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ELEMENTARY PARTICLE PHYSICS AND FIELD THEORY

EINSTEIN-MAXWELL EQUATIONS FOR HOMOGENEOUS SPACES

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The paper studies the energy-momentum tensor components for admissible electromagnetic fields in a nonholonomic system given by the group operation in homogeneous spaces. Compact expressions are obtained for Maxwell field equations.

Keywords: homogeneous space, Maxwell field equations, Einstein–Maxwell equations.

INTRODUCTION

Gravitational, electromagnetic and other physical fields, in which equations of motion admit linear and quadratic integrals of motion, are rather interesting for gravitation and relativistic quantum theories, since the study of motion based on these integrals, allows to receive important information about both fields and field processes. Today, the classification of space-time metrics and admissible electromagnetic fields invariant with respect to the motion group G_r (N), where $r \leq 4$, acting on the hypersurface V_4, is complete. This allows to formulate a new classification problem, i.e., enumeration of all non-equivalent solutions of Einstein–Maxwell equations matching these symmetries. This problem is preceded by already solved problem of classifying Maxwell vacuum equations in works [1–3], which give all non-equivalent solutions of Maxwell vacuum equations for homogeneous spaces and admissible electromagnetic fields. According to [4], admissible fields are invariant with respect to the motion group $G_3(N)$ for homogeneous space.

The purpose of this work is in the form convenient for further application, study the energy-momentum tensor for the electromagnetic field and derive Maxwell vacuum equations for the admissible electromagnetic field in homogeneous space. Tetrad reference vectors used in semi-geodesic holonomic coordinates, are defined by operators of the group $G_3(N)$.

Note that the problem of studying spaces with a full set of Killing vector fields, is still relevant to the gravitation theory and cosmology. There is a large number of papers [5–34], in which the symmetry of these fields is used to solve various problems in the gravitation theory and cosmology and in the relativistic quantum theory.

TETRAD COMPONENTS OF ENERGY-MOMENTUM TENSOR

Let us consider a four-dimensional pseudo-Riemannian manifold g with the space-like hypersurface V_3 under a simply transitive action of the group $G_3(N)$. The Bianchi classification is used for such homogeneous spaces of the type N. In the semi-geodesic coordinate system $\{u^i\}$, the metric on the homogeneous space takes the form:

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$$ds^2 = -du^{0^2} + \eta_{\alpha\beta}\left(u^0\right)e_a^\alpha e_b^\beta du^a du^b \,, \tag{1}$$

$$X_\alpha = e_\alpha^a \partial_a, \qquad \left|X_\alpha, X_\alpha\right| = C_{\alpha\beta}^\gamma X_\alpha, \qquad e_{a,0}^\alpha = 0,\ e_a^\alpha e_\beta^a = \delta_\beta^\alpha \,. \tag{2}$$

Coordinate indices of tensor quantities in the semi-geodesic coordinate system are indicated by small letters i, j, k, l = 0 to 3 and a, b = 1 to 3. Indices in the nonholonomic frame of the reference on V_3 hypersurface are indicated by Greek letters α, β, γ, σ, τ = 1 to 3. Repeated upper and lower indices are summarized within the change range. The V_3 hypersurface geometry is given by reference vectors of the nonholonomic triad e_α^a and nonholonomic components of the metric tensor $\eta_{\alpha\beta}$. The reference vector tetrad in V_4 hypersurface is written as $e_j^i = \left[\delta_0^i, e_\alpha^i\right]$.

The admissible electromagnetic potential components in holonomic coordinates can be represented by $A_0 = 0,\quad A_a = e_a^\alpha a\left(u^0\right)$. Tetrad components of the electromagnetic field tensor are as follows:

$$f_{\alpha\beta} = e_\alpha^a e_\beta^b F_{ab} = C_{\beta\alpha}^\gamma a_\gamma = \omega^1 \varepsilon_{\alpha\beta}^{23} + \omega^2 \varepsilon_{\alpha\beta}^{31} + \omega^3 \varepsilon_{\alpha\beta}^{12} \,. \tag{3}$$

Here

$$\omega^1 = C_{23}^\gamma a_\gamma, \qquad \omega^2 = C_{31}^\gamma a_\gamma, \qquad \omega^3 = C_{12}^\gamma a_\gamma. \qquad \varepsilon_{\alpha\beta}^{\gamma\sigma} = \delta_\alpha^\gamma \delta_\beta^\sigma - \delta_\beta^\gamma \delta_\alpha^\sigma \,.$$

Let us find nonholonomic components of the energy-momentum tensor $\tilde{T}_j^i = 16\kappa T_j^i$:

$$T_j^i = d_j^i F^{ij} F_{ij} - 4F_i^k F_{jk} \,.$$

Using Eq. (3), functions $f^{\alpha\gamma} f_{\beta\gamma} \quad \left(f^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\sigma} f_{\gamma\sigma}\right)$ can be obtained from

$$\eta f^{\alpha\gamma} f_{\beta\gamma} = \delta_\beta^\alpha \eta_{\sigma\gamma}\omega^\gamma\omega^\sigma - \eta_{\beta\gamma}\omega^\alpha\omega^\gamma \Rightarrow \eta f^{\alpha\gamma} f_{\alpha\gamma} = 2\eta_{\sigma\gamma}\omega^\gamma\omega^\sigma, \qquad \eta = \det\left\|\eta_{\alpha\beta}\right\| \,. \tag{4}$$

Let us introduce functions $\beta^\alpha = f^{0\alpha} = e_a^\alpha F^{0a} = \eta^{\alpha\beta} e_\beta^a \dot{a}_a$. The dot denotes a derivative with respect to the variable u^0. Tetrad components of the energy-momentum tensor $\tau_j^i = e_\kappa^i e_j^l T_l^\kappa$ can thus be written as follows:

$$\tau_\beta^\alpha = 2\left(\frac{1}{\eta}\left(2\eta_{\beta\gamma}\omega^\alpha\omega^\gamma - \delta_\beta^\alpha \eta_{\sigma\gamma}\omega^\gamma\omega^\sigma\right) + 2\eta_{\beta\gamma}\beta^\alpha\beta^\gamma - \delta_\beta^\alpha \eta_{\sigma\gamma}\beta^\gamma\beta^\sigma\right), \tag{5}$$

$$\tau_\alpha^0 = 4\varepsilon_{\alpha\beta\gamma}\omega^\beta\beta^\gamma, \quad \tau_0^0 = 2\left(\frac{1}{\eta}\eta_{\sigma\gamma}\omega^\gamma\omega^\sigma + \eta_{\sigma\gamma}\beta^\gamma\beta^\sigma\right).$$

MAXWELL FIELD EQUATIONS

Let us obtain Maxwell equations in the chosen nonholonomic tetrad. Thus, the function $\Im^j$ is obtained for this purpose:

$$\Im^{j} = \frac{1}{\sqrt{g}}\left(\sqrt{g}\, e^{j}_{\kappa} F^{\kappa i}\right)_{,i} \qquad \left(g = \det\left\| g_{ab} \right\|\right). \tag{6}$$

Let $g = \left(e e e_0\right)^2$, where $e_0^2 = \det\left\|\eta_{\alpha\beta}\right\|$, $e = \det\left\|e^{\alpha}_{a}\right\|$, $f_{|\alpha} = e^{a}_{\alpha} f_{,a}$, then the function $\Im^{\gamma}$ is as follows:

$$\Im^{\gamma} = f^{\alpha\beta}\left(\delta^{\gamma}_{\alpha}\left(\frac{e_{|\beta}}{e} + e^{a}_{\beta_{|a}}\right) + e^{\gamma}_{a} e^{a}_{\alpha_{|\beta}}\right) + \frac{1}{e_0}\left(e_0 \beta^{\alpha}\right)_{,0} = C^{\sigma}_{\sigma\beta} f^{\gamma\delta} - C^{\gamma}_{\alpha<\beta} f^{\alpha\beta} + \frac{1}{e_0}\left(e_0 \beta^{\alpha}\right)_{,0}. \tag{7}$$

Using Eq. (3), we get

$$\Im^{\alpha} = \frac{1}{e_0}\left(e_0 \beta^{\alpha}\right)_{,0} - \frac{1}{e_0^2}\eta_{\alpha\beta}\omega^{\beta} c^{\alpha\gamma}, \qquad c^{\alpha 1} = C^{\alpha}_{23}, \qquad c^{\alpha 2} = C^{\alpha}_{31}, \qquad c^{\alpha 3} = C^{\alpha}_{12}. \tag{8}$$

The function $\Im^{0}$ is defined by

$$\Im^{0} = -\beta^{\alpha}\left(e^{a}_{\alpha,a} + \frac{e_{|\alpha}}{e}\right) = \beta^{\alpha} C^{\gamma}_{\alpha\gamma}. \tag{9}$$

Thus, Maxwell's vacuum equations take the form

$$e_0\left(e_0 \beta^{\alpha}\right)_{,0} = \eta_{\alpha\beta}\omega^{\beta} C^{\alpha\gamma}, \quad \beta^{\alpha} C^{\gamma}_{\alpha\gamma} = 0. \tag{10}$$

Tetrad components of the Ricci curvature tensor are given in [35]. Equating them to tetrad components of the energy-momentum tensor (Eq. (5)), remaining equations are derived from the system of Einstein-Maxwell equations.

COMPLIANCE WITH ETHICAL STANDARDS

Conflicts of interest

The authors declare no conflict of interest.

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REFERENCES

1. V. V. Obukhov, Symmetry, **14**, No. 12, 2595 (2022); doi.org/10.3390/sym14122595.
2. V. V. Obukhov, Symmetry, **15**, No. 3, 648 (2023); doi.org/10.3390/sym15030648.
3. V. V. Obukhov, Axioms, **12**, 135 (2023); doi.org/10.3390/axioms12020135.
4. V. V. Obukhov, J. Math. Phys., **63**, No. 2, (2022); doi.org/10.1063/5.0080703.
5. A. A. Magazev, Theor. Math. Phys., **173**, No. 3, 1654–1667 (2012); doi.org/10.1007/s11232-012-0139-x, arxiv.org/abs/1406.5698.
6. A. A. Magazev, I. V. Shirokov, and Yu. A. Yurevich, Theor. Math. Phys., **156**, No. 2, 1127–1141 (2008); doi.org/10.4213/tmf6240.
7. V. V. Obukhov, S. V. Chervon, and D. V. Kartashov, IJGMMP, (2024).
8. S. Capozziello, M. De Laurentis, and S. D. Odintsov, Eur. Phys. J. C, **72**, 2068 (2012); doi:10.1140/epjc/s10052-012-2068-0.
9. E. K. Osetrin, K. E. Osetrin, and A. E. Filippov, Russ. Phys. J., **63**, No. 3, 410–419 (2020); doi.org/10.1007/s11182-020-02051-1.
10. E. K. Osetrin, K. E. Osetrin, and A. E. Filippov, Russ. Phys. J., **63**, No. 3, 403–409 (2020); doi.org/10.1007/s11182-020-02050-2.
11. J. de Haro, S. Nojiri, S. D. Odintsov, V. K. Oikonomou, and S. Pan, Phys. Rept., **1034**, 1–114 (2023); doi:10.1016/j.physrep.2023.09.003.
12. S. Nojiri and S. D. Odintsov, Phys. Rept., **505**, 59–144 (2011); doi:10.1016/j.physrep.2011.04.001.
13. V. V. Obukhov, Symmetry, **14**, 346 (2022); doi.org/10.3390/sym14020346.
14. I. Brevik, V. V. Obukhov, and A. V. Timoshkin, MPLA, **31**, No. 18, 1650105 (2016); doi.org/10.1142/S0217732316501054.
15. E. Elizalde, A. N. Makarenko, V. V. Obukhov, and S. D. Odintsov, Astrophys. Space Sci., 344, 479–488 (2013); doi.org/10.1007/s10509-012-1339-4.
16. U. Camci, Symmetry, **16**, 115 (2024); doi.org/10.3390/sym16010115.
17. A. V. Shapovalov, Symmetry, **15**, No. 4, 905 (2023); doi.org/10.3390/sym15040905.
18. E. Osetrin and K. Osetrin, J. Math. Phys., **58**, No. 11, 112504 (2017); doi.org/10.1063/1.5003854.
19. K. Osetrin, A. Filippov, E. Osetrin, Mod. Phys. Lett., **31**, No. 6, 1650027 (2016); doi.org/10.1142/S0217732316500279.
20. K. Osetrin, A. Filippov, E. Osetrin, Mod. Phys. Lett., **35**, 2050275 (2020); doi.org/10.1142/S0217732320502752.
21. V. V. Obukhov, J. Math. Phys., **64**, 093507 (2023); doi.org/10.1063/5.0158054.
22. E. Osetrin, K. Osetrin, A. Filippov, and I. Kirnos, IJGMMP, **17**, No. 12, 2050184 (2020); doi.org/10.1142/S0219887820501844.
23. V. Epp and M. A. Masterova, Astrophys. Space Sci., **353**, No. 2, 473–483 (2014).
24. V. Epp and O. N. Pervukhina, MNRAS, **474**, No. 4, 5330–5339 (2018).
25. K. Osetrin, E. Osetrin, and E. Osetrina, Eur. Phys. J. C, **82**, 894 (2022); doi.org/10.1140/epjc/s10052-022-10852-6.
26. K. Osetrin, E. Osetrin, and E. Osetrina, Eur. Phys. J. Plus, **137**, 856 (2022); doi.org/10.1140/epjp/s13360-022-03061-3.
27. K. E. Osetrin, A. E. Filippov, and E. K. Osetrin, Russ. Phys. J., **61**, No. 8, 1383–1391 (2018).
28. E. Osetrin, and K. Osetrin, J. Math. Phys., **58**, No. 11, 112504 (2017); doi.org/10.1063/1.5003854.
29. I. V. Fomin and S. V. Chervon, Phys. Rev., **D 100**, No. 2, 023511 (2019); doi.org/10.1103/PhysRevD.100.023511.
30. I. V. Fomin and S. V. Chervon, Russ. Phys. J., **60**, No. 30, 427–440 (2017); doi.org/10.1007/s11182-017-1091-x.
31. K. Osetrin, E. Osetrin, and E. Osetrina, Eur. Phys. J. C, **82**, 894 (2022); doi.org/10.1140/epjc/s10052-022-10852-6.

32. K. Osetrin, E. Osetrin, and E. Osetrina, Eur. Phys. J. Plus, **137**, 856 (2022); doi.org/10.1140/epjp/s13360-022-03061-3.
33. K. E. Osetrin, A. E. Filippov, and E. K. Osetrin, Russ. Phys. J., **61**, No. 8, 1383–1391 (2018); doi.org/10.1007/s11182-018-1546-8.
34. K. Osetrin and E. Osetrin, Symmetry, **12**, No. 8, 1372 (2020); doi.org/10.3390/SYM12081372.
35. L. D. Landau and E. M. Lifshits, The Classical Theory of Fields [in Russian], Vol. 2, 7th ed., Nauka, Moscow (1988).