values. Jacob Bernoulli (1654–1705) and Johan Bernoulli (1667–1748) did a great deal of work with series for the purposes of obtaining areas under curves between 1689 and 1704. Wallis (1995) obtained from Mercator's series the next one:

$$\ln\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right)(1<x<1) \tag{6}$$

We note that any positive real number can be written as $(1 + x)/(1 - x)$ for some $x \in (-1, 1)$; then by (6) we can approximate the natural logarithm of any positive real number. Setting $x = 1/3$ in (6) gives rise to the rapidly converging series:

$$\ln 2=2\left(\frac{1}{3}+\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}+\frac{1}{7\cdot 3^7}+\frac{1}{9\cdot 3^9}+\cdots\right) \tag{7}$$

The sum of the first seven terms of this series gives the approximation $\ln 2 \approx 0.693147170$, which is correct to seven places.

Once we have a method of approximating ln 2 as accurately as desired, it is possible to outline an algorithm for approximating the natural logarithm of any number. Indeed, if we replace x by $1/(2\,m + 1)$, $m \in N$, in (6), we obtain:

$$\ln(m+1)-\ln m=2\left\{\frac{1}{2m+1}+\frac{1}{3(2m+1)^3}+\frac{1}{5(2m+1)^5}+\cdots\right\} \tag{8}$$

Thus, knowing ln2, we can approximate inductively by (8) the natural logarithm of any positive integer. Moreover, the partial sums of the foregoing examples are easily computed on a calculator and can quickly be compared with the values obtained by the calculator's natural logarithm button. *It certainly should help the students to demystify what the calculator does 'behind the curtains' and as well to become aware of the limitations of a calculator or a computer.*

While the previous investigations and many others similar to them led to greatly improved methods of computing logarithms by infinite series, no modification of the logarithmic concept resulted from these researches. However, in the 1660's, we can say that areas under the hyperbola have been generally considered having the logarithmic property. Isaac Newton (1642–1727) wrote in his *Waste Book* in 1664–1665 (Whiteside 1967–1976: 457):

> *In y^e Hyperbola y^e area of it bears y^e same respect to its Asymptote w^{ch} a logarithme dot[h its] number.*

Newton, in a manuscript probably written in 1667, obtained equation (5) from (4) by simple term by term integration (Coolidge 1950; Whiteside 1967–1976, vol. II: 184–189). It is fair to say that although Newton gives Mercator's formula, he gives it as the area under the hyperbola, with no mention of Mercator or of logarithms, and writes $A\,(1 + x)$ instead of $\log\,(1 + x)$ for the area under the hyperbola $y = 1/(1 + x)$ and over the interval $[0, x]$ (or the negative of this area over the interval $[x, 0]$ if $-1 < x < 0$). Although Newton does not refer to $A(1 + x)$ as a logarithm, he asserts that:

$$\begin{aligned} &A((1+x)(1+y)) = A(1+x) + A(1+y), \\ &A\left(\frac{1+x}{1+y}\right) = A(1+x) - A(1+y), \quad (x, y > -1) \end{aligned} \tag{9}$$

In other words he asserts that *all the hyperbolic-areas have the same properties as the logarithms*. On the basis of these formulas he proceeds to calculate a small table of logarithms of integers to 57 decimal places! To check the accuracy of his computations, Newton calculates A (0.9984) in two different ways: first by (5) and then by the formulas (9). He finds that the two results agree to more than 50 decimal places (Edwards 1979: 160). A result, quite convincing for the validity of the relations (9)!

Charles-René Reyneau (1656–1728) is the author of *L' Analyse démontrée, ou la mèthode de rèsoudre facilement des problèmes et d'apprendre facilement ces sciences* (Paris, 1708). Reyneau considers the Fig. 3 with the abscissas in geometric progression with $a = OK = 1$ and "*whose ratio differs from the unit by an infinitely small quantity*" (note $1 + \varepsilon$ this ratio). In addition, the author does not doubt that "*one can conceive all the numbers in this progression*" (cited in Lubet 2006: 237). The logarithms then are defined as in Table 5 where the entries in the second row are the numbers $(1 + \varepsilon)^n$, $n \in N$. Certainly, Reyneau should have clarified some issues like the following:

> *Can all the numbers be reached by a geometric progression of ratio (1 + ε)*? To *what numbers is he referring*? *Which is the role of infinity in Analysis*?

From a theoretical viewpoint, the model of hyperbola-area made visible the end of the road in the search for a denser and denser geometric progression. Thus, to attain the densest geometric progression we have to leave the realm of the geometric progressions. The densest geometric progression "is" the continuum; logarithms have to be defined on the continuum. It is necessary to adopt log 1 = 0. This determines the place from which areas are measured and guarantees that log AB = log A + log B. But, *which is the base, if there is at all, of the logarithmic system defined by this densest geometric progression*? The base of the logarithms is determined by the hyperbola, which will be taken as $y = 1/x$, with no other justification at the moment than its simplicity. It is known that different hyperbolic curves illustrate different logarithmic systems; the hyperbola $y = 1/(\ln R)x$ gives logarithms to the base R – later on, that could be the object of another activity. The definition $\log|x| = \int_1^x \frac{1}{t}dt$, given in some modern textbooks, is the result of the above analysis.

It took a further seventy years or so for the union of the logarithmic and exponential concepts. By the end of the seventeenth century it was recognized that logarithms could be defined as exponents. However, we find the definition and the first systematic exposition of logarithms as exponents in the introduction to William Gardiner's *Tables of Logarithms* (London, 1742). According to Cajori (1913, p.46), the presentation is due to William Jones (1675–1749). The one whose influence was greatest in emphasising the new view was Euler, who, in his *Introductio in analysin infinitorum* (Berlin 1748), introduced the logarithm of x with base a as that exponent y such that $a^y = x$ (Euler 1748/1990: Book I, Ch. VI). In Ch. VII Euler develops the infinite series expansions for the exponential and logarithmic functions. One can interpret these calculations, which are presented for example by Edwards (1979: 272–274), to make them close to the text of Reyneau: a geometric progression of ratio $(1 + k\varepsilon)$ is associated to an arithmetic progression of common difference ε, the number x is a term in this progression, he holds the 'rank' N, namely $x = N\varepsilon$. Euler writes:

$$a^x = (1 + k\varepsilon)^N = \left(1 + k\frac{x}{N}\right)^N, \log_a\left((1 + k\varepsilon)^N\right) = x$$

But Euler does not use the terminology of progressions, and he does not need to specify, as did Reyneau, that all numbers are contained among the terms of a sequence. The binomial formula plays a key role. Its use for any exponent realizes a mild passage from discrete to continuous. In addition, the calculation is independent of any geometric representation.

The track inaugurated by Euler translates into analytical terms the '*densification*' of a geometric progression. The legitimacy of Euler's calculations is quickly called into question when one considers them in details. But the fundamental idea that underlies them remains fertile. When the properties of convergence and continuity are cleared up, they are nicely adjusting to the approach followed by Euler in his *Introduction* and yield (with $k = 1$), the well-known results:

$$e^x = \lim_{n\to+\infty}\left(1 + \frac{x}{n}\right)^n \quad \text{and} \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

The wider adoption of the definition of logarithms as exponents, in school books, was due largely to the influence of Euler, who gave it in his *Complete Introduction of Algebra* (Sect. 1).

Next, we present a part of the standard material about logarithms. The discussion in this section (Sect. 2.4) will be used to motivate the students, explain the origin of basic propositions and definitions and design some interesting activities for the students.

3 Formal Presentation of Theory

3.1 Natural Logarithms

We begin with that historical proposition of Gregory of St. Vincent in which he actually proved the logarithmic property of hyperbola (Sect. 2.4). Here we formulate the proposition in an equivalent way that is used more effectively in what follows.

Proposition of Gregory of St.Vincent *For* $0 < a < b$ *let* $E_{a,b}$ *be the area enclosed by the hyperbola* $y = 1/x$*, the straight lines* $x = a$, $x = b$ *and the x-axis (Fig. 4). Then for any* $k > 0$:

$$E_{ka,kb} = E_{a,b}$$

Proof Intuitively it seems right that the area remains the same when we move along the x-axis and under the curve, provided that we stretch the length to the same ratio with the one we contract the height.

In order to compare analytically the two areas, we divide each interval [a, b] and [ka, kb] into n subintervals of equal length and will approach each area with rectangles (Fig. 4). Let $T_{a,b}$ and $T_{ka,kb}$ be the areas of the figures consisting of the rectangles belonging to the two intervals. Each rectangle in $T_{a,b}$ has width $(b - a)/n$ and each one in $T_{ka,kb}$ has width $(kb - ka)/n = k(b - a)/n$. We now compare rectangle for rectangle. The first one from the left in $T_{a,b}$ has height $\frac{1}{a+(b-a)/n}$; hence it has area

$$\frac{b-a}{n}\frac{1}{a+(b-a)/n} \tag{10}$$

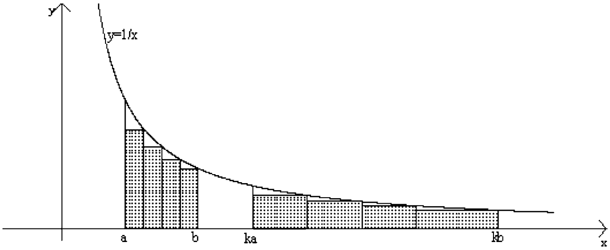

Fig. 4 The area under the hyperbola and between a, b is equal to that between ka and kb ($1 < a < b$)

The corresponding first one from the left in $T_{ka,kb}$ has height $\frac{1}{ka+k(b-a)/n}$; hence it has area

$$\frac{k(b-a)}{n}\frac{1}{ka+k(b-a)/n} \tag{11}$$

Consequently, from (10) and (11), the two rectangles have the same area. This continues to be true for all the n rectangles. And it follows that $T_{a,b} = T_{ka,kb}$, and, as n increases, $E_{a,b} = E_{ka,kb}$. (*Note*: We have treated the limits intuitively since at the level of 11th graders the concept of limit and its properties has not yet been introduced in a rigorous way).

Corollary If the segments *OK*, *OL*, *OM*, *ON* are consecutive terms of a geometric progression, then the areas (*ABLK*), (*ACMK*), (*ADNK*) are consecutive terms of an arithmetic progression (Fig. 3).

Proof Suppose that $OK = 1$, $OL = r$, $OM = r^2$, $ON = r^3$ and apply Gregory's proposition.

All our discussion until now justifies the next definition of the natural logarithm (Fig. 5).

Definition

$$\ln x = \begin{cases} E_{1,x}, & \text{if} \quad x > 1 \\ 0, & \text{if} \quad x = 1 \\ -E_{x,1}, & \text{if} \quad 0 < x < 1 \end{cases}$$

We prove the basic property of the logarithmic function that is *the property of turning the multiplication into addition.*

Proposition If $x, y > 0$, then $\ln(x\,y) = \ln x + \ln y$

Proof Let $1 < x \leq y$.

Then $xy > x$ and $xy > y$ (Fig. 6). Therefore:

$$\ln(xy) = E_{1,xy} = E_{1,y} + E_{y,xy} = \ln y + E_{y,xy} \tag{12}$$

However, from Gregory's proposition we have that:

$$E_{y,xy} = E_{1,x} \tag{13}$$

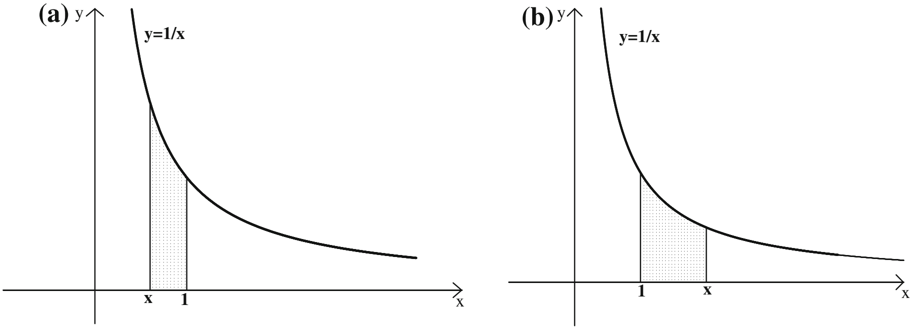

Fig. 5 The definition of the natural logarithm

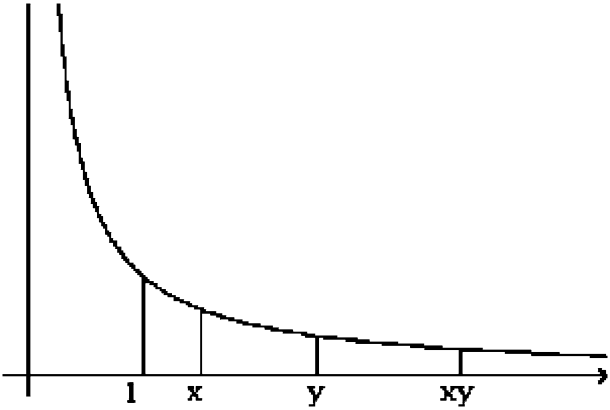

Fig. 6 $\ln(xy) = \ln x + \ln y$

From the above Eqs. 12 and 13 we take

$\ln(xy) = \ln x + \ln y$

The other cases $0 < x \le y \le 1$ and $0 < x \le 1 \le y$ are proved similarly.

Corollary

(i) *If* $x > 0$ *and* r *is a rational number, then* $\ln(x^r) = r \ln x$.
(ii) *If* $x, y > 0$*, then* $\ln(x/y) = \ln x - \ln y$

Proof is left to the reader.

The graph of the logarithmic function It follows from the definition that: the logarithmic function $y = \ln x$ is strictly increasing and 1 – 1, $\ln 1 = 0$, $\ln x > 0$ if $x > 1$ and $\ln x < 0$ if $0 < x < 1$.

We observe also that $\ln(2^n) = n \ln 2$ and $\ln(2^{-n}) = -n \ln 2$ for each natural number n. Since $\ln 2 > \ln 1 = 0$, this means that there exist arbitrarily large and small logarithmic values. Consequently, as x becomes large or as we say tends to $+\infty$, $\ln x$ grows also to $+\infty$. Respectively, when x becomes small tending to 0, $\ln x$ also becomes small tending to $-\infty$.

To obtain a better image, we will illustrate the calculation of ln 2, that is the area between $x = 1$ and $x = 2$, and so we will give also an idea for the calculation of logarithms via calculating areas. An application of the *composite trapezoidal rule* for n = 3 (e.g., Atkinson 1989: 253) gives 0.7 as an upper estimate for ln 2. Similarly, applying the *composite midpoint rule* for n = 3 (*idem.*: 269) we obtain 0.689 as a lower approximation for the ln2. Hence $0.689 < \ln 2 < 0.7$ and consequently 0.7 approximates ln 2 with error < 0.011.

Taking $\ln 2 = 0.7$ and drawing certain points with the help of the relation $\ln 2^n = n \ln 2$ we find the graph of $y = \ln x$ (Fig. 7). Since the logarithmic function is strictly increasing, it is explicit from the figure that the graph cuts each horizontal line $y = y_0$ in precisely one point. Consequently for any given number y_0, there exists precisely one number x_0 such that $\ln x_0 = y_0$.

Definition of the number e It follows from the paragraph above that there exists for $y = 1$ only one number whose logarithm is equal to 1. This number should be the base of the natural logarithms. It seems that Euler was the first one to recognize the importance of this number and used the special symbol *e* for its notation around 1730 (e.g., Kline 1972: 258). Therefore, *e* is the number for which $\ln e = 1$, that is to say the area $E_{1,e}$ is equal to 1 (Fig. 8).

An initial estimate for the number *e* can be obtained as follows: It is known from the previous discussion that $\ln 2 < 1$ and it is easy to see that $\ln 3 > 1$ (apply the midpoint rule for $f(x) = 1/x$, over the interval [1, 3]). Thus:

$$2 < e < 3$$

Textbooks usually introduce e as the limit of the sequence $a_n = (1 + 1/n)^n$, $n \in N^*$, which appears through a study of compound interest. Next, they inform students that the mysterious number e is particularly important and, used as a base, gives rise to a particularly important class of logarithms (e.g., Brown 1992: 187, 193). The students, unable to see any connection of that number with logarithms, feel that the whole system is artificial. This unmotivated development disturbs students and it is particularly harmful to mathematics teaching.

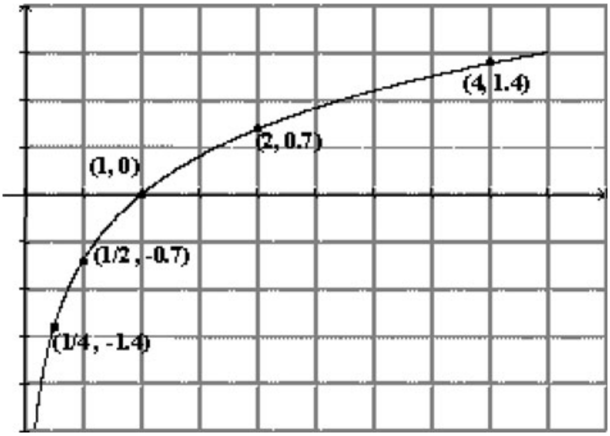

Fig. 7 The graph of $y = \ln x$

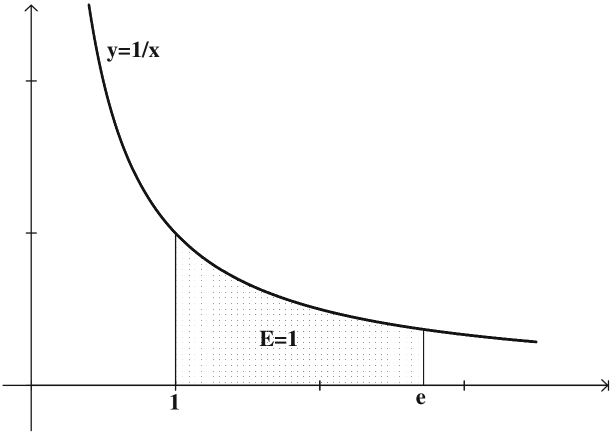

Fig. 8 The definition of e

The presentation followed in this paper leads to a quite reasonable explanation of the equality:

$$e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n$$

From the definition, e is a number such that $E_{1,e} = 1$. We divide the interval $[1, e]$ into n subintervals and approach the area $E_{1,e}$ with rectangles (Fig. 9, for $n = 4$). The ends of the subintervals are determined by the points $\{1, r, r^2, \ldots, r^n = e\}$, which are in geometric progression with common ratio $r = \sqrt[n]{e}$. The considered rectangles have their top left corners on the graph of $1/x$. Consequently, their heights are: $1, 1/r, 1/r^2, \ldots, 1/r^{n-1}$. Then, we calculate easily that the area of the figure consisting of the sum of all the rectangles is $E_n = n\ (r - 1)$, that is $E_n = n(\sqrt[n]{e} - 1)$.

As n becomes very large we can consider that: $E_n \approx E_{1,e}$. In other words, $E_n \approx 1$

or $n(\sqrt[n]{e - 1}) \cong 1$
or $\sqrt[n]{e} - 1 \cong 1/n$
or $\sqrt[n]{e} \cong 1 + 1/n$
or $e \cong (1 + 1/n)^n$

and from this we can conclude that:

$$e = \lim_{n \to +\infty} (1 + 1/n)^n.$$

Now we can compute a more accurate value of e though the convergence of the sequence is too slow. Euler gave an approximation for e to 18 decimal places: $e = 2.718281828459045235$. Note that we can easily remember the first 9 decimal digits, but never right on the board $e = 2.718281828$ since the students tend to think that e is rational.

We shall not enter into a discussion of logarithms to any base a ($a > 0$ and $a \neq 1$). There are certain basic propositions that we encounter in the curriculum of 12th grade (as they are the basic limit: $\lim_{x \to 1} \ln x/(x - 1) = 1$, the continuity and differentiability of the logarithmic function) and are usually omitted in most school books, with the remark that they are beyond the possibilities of the students. However, with the new definition of the

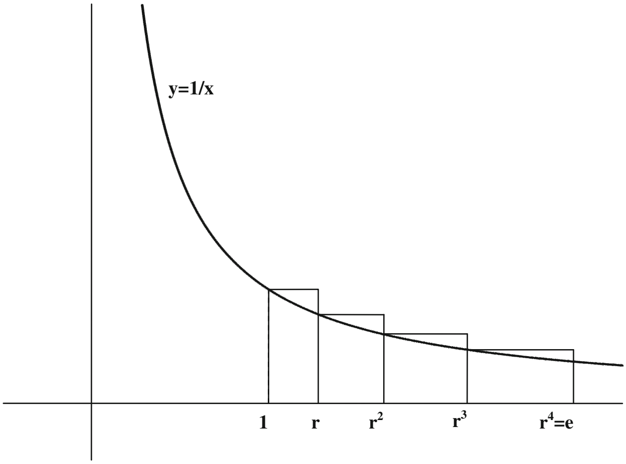

Fig. 9 *e* is the limit of the sequence $(1 + 1/n)^n$, $n \in N$

logarithm these proofs become simple and comprehensible. We shall give only the proof of the basic limit.

$$\textbf{Basic inequality}.\ \frac{x-1}{x} \le \ln x \le x-1 \quad \text{for any} \quad x > 0.$$

Proof The shaded area is between the rectangles that have as base the interval $[1, x]$ and heights $1/x$ and 1 respectively (Fig 10). Hence $(x - 1)/x < \ln x < (x - 1) \cdot 1$. Thus we have proved the inequality for $x > 1$. The case for $0 < x < 1$ is proved similarly and the case $x = 1$ is trivial.

$$\textbf{Basic limit}.\ \lim_{x\to 1}\frac{\ln x}{x-1} = 1$$

Proof The previous inequality for $x > 1$ becomes $\frac{1}{x} \le \frac{\ln x}{x-1} \le 1$, and therefore $\lim\limits_{x\to 1^+} \frac{\ln x}{x-1} = 1$. We proceed in a similar way for $x < 1$.

3.2 The Exponential Function e^x

The exponential function will be defined as the inverse function of ln x. Usually, the concept of the inverse of a function has not yet been taught at the level of 11th grade. It is possible to avoid any mention to the inverse functions by working as follows: First we ask the students to check that the points (ξ, n) and (n, ξ) are symmetrical about the line $y = x$ in the first quadrant. Then we ask them to construct the symmetrical of the graph of the function $y = \ln x$ about this line and we help them notice that the result must be the graph of some function *f* since any straight line vertical to the axis of abscissas cuts it at most to one point (Fig. 11).

This function *f* will be called the *exponential function* (the reason for this name will be apparent shortly) and it is defined by the equivalence:

$$y = f(x) \Leftrightarrow \ln y = x, \quad x, y \in R \text{ and } y > 0 \tag{14}$$

From the definition it results immediately that:

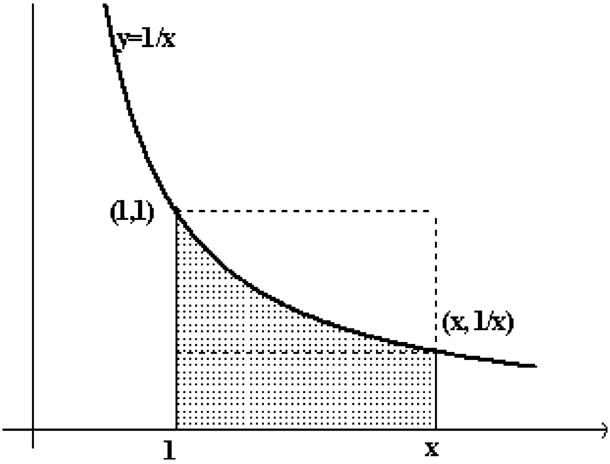

Fig. 10 (*x*-1)/*x* < ln *x* < *x*−1

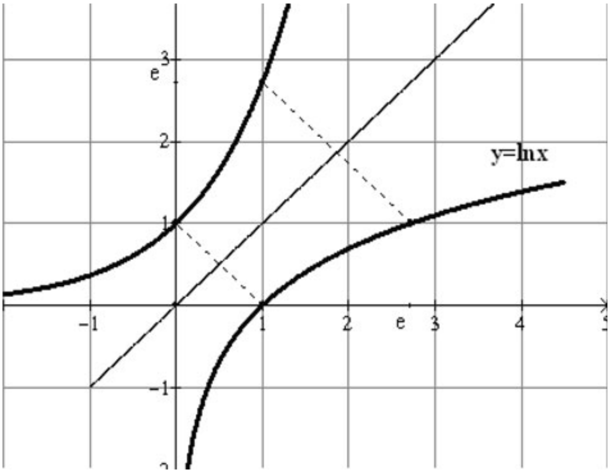

Fig. 11 The exponential function is the inverse of the logarithmic

$$\text{(i)} f(\ln x) = x, \quad x > 0 \quad \text{(ii)} \ln f(x) = x, \quad x \in R \quad \text{(iii)} f(0) = 1, \quad f(1) = e$$

Next we prove the function f coincides with $y = e^x$ in the domain of rational numbers.

First, from the logarithmic property ln (ab) = ln a + ln b we get easily that f has the next basic property for any real numbers x, y:

$$f(x + y) = f(x) \cdot f(y)$$

Repeated applications of this last property give, at least for integral n, m,

$$f(n) = e^n \quad \text{and} \quad f(1/m) = e^{1/m}$$

It follows that $f\ (n/m) = f\ (n(1/m)) = e^{n/m}$. So, the function f coincides with e^x in the domain of rational numbers:

$$f(r) = e^r,$$

where r is any rational number. In view of that, we define also e^x for irrational numbers by the equality:

$$e^x = f(x) \quad \text{for any real number } x.$$

Therefore the equivalence (14) is written:

$$y = e^x \Leftrightarrow \ln y = x, \quad x, y \in R \quad \text{and} \quad y > 0 \tag{15}$$

Now the exponential law $e^x \cdot e^y = e^{x+y}$ for all real numbers x, y, is nothing but a restatement of the property $f(x + y) = f(x) \cdot f(y)$. The other exponential properties are also easily derived from the logarithmic properties. The function $y = f(x)$ will be symbolized from now on by $y = e^x$. The name "*exponential function*" is now very clear.

We have not yet defined a^x, if $a \neq e$. A reasonable way to do this is to change the base a for e, provided that $a > 0$. From property (i) above, we know that $a = e^{\ln a}$. Hence $a^x = (e^{\ln a})^x = e^{x\ln a}$. The requirement $a > 0$ is necessary, in order that ln a be defined.

As we noted earlier (Sect. 2.2) it was surprising that logarithms were invented long before exponents. But, as Cajori (1913: 174) remarks, another surprise follows. Namely, every interpretation of the general power a^b, where a, b are complex numbers, involves previously proved results on logarithms (e.g., Churchill et al. 1974: 68). Thus, it seems that the logarithmic concept is the more primitive.

4 Conclusions

The value of history of mathematics in teaching has been pointed out for many years. Fauvel (1991) for example mentions at least fifteen reasons for applying the history to teaching and learning mathematics. The interest among teachers in the application of history to mathematics education is growing as is evident from the growing number of books and articles of historical content and from the growing number of research groups in this field (e.g., Fauvel and van Maanen 2000; Gulikers and Blom 2001). The positive contribution of history of mathematics to the teaching of mathematics is located mainly in the next three arguments (*idem.*):

- History of mathematics can help students understand better the mathematical concepts, methods and proofs showing them how they were discovered and developed.
- History of mathematics can help students realize that mathematics is a human and dynamic activity influenced by social and cultural factors and is shaped according to the utilitarian and intellectual needs of each era.
- History of mathematics can help stimulate students' interest for learning and improve their perceptions of mathematics and attitudes towards it.

However the previous arguments are not taken into consideration in the most school books. These never describe the efforts and the failures that led to the concepts they describe. The presentation in the form Definition–Theorem–Proof–Corollary can be elegant and can save time but the students remain with the query: How did the idea for these definitions and theorems come? According to Freudenthal (1973: 107):

> the basic definitions should not appear in the beginning of an exploration, because in order to define something one should know what this is and also in what it is useful.

Then, what we can do for them? We can make them see why and how we have found what they are taught today. The procedure that originally led to a mathematical result is considered indispensable for its understanding; there is a connection between student's mistakes, cognitive obstacles, and problems in the historical development of mathematics (e.g., Brousseau 1997: 87–88; Hadamard 1954: 104). This process includes important aspects of doing mathematics, such as the role of intuitions, heuristics, misconceptions, contradictions, refutations, alternative approaches, and as well as the motivation and the problems for which a mathematical concept was created (e.g., Lakatos 1976: Introduction and Appendices). It also conveys to our students the basic attributes of mathematics as a part of human culture (e.g., Bishop 1991). However, we have made it clear in Sect. 2.4 that this does not imply that the learning of mathematics should be dictated by an *ontogeny recapitulates phylogeny* argument. Thus, it turns out that using history in the teaching of mathematics is not (only) a matter of *content* but it is a matter of *attitude* and a problem of *image* of the mathematical science.

Answering the question why we should apply the history of mathematics in its teaching is one thing; how to do so is another thing. The lack of supporting teaching material such as didactical guidelines and empirical descriptions for teachers on how to use available historical material in their lessons is one of the difficulties that have been raised in integrating history in the teaching of mathematics (e.g. Fauvel 1991; Fauvel and van Maanen 2000: 212; Fowler 1992; Gulikers and Blom 2001: 231). Nevertheless, the amount of general articles outnumbers the practical essays which contain suggestions for resources or lessons.

The present article focuses on converting the history of logarithms into material appropriate for teaching students of 11th grade. It is unquestionable that the abstract definition of the logarithmic function as the inverse of the exponential function has a certain practical value into the stifling analytic program; it allows the easy and direct proof of the logarithmic properties from the properties of powers. But, this gives the impression that mathematics were created ready made. As Klein (1945: 146) has put it:

> this is despicable utilitarianism which is scornful of every higher principle of instruction, and which we must surely and severely condemn.

Students will begin to recognize the significance of logarithms through the various ways of recapturing their history. Teaching the theory of logarithms by following the historical development in its basic lines presents the following specific advantages:

(a_1) It justifies what is 'natural' in the natural logarithms as well as their relation with number e.

(a_2) It justifies the origin of the term logarithm. Also, the consideration of the trigonometric numbers as lengths, and not as ratios, allows the explanation of the origin of the words sine, cosine, tangent and secant (e.g., Smith 1923/1958: 618–623).

(a_3) It allows the proof of important properties of the logarithmic function with the help of a geometric model. The current studies on visualization (e.g., Dreyfus 1991a) leave little doubt as to the effectiveness of graphical representations in the learning of abstract subjects. Once again we employ geometry to support the science of computation; we should remember that the Greeks gave numerical computations a geometrical interpretation (e.g., Heath 1956).

(a_4) It connects creatively the new unit with the previous one, that of the progressions.

(a_5) It offers a first class opportunity to stress the usefulness of trigonometric formulas that they convert products into sums in the discussion of the method of prosthaphaeresis.

This example has much more value than the barren use of formulas in the solution of routine exercises because it really contributes in enlarging students' understanding of trigonometry and increases their estimate to this course.

(a_6) It connects the new unit with the study and the graphic representation of the function $y = 1/x$.

(a_7) It makes worthy the knowledge about the conic sections.

(a_8) It ensures a unique foundation that can support later the teaching of integral calculus.

Moreover, the topic is useful for drawing attention to some general characteristics of mathematics, which would help students better understand the nature and potential of what they are studying.

(b_1) It helps students to realize that mathematics is not a given system of results, by emphasizing the existence of a motivation for the introduction of a new concept and the construction of knowledge out of the activity of problem solving. Epistemologist Bachelard (1938/1983: 14) has written:

All knowledge is a response to a question. If there had not been any questions, it would not have been possible to have scientific knowledge.

(b_2) It reveals the evolutionary nature of mathematics in its form. Advances in form have often made it easier to learn mathematics (although they can also give rise to difficulties that can bar the way to effective understanding).

In the case of logarithms, students may appreciate the power of a good notation for the advancement of mathematics. Difficulty in developing facility with the meaning and use of symbols is one of the major obstacles to the learning of algebra in school. It is still an important goal of mathematics education to show the merit of suitable symbolism. A beneficial activity could be here a project for the development of notation, in particular of the exponential, from the time of Chuquet until the middle of eighteenth century (e.g., Cajori 1913, pp. 13, 35–37; Kline 1972, pp. 259–263).

(b_3) It reveals the evolutionary nature of mathematics in its content. In this way, students can gain a feeling for the nature, growth processes, and aliveness of mathematics.

Logarithms are a good and accessible example of something fundamentally changing its conceptual role within mathematics. The initial usefulness of logarithms in the simplification of numerical calculations has diminished in importance, whereas the use of logarithmic function as an intrinsic ingredient in the solution of problems continues to increase. A device for easing the activity of calculating turns out to be a function of immense power.

Comparison of our present-day mathematics with older techniques enables us to determine the value of our modern mathematics, and to point out this value to our students. One can prepare a worksheet where students will have to multiply two many-digits numbers in three different ways—by means of prosthaphaeresis, logarithms and a pocket calculator—and pose the questions: *Which method is simpler*? *How many distinct operations are required to compute products by the prosthaphaeretic rule*? *How many are required by using logarithms*? A project would also be given about the evolution of calculating devices and machines from early seventeenth century to 1942; for example: Napier's rods, slide rule—Pascal's or Leibniz's machines—Babbage's difference and analytical engines (Cajori 1994; Smith 1929/1959, 1923/1958; Swade 1991). Instruments as Napier's rods and the slide rule can be introduced in the mathematics classroom.

(b_4) It provides opportunities to unfold and emphasize important pedagogical issues.

It is known that the variety of equivalent formulations of a concept is a source of richness and power. The concept of logarithm, defined as area and as an exponent, let this idea to be understood, not as an artificial construction but as a response to problems and inquiries within the practice of mathematics. The importance to have many representations of a concept and the process of switching representations is discussed by Dreyfus (1991b: 32–33).

Moreover, the proposed presentation offers students a good opportunity to get the experience of changing their perspective while solving problems, because of the variety of methods from various periods that have been applied to solve the original problem of performing calculations with large numbers. This is the *flexibility* component of creative mathematical thinking that refers to a learner's ability to propose a variety of approaches to a specific problem. The core features of creativity (fluency, flexibility, elaboration and originality) can also be used to improve mathematical teaching and learning (e.g., Silver 1997; Sternberg 1988).

(b_5) It also suggests that different mathematical domains reinforce each other.

The historical development of the logarithmic ideas pulls together ideas from different areas of mathematics such as arithmetic, algebra, trigonometry, conic sections and calculus. So, it helps to prevent the view of mathematics as a subject divided into non-overlapping compartments. This belief is unproductive in the teaching and learning process, adversely influencing the attitudes of students.

(b_6) It presents the development of mathematics as a human activity influenced by the social and cultural milieu and being related to other disciplines.

Mathematics is a body of knowledge that is constantly evolving in response to societal conditions and, as they change, so does the nature of the mathematics that serves them. Logarithms came out as a response to needs that are both material and intellectual. Especially suitable projects to display the cultural and scientific setting in which the development of logarithms took place should be assigned around the question: who needed big numbers anyway? Examples of such projects could be: "*The navigational problems of seafarers of the fifteenth century, considered from the mathematical stand point*", and "*The cartographers' problems of course plotting and of determining the most efficient paths to navigate, studied from the mathematical viewpoint*" (e.g., Resnikoff and Wells 1984). The historical treatment of logarithms provides also examples with discussions about ideological concepts and value judgments about the structure of society. One may give as homework assignments: "Why *did the emergence of mercantile capitalism in forteenth century Italy spawn a mathematical Renaissance*"? (Swetz 1987), and "*Why did Columbus sail under the Spanish flag rather than the Portuguese*"? (e.g., Resnikoff and Wells 1984: 175–176).

(b_7) It helps to develop a multicultural approach in the classroom and give rise to the consolidation of a scientific world view.

Many questions which arise in today's courses were considered by other peoples years ago. What does differ is the approach. Take for example the number systems of the Egyptians and the Babylonians and compare these systems with ours. One can design some activities around the following issues: *How did Egyptians multiply numbers*? *How did Babylonians multiply numbers*? *How did our modern algorithms for the basic operations evolve*? Students might appreciate the crucial role representations play and the contrast of our own numeration system with other systems can be effective for a better understanding of its particular features (e.g., Heiede 2000; Ofir and Arcavi 1992; Smith 1923/1958; Swetz 1984, 1995).

Depending on the available time, it is beneficial but also enjoyable to play around with the Babylonian hexadecimal system. For the students, who have a certain experience with investigations of the law (model, pattern) that shapes a sequence of numbers, there exist interesting examples in the plate Plimpton 322 (e.g., Buck 1980; Polya 1954/1990).

(b_8) It promotes positive students' affective outcomes toward mathematics.

Attitudes are central to the educative process both as ends and as means. One technique that has been suggested as a means to improve students' affective outcomes toward mathematics and that has been endorsed by the National Council of Teachers of Mathematics (Hallenberg 1969/1989) is that of incorporating the history of mathematics into classroom discussions of mathematical topics. From the historical development of logarithms in this paper we may infer that history may influence positively the students' affect toward mathematics in several ways, some of which are the following:

- It helps to increase students' interest for learning by exposing our students to the affective aspect of doing mathematics. If in teaching we give students only the rigorous part of mathematics, we give them the wrong image of mathematics as a boring and finished subject. History can induce us to create a classroom climate of search and investigation and not just of conveying information. Reconstructing relationships is often considered among the most effective ways for students to learn mathematics (e.g., Dreyfus 1991b: 40). This effectiveness may be attributed to the psychological aspects of the process of discovery: the personal involvement, the intensity of attention, the feeling of achievement and success, the pure joy of discovery.
- It helps to increase students' appreciation of the importance, usefulness and value of mathematics and motivate them for studying mathematics. Knowing the origin of problems, concepts and proofs, how ideas were perceived, refined, and developed into useful theories and realizing that they were invented as answers to concrete questions posed by human beings has a motivational effect. Teaching mathematics as a "dull drill" subject will produce students who perceive mathematics as an incomprehensible collection of rules and formulas that land on the blackboard. These students build psychological barriers to true mathematical understanding and develop anxieties about the learning and use of mathematics.
- It helps to increase students' confidence and perseverance and decrease anxiety. Mathematics is not only for the geniuses. Mathematical results are the fruition of centuries of thought and development, and on the way to final achievement there are bound to be doubts, mistakes and failures. Students derive comfort from realizing that they are not the only ones with problems so that they get less discouraged by their own misunderstandings and mistakes. And finally,
- It helps us to interject human interest, a factor of affect improvement, in mathematics lessons from which such aspect is often absent. There is no question that biographical material enlivens classroom teaching. They are full of emotion and possess all the facets of human life that capture the imagination and perpetuate interest: successes, failures, perseverance, adventures, ingenuity, social and sex discriminations, foible, disputations, unfairnesses, intrigues, scandals and so on. The story of Napier's perseverance for years in trying to shorten the labor of tedious calculations is educationally enlightening. This is also an example of success but we have to give our

students some feeling for the pains of failure in mathematics: Gregory St. Vincent was one of the great mathematicians who made repeated unsuccessful attempts to solve the problem of squaring the circle. It also provides motivation seeing mathematics as a part of the social and cultural milieu (see above b_6 and b_7).

A lot of favorable testimony has been accumulated about the use of history in mathematics teaching. This opinion, which is common to the majority of the cases, usually comes from subjective impressions and not from regular and systemic studies of the outputs. We have also undertaken the proposed teaching sequence for the concept of logarithms in two high school classrooms. Our experience was positive. More importantly, students responded positively with a good and real feeling that they had been engaged in an actual mathematical investigation to tackle a problem. There are, however, some systemic case studies that emphasise the positive effect the history of mathematics had on the learning of mathematics and attitudes toward mathematics (Fauvel and van Maanen 2000: Sects. 5.2–5.3; Gulikers and Blom 2001: 233–235; McBride and Rollins 1977).

Felix Klein has proposed the use of the following equality:

$$\int_1^a \frac{dx}{x} = \int_c^{ca} \frac{dx}{x}$$

for the definition of logarithms. He himself (Klein 1945: 156) added:

> I wish very much that some one would give this plan a practical test in the schools. Just how it should be carried out in detail must, of course, be decided by the experienced school man.

The proposition of Gregory of St Vincent is precisely the translation of this integral relation to the level of 11th graders. The present paper is our attempt to apply Klein's recommendation. During the teaching sequence we have proposed a lot of interesting activities broadly dispersed within classroom drills and homework assignments. Certainly a single teacher could not do all these activities but one may choose among them the most appropriate for her/his class. There is a reasonable amount of information in the article so that one can get acquainted with the subject and feel comfortable while teaching it.

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References

Alexanderson, G. L. (2003). Major donation announced for new MAA conference center. *Focus, 23*(2), 3–5.

Atkinson, K. E. (1989). *An introduction to numerical analysis*. New York: Wiley.

Ayoub, R. (1993). What is a Napierian logarithm? *American Mathematical Monthly, 100*, 351–364.

Bachelard, G. (1938/1983). *La Formation De l' Esprit Scientifique*. Paris: J. Vrin.

Bartolini-Bussi, M. G., & Bazzini, L. (2003). Research, practice and theory in didactics of mathematics: Towards dialogue between different fields. *Educational Studies in Mathematics, 54*, 203–223.

Bartolini-Bussi, M. G., & Sierpinska, A. (2000). The relevance of historical studies in designing and analysing classroom activities. In J. Fauvel & J. Van Maanen (Eds.), *History in mathematics education, The ICMI Study* (pp. 154–161). Dordrecht: Kluwer.

Bishop, A. J. (1991). *Mathematical enculturation*. Dordrecht: Kluwer.

Boyé, A. (2006). Des Logarithmes Ordinaires aux Logarithmes Naturels. In *Histoire de Logarithmes* (pp. 217–232). Commission Inter-IREM, Paris: Ellipses

Brousseau, G. (1983). Les obstacles epistemologiques et les problemes en mathematique. *Recherches en Didactiques des Mathematiques, 4*(2), 164–198.

Brousseau, G. (1997). *Theory of didactical situations*. Dordrecht: Kluwer.

Brown, R. G. (1992). *Advanced mathematics*. Boston: Houghton Mifflin Company.

Buck, R. C. (1980). Sherlock Holmes in Babylon. *The American Mathematical Monthly, 87*, 338–345.

Burn, B. (1998). Napier's logarithms. *Mathematics in School, 27*, 32–33.

Burn, B. (2000). Gregory of St. Vincent and the rectangular hyperbola. *The Mathematical Gazette, 84*, 480–485.

Burn, R. P. (2001). Alphonse Antonio de Sarasa and Logarithms. *Historia Mathematica, 28*, 1–17.

Cajori, F. (1913). History of the exponential and logarithmic concepts. *The American Mathematical Monthly, 20*(1), 5–14; *20*(2), 35–47; *20*(6), 173–182.

Cajori, F. (1994). *History of the logarithmic slide rule and allied instruments*. Mendham, NJ: Astragal Press.

Chevallard, Y. (1985). *La transposition didactique: Du savoir savavt au savoir enseigne*. Grenoble: La Pensée Sauvage.

Churchill, R. V., Brown, J. W., & Verhey, R. F. (1974). *Complex variables and applications*. London: Mc Graw-Hill.

Coolidge, J. L. (1950). The number e. *American Mathematical Monthly, 57*, 591–602.

Coolidge, J. L. (1990). *The mathematics of great amateurs*. Oxford: Claredon.

Dhombres, J. (1993). Is one proof enough? Travels with a mathematician of the Baroque period. *Educational Studies in Mathematics, 24*, 401–419.

Dreyfus, T. (1991a). On the status of visual reasoning in mathematics and mathematics education. In F. Furinghetti (Ed.), *Proceedings of the fifteenth international conference for the psychology of mathematics education* (Vol. 1, pp. 32–48). IPC, PME 15, Assisi, Italy.

Dreyfus, T. (1991b). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25–41). Dordrecht: Kluwer.

Edwards, C. H. (1979). *The historical development of the calculus*. New York: Springer.

Euler, L. (1748/1990). *Introduction to analysis of the infinite, books I and II* (J. D. Blanton, Trans.). New York: Springer.

Euler, L. (1770/1984). *Elements of algebra*, (John Hewlet, Trans.). New York: Springer.

Fauvel, J. (1991). Using history in mathematics education. *For the Learning of Mathematics, 11*(2), 3–6.

Fauvel, J. (1995). Revisiting the history of logarithms. In F. Swetz, J. Fauvel, O. Bekken, B. Johanson, & V. Katz (Eds.), *Learn from the masters!* (pp. 39–48). USA: MAA Notes.

Fauvel, J., & Gray, J. (1987). *The history of mathematics: A reader*. London: Macmillan.

Fauvel, J., & van Maanen, J. (2000). *History in mathematics education, The ICMI Study*. Dordrecht: Kluwer.

Flegg, H. G., Hay, C. M., & Moss, B. (Trans.) (Ed.) (1985). *Nicolas Chuquet, renaissance mathematician*. Boston: Reidel.

Fowler, D. H. (1992). A final-year university course on the history of mathematics: Actively confronting the past. *American Mathematical Monthly, 76*, 46–48.

Freudenthal, H. (1973). What groups mean in mathematics and what they should mean in mathematical education. In A. G. Howson (Ed.), *Developments in mathematical education* (pp. 101–114). Cambridge: Cambridge University Press.

Freudenthal, H. (1984). The implicit philosophy of mathematics history and education. In *Proceedings of the ICMI* 1983 (Vol. 2, pp. 1695–1709). Warszawa/Amsterdam: PNW/North Holland.

Friedelmeyer, J.-P. (1990). Teaching 6th form mathematics with a historical perspective. In J. Fauvel (Ed.), *History in the mathematics classroom: The IREM papers* (pp. 1–16). Leicester: Mathematical Association.

Furinghetti, F., & Radford, L. (2002). Historical conceptual developments and the teaching of mathematics: From phylogenesis and ontogenesis theory to classroom practice. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 631–654). New Jersey: Lawrence Erlbaum.

Guedj, D. (1998). *Le théorème du perroquet*. Paris: Editions du Seuil. (F. Wynne, Trans., 2000: *The Parrot's theorem: A novel*. New York: St. Martin's Press).

Gulikers, I., & Blom, K. (2001). A historical angle, a survey of recent literature on the use and value of history in geometrical education. *Educational Studies in Mathematics, 47*, 223–258.

Hadamard, J. (1954). *The psychology of invention in the mathematical field*. New York: Dover.

Hallenberg, A. (Ed.) (1989/1969). Historical topics for the mathematics classroom. In *Thirty-first yearbook of NCTM*. Washington, DC: The Council.

Heath, T. L. (1956). *The thirteen books of the elements* (Vol. 1, 2, 3). New York: Dover.

Heiede, T. (2000). A Babylonian tablet. In J. Fauvel & J. Van Maanen (Eds.), *History in mathematics education, The ICMI Study* (pp. 255–257). Dordrecht: Kluwer.

Jahnke, H. H., Knoche, N., & Otte, M. (1996). *History of mathematics and education: Ideas and experiences*. Göttingen: Vandenhoeck und Ruprecht.

Katz, V. (1995). Napier's logarithms adapted for today's classroom. In F. Swetz, J. Fauvel, O. Bekken, B. Johanson, & V. Katz (Eds.), *Learn from the masters!* (pp. 49–55). USA: MAA Notes.

Klein, F. (1945). *Elementary mathematics from an advanced standpoint: Arithmetic – algebra – analysis* (E. R. Hedrick & C. A. Noble, Trans.). New York: Dover Publications.

Kline, M. (1972). *Mathematical thought from ancient to modern times*. New York: Oxford University Press.

Kouteynikoff, O. (2006). Inventions de Nombres: Calculs ou Resolutions? In *Histoire de Logarithmes* (pp. 11–37). Commission Inter-IREM, Paris: Ellipses.

Kronfellner, M. (1996). The history of the concept of function and some implications for classroom teaching. In R. Calinger (Ed.), *Vita mathematica: Historical research and integration with teaching* (pp. 317–320). Washington: MAA.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge: Cambridge University Press.

Lubet, J-P. (2006). Dans le Traités d' Analyse: Logarithmes et Exponentielles au Gré de la Rigueur. In *Histoire de Logarithmes* (pp. 233–268).Commission Inter-IREM, Paris: Ellipses.

McBride, C. C., & Rollins, J. H. (1977). The effects of history of mathematics on attitudes toward mathematics of college algebra students. *Journal for Research in Mathematics Education, 8*(1), 57–61.

Menghini, M. (2000). On potentialities, limits and risks. In J. Fauvel & J. Van Maanen (Eds.), *History in mathematics education, The ICMI Study* (pp. 86–90). Dordrecht: Kluwer.

Ofir, R., & Arcavi, A. (1992). Word problems and equations: An historical activity for the algebra classroom. *The Mathematical Gazette, 76*(475), 69–84.

Pierce, R. C. (1977). Sixteenth-century astronomers had prosthaphaeresis. *The Mathematics Teacher, 70*, 613–614.

Pólya, G. (1954/1990). *Induction and analogy in mathematics*. Princeton: Princeton University Press.

Pólya, G. (1981). *Mathematical discovery* (Vol. II). New York: Wiley.

Radford, L., & Guérette, G. (1996). Second degree equations in the classroom: A Babylonian approach. In V. Katz (Ed.), *Using history to teach mathematics: An international perspective* (pp. 69–75). Washington: MAA.

Resnikoff, H. L., & Wells, R. O., Jr. (1984). *Mathematics in civilization*. New York: Dover.

Rowe, D. E. (1996). New trends and old images in the history of mathematics. In R. Calinger (Ed.), *Vita mathematica: Historical research and integration with teaching* (pp. 3–16). Washington: MAA.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational studies in mathematics, 22*, 1–36.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behavior, 14*, 15–39.

Sierpinska, A. (1994). *Understanding in mathematics*. London, Washington, DC: The Falmer.

Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *Zentrallblatt fur Didactik der Mathematik, 3*, 75–80.

Smith, D. E. (1915). The law of exponents in the works of the sixteenth century. In C. G. Knott (Ed.), *Napier tercentenary memorial volume* (pp. 81–91). London: Longmans Green and Co.

Smith, D. E. (1923/1958). *History of mathematics* (Vol. 2). New York: Dover Publications.

Smith, D. E. (1929/1959). *A source book in mathematics*. New York: Dover Publications.

Sternberg, R. J. (1988). *The nature of creativity: Contemporary psychological perspectives*. New York: Cambridge University Press.

Struik, D.J. (1948/1987). *A concise history of mathematics*. New York: Dover Publications.

Swade, D. (1991). *Charles Babbage and his calculating engines*. London: Science Museum.

Swetz, F. (1984). Seeking relevance? Try the history of mathematics. *Mathematics Teacher, 77*(1), 54–62.

Swetz, F. (1987). *Capitalism and arithmetic: The new math of the 15th century*. La Salle: Open Court.

Swetz, F. (1995). To know and to teach: Mathematical pedagogy from a historical context. *Educational Studies in Mathematics, 29*, 73–88.

Van der Waerden, B. L. (1961). *Science awakening*. New York: Holt, Rinehart and Winston.

Van Looy, H. (1984). A chronology and historical analysis of the mathematical manuscripts of Gregorius a Sanctio Vincentio. *Historia Mathematica, 11*, 57–75.

Vasco, C. E. (1995). History of mathematics as a tool for teaching mathematics for understanding. In D. N. Perkins, et al. (Eds.), *Software goes to school* (pp. 56–69). New York: Oxford University Press.

Waldeg, G. (1997). Histoire, Epistemologie et Methodologie dans la Recherche en Didactique. *For The Learning of Mathematics, 17*(1), 43.

Whiteside, D. T. (1960–1962). Mathematical thought in the later 17th century. *Archive for the History of Exact Science, 1*, 179–387.

Whiteside, D. T. (1967–1976). *The mathematical papers of Isaac Newton* (Vol. I, II). Cambridge: Cambridge University Press.