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COSMOLOGICAL MODELS WITH BIANCHI TYPE V METRIC

E. V. Kuvshinova,[1] O. V. Sandakova,[2] and D. M. Yanishevsky[1] UDC 530.12:531.551

Within the framework of General relativity, two cosmological models are constructed for type V metric according to the Bianchi classification. In the first model, an anisotropic liquid is used as the source of gravity. This model is non-stationary and non-rotating. In the second cosmological model, sources of gravity are anisotropic liquid and pure radiation. All kinematic parameters of the models have been found. Both models are examined for causality.

Keywords: cosmological model, Einstein equations, dark energy.

INTRODUCTION

Nowadays, small anisotropy of the Universe, including anisotropy caused by cosmological rotation, is not excluded; therefore, models in homogeneous but non-isotropic metrics are of interest related to the past of the Universe and the behavior of matter in the vicinity of a singular state. Thus, V. A. Korotkii [1] proposed two experiments to detect rotation of the homogeneous Universe with the geometry correct in the causal sense. Using the symmetry of the problem, he found the first integrals of the geodesics corresponding to isotropic vectors – trajectories of light beams in the geometric optics approximations. In the Bianchi type II metric considered in [1], these trajectories – null geodesics – are screw lines of constant radius and variable step passing through any observer and in some directions degenerating into circles. V. A. Korotkii noted that the presence of closed light rays passing through the observer could be the proof of global rotation. The matter is that if we assume that the closed light ray emitted by the observer's *home* galaxy has not crossed anywhere other galaxies and in time T has returned back, two images of the initial galaxy, but T years younger, will be observed in opposite sides of the celestial sphere. It is the first proposed experiment. The second experiment consists in observation of a remote galaxy, light from which reaches the observer unhindered, along the closed index-zero geodesics from both sides. In this case, the observer will observe the remote galaxy images, the ages of which can be equal or significantly different, in opposite sides of the celestial sphere. The implementation of such experiment implies the study of the catalog of galaxy images to elucidate opposite directions of their observations and to identity their shapes and sizes [1].

The Bianchi type V metric was successfully used by different authors in a number of works to elucidate relationships between relic radiation and isotropic cosmological model parameters [2], to construct a model with the dynamics similar to the Friedman one [3], to consider spinor fields in cosmology to study the corresponding cosmological modes [4], and for some other reasons [5]. Here we construct the cosmological models with and without rotation.

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[1]Perm State University, Perm, Russia, e-mail: kuvlenka@narod.ru; ydm86@yandex ru; [2]Perm State Agro-Technological University named after Academician D. N. Pryanishnikov, Perm, Russia, e-mail: o_sandakova@list ru.
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MODEL WITHOUT ROTATION

The metric of Bianchi type V has the form

$$ds^2 = \eta_{\alpha\beta}\theta^\alpha\theta^\beta, \alpha,\beta = \overline{0,3}, \tag{1}$$

where $\eta_{\alpha\beta}$ are elements of the diagonal Lorentz matrix and θ^α are the orthonormalized 1-forms expressed as follows:

$$\theta^0 = dt - R\nu_A e^A,\ \theta^A = RK_A e^A,\ \ K_A = \{a,b,c\},\ \nu_A = \{d,0,0\},\ A = 1,2,3. \tag{2}$$

The 1-forms of e^A are

$$\begin{aligned} e^1 &= dx, \\ e^2 &= e^x dy, \\ e^3 &= e^x dz. \end{aligned} \tag{3}$$

For metrics (1)–(3), we are looking for a solution of the Einstein equations

$$R_{ik} - \frac{1}{2}Rg_{ik} = T_{ik}. \tag{4}$$

Here $c = 1$, $\hbar = 1$, and $8\pi G = 1$, where G is Newton's gravitational constant. We are looking for solution of Eq. (4) in the tetrad formalism. In this case, the gravitation source is the anisotropic liquid which describes the dark energy.

The energy-momentum tensor of the anisotropic liquid has the form

$$T_{ik} = (p+\rho)u_i u_k + (\sigma - p)\chi_i\chi_k - p\eta_{ik}, \tag{5}$$

where $u_i = \delta_0^i$ is the 4-velocity vector of the anisotropic liquid projected onto the tetrad, $\chi_i = \{0,1,0,0\}$ is the anisotropy vector, ρ is the liquid energy density, and p,σ are the anisotropic pressure components. From Eq. (4) for metrics (1)–(3), we obtain the system of equations:

$$\begin{cases} \dfrac{-3 - 4d\dot{R} + (3a^2 - d^2)\dot{R}^2 - 2d^2 R\ddot{R}}{a^2R^2} = \rho, \\[2ex] \dfrac{1 + 4d\dot{R} - (a^2 - 3d^2)\dot{R}^2 - 2a^2 R\ddot{R}}{a^2R^2} = \sigma, \\[2ex] \dfrac{1 + 2d\dot{R} + (d^2 - a^2)\dot{R}^2 + 2(d^2 - a^2)R\ddot{R}}{a^2R^2} = p, \\[2ex] \dfrac{2d(\dot{R}^2 - R\ddot{R})}{aR^2} = 0. \end{cases} \tag{6}$$

System (6) has the non-static solution:

$$R = R_0 e^{Ht}, \tag{7}$$

$$\rho = \frac{3(a^2-d^2)H^2}{a^2} - \frac{4Hd}{a^2 R_0 e^{Ht}} - \frac{3}{a^2 R_0{}^2 e^{2Ht}}, \tag{8}$$

$$\sigma = -\frac{3(a^2-d^2)H^2}{a^2} + \frac{4Hd}{a^2 R_0 e^{Ht}} + \frac{1}{a^2 R_0{}^2 e^{2Ht}}, \tag{9}$$

$$p = -\frac{3(a^2-d^2)H^2}{a^2} + \frac{2Hd}{a^2 R_0 e^{Ht}} + \frac{1}{a^2 R_0{}^2 e^{2Ht}}. \tag{10}$$

The energy of the anisotropic liquid is positive when the conditions $a^2 > d^2$ and $HR_0 \geq \frac{2d + \sqrt{9a^2 - 5d^2}}{3(a^2 - d^2)}$ are satisfied, and at greater times, its asymptotic izotropization takes place, and the state equations become vacuum-like.

Determine whether the non-static model with metrics defined by conditions (1)–(3) is causal. For this purpose, we assume the existence of closed time-like curves with the point on each curve satisfying to the condition $dt / ds = 0$, whereas $u_\mu u^\mu > 0$ due to time similarity. To satisfy to these two conditions, the quadratic form comprising components of the tangent vector with the matrix of the coefficients of spatial metric tensor components should be positively defined. The matrix of the spatial component of the examined metric has the form

$$\begin{pmatrix} (d^2-a^2)R^2 & 0 & 0 \\ 0 & -b^2 e^{2x} R^2 & 0 \\ 0 & 0 & -c^2 e^{2x} R^2 \end{pmatrix}. \tag{11}$$

The form of this matrix, taking into account the positive energy density, implies the non-positively defined quadratic form. Thus, the contradiction arises with the hypothesis on the existence of closed time-like lines; hence, the examined model is causal. Here the expansion parameter $\Theta = \frac{3\dot{R}}{R}$, acceleration $A = \frac{d\dot{R}}{aR}$, rotation, and shift are absent.

MODEL WITH ROTATION

Let the Bianchi type V metric has the same form as Eq. (1), but the orthonormalized 1-forms can now be written as

$$\theta^0 = dt - R\nu_A e^A,\ \theta^A = RK_A e^A,\ K_A = \{a,b,b\},\ \nu_A = \{d,d,0\},\ A = 1,2,3. \tag{12}$$

Gravitation sources for this model are the anisotropic liquid and pure radiation. The energy-momentum tensor of the anisotropic liquid is

$$T_{ik}^{(1)} = (p+\rho)u_i u_k + (\sigma - p)\chi_i \chi_k + (\pi - p)\xi_i \xi_k - p\eta_{ik}, \tag{13}$$

where $p, \pi, \text{and}\ \sigma$ are the anisotropic liquid pressure components, ρ is the anisotropic liquid energy density, $\chi_i = \{0,0,1,0\}$ and $\xi_i = \{0,0,0,1\}$ are the anisotropy vectors, and $u^i = \delta_0^i$ is the 4-velocity vector of the accompanying anisotropic liquid. The pure radiation field with the energy-momentum tensor

$$T_{ik}^{(2)} = wk_i k_k, (w > 0),\ k_i = \{k_0, k_1, k_{1,}, 0\} \tag{14}$$

is also among the material sources. In system of Einstein equations (4), we accept that

$$T_{ik} = T_{ik}^{(1)} + T_{ik}^{(2)}. \tag{15}$$

Then the system of the Einstein equations takes the form

$$-\frac{12b^2 - 3d^2 + 4\dot{R}\left(4b^2 d + \left(-3a^2 b^2 + (a^2+b^2)d^2\right)\dot{R}\right) + 8\left(a^2+b^2\right)d^2 R\ddot{R}}{4a^2 b^2 R^2} = \rho + wk_0{}^2,$$

$$\frac{4b^2 + d^2 + 4\dot{R}\left(4b^2 d + \left(3d^2 b^2 + (d^2 - b^2)a^2\right)\dot{R}\right) + 8\left(d^2 - b^2\right)a^2 R\ddot{R}}{4a^2 b^2 R^2} = p + wk_1{}^2,$$

$$\frac{d\left(\dot{R}(d + 2b^2\dot{R}) - 2b^2 R\ddot{R}\right)}{ab^2 R^2} = wk_0 k_1, \tag{16}$$

$$-\frac{d\left(1 + 2\dot{R}(d - 2a^2\dot{R}) + 4a^2 R\ddot{R}\right)}{2a^2 b R^2} = wk_0 k_1,$$

$$\frac{d\left(\dot{R} + 2d\dot{R}^2 - 2dR\ddot{R}\right)}{abR^2} = wk_1{}^2,$$

$$\frac{4b^2 + d^2 + 8b^2 d\dot{R} + 4\left(-a^2 b^2 + \left(3a^2 + b^2\right)d^2\right)\dot{R}^2 + 8b^2\left(d^2 - a^2\right)R\ddot{R}}{4a^2 b^2 R^2} = \sigma + wk_1{}^2,$$

$$\frac{4b^2 - d^2 + 8b^2 d\dot{R} + 4\left(-a^2 b^2 + \left(a^2 + b^2\right)d^2\right)\dot{R}^2 + 8\left(-a^2 b^2 + (a^2+b^2)d^2\right)R\ddot{R}}{4a^2 b^2 R^2} = \pi.$$

From Eq. (16) we obtain the equation for the scale factor

$$4a(a-b)\dot{R}^2 - 4a(a-b)R\ddot{R} - 2\left(\frac{a+b}{b}\right)d\dot{R} = 1 \tag{17}$$

and the parameters of matter

$$\rho = \frac{3d^2 - 12b^2 - 4d\left(4b^2 + \sqrt{2}ad\dot{R}\right) - 4\left(2\sqrt{2}ab^2d + b^2d^2 + a^2\left(-3b^2 + d^2\right)\right)\dot{R}^2}{4a^2b^2R^2}$$

$$+\frac{8d\left(\sqrt{2}ab^2 - a^2d - b^2d\right)R\ddot{R}}{4a^2b^2R^2}, \tag{18}$$

$$k_0{}^2 = \frac{2d\left(\dot{R} + 2d\dot{R}^2 + 4a^2R\ddot{R}\right)}{wabR^2}, \tag{19}$$

$$k_1{}^2 = \frac{d\left(\dot{R} + 2d\dot{R}^2 + 4a^2R\ddot{R}\right)}{wabR^2} \tag{20}$$

$$p = \frac{4b^2 + d^2 - 4(a - 4b)bd\dot{R} - 4\left(2abd^2 - 3b^2d^2 + a^2(b^2 - d^2)\right)\dot{R}^2 + 8a\left(ad^2 + bd^2 - ab^2\right)R\ddot{R}}{4a^2b^2R^2}, \tag{21}$$

$$\sigma = \frac{4b^2 + d^2 + 4(2b - a)bd\dot{R} - 4\left(2abd^2 - 3b^2d^2 + a^2(b^2 - 3d^2)\right)\dot{R}^2 + 8b\left(ad^2 + bd^2 - ab^2\right)R\ddot{R}}{4a^2b^2R^2}, \tag{22}$$

$$\pi = \frac{4b^2 - d^2 + 8b^2d\dot{R} + 4\left(d^2(a^2 + b^2) - a^2b^2\right)\dot{R}^2 + 8\left(d^2(a^2 + b^2) - a^2b^2\right)R\ddot{R}}{4a^2b^2R^2}. \tag{23}$$

If $a \neq b$, using replacement $\dot{R} = y(R)$, we reduce Eq. (17) to

$$\frac{dR}{R} = \frac{ydy}{y^2 - 2\frac{\beta}{\alpha}y - \frac{1}{\alpha}},$$

where $\alpha = 4a(a - b)$ and $\beta = \frac{(a + b)d}{b}$. As a result, we obtain

$$\ln R = \frac{1}{2}\ln\left|y^2 - 2\frac{\beta}{\alpha}y - \frac{1}{\alpha}\right| + \frac{\beta}{2\sqrt{\beta^2 + \alpha}}\ln\left|\frac{y - \frac{\beta}{\alpha} - \sqrt{\frac{\beta^2 + \alpha}{\alpha^2}}}{y - \frac{\beta}{\alpha} + \sqrt{\frac{\beta^2 + \alpha}{\alpha^2}}}\right| - \ln A^2. \tag{24}$$

Designating $y_1 = \frac{\beta}{\alpha} + \sqrt{\frac{\beta^2 + \alpha}{\alpha^2}}$ and $y_2 = \frac{\beta}{\alpha} - \sqrt{\frac{\beta^2 + \alpha}{\alpha^2}}$, we can rewrite Eq. (24) in the form

$$\ln R^2 = \ln\left|(y - y_1)(y - y_2)\right| + \frac{\beta}{\sqrt{(\beta^2 + \alpha)}}\ln\left|\frac{y - y_1}{y - y_2}\right| - \ln A^2. \tag{25}$$

Then we obtain that the functions $z_1 = \ln|(y-y_1)(y-y_2)|$ and $z_2 = \frac{\beta}{\sqrt{\beta^2+\alpha}}\ln\left|\frac{y-y_1}{y-y_2}\right|$ increase for $y > y_1$.

Since we are interested in the expansion, we restrict ourselves to the requirement that $y > y_1$; in this case, z_1 and z_2 increase. In addition, $\frac{y-y_1}{y-y_2} = 1 + \frac{y_2 - y_1}{y-y_2} < 1$ for any $y > 0$. Then taking into account that $\dot{R} > 0$, under the condition that $\frac{\beta}{\alpha} + \sqrt{\left(\frac{\beta}{\alpha}\right)^2 + \frac{1}{\alpha} + A^2R^2} >> \frac{\beta}{\alpha} + \sqrt{\left(\frac{\beta}{\alpha}\right)^2 + \frac{1}{\alpha}}$ at large times, we approximately obtain $R \approx R_0 e^{Ht}$ and $H = A$. In this case, we obtain the following parameters:

$$\rho = \frac{-d^2\left(-3+4\sqrt{2}aR_0He^{Ht}+12a^2R_0{}^2H^2e^{2Ht}\right)-4b^2\left(3+4R_0dHe^{Ht}-3R_0{}^2\left(a^2-d^2\right)H^2e^{2Ht}\right)}{4a^2b^2R_0{}^2e^{2Ht}}, \quad (26)$$

$$p = \frac{-4abdR_0He^{Ht}+d^2\left(1+12a^2R_0{}^2H^2e^{2Ht}\right)+4b^2\left(1+4R_0dHe^{Ht}-3R_0{}^2\left(a^2-d^2\right)H^2e^{2Ht}\right)}{4a^2b^2R_0{}^2e^{2Ht}}, \quad (27)$$

$$\sigma = \frac{-4abdR_0He^{Ht}+d^2\left(1+12a^2R_0{}^2H^2e^{2Ht}\right)+4b^2\left(1+2R_0dHe^{Ht}-3R_0{}^2\left(a^2-d^2\right)H^2e^{2Ht}\right)}{4a^2b^2R_0{}^2e^{2Ht}}, \quad (28)$$

$$\pi = \frac{4b^2-d^2+8b^2dR_0He^{Ht}+4\left(d^2(a^2+b^2)-a^2b^2\right)R_0{}^2H^2e^{2Ht}}{4a^2b^2R_0{}^2e^{2Ht}} + \frac{8\left(d^2(a^2+b^2)-a^2b^2\right)R_0{}^2H^2e^{2Ht}}{4a^2b^2R_0{}^2e^{2Ht}}, \quad (29)$$

$$k_0{}^2 = \frac{d^2He^{-Ht}}{ab^2R_0}. \quad (30)$$

The matrix of the spatial component of the metric

$$\begin{pmatrix} (d^2-a^2)R^2 & d^2e^xR^2 & 0 \\ d^2e^xR^2 & (d^2-b^2)e^{2x}R^2 & 0 \\ 0 & 0 & -b^2e^{2x}R^2 \end{pmatrix}$$

is negatively defined provided that $a^2 > d^2$, what entails the space-time causality.

The dark energy (anisotropic liquid) kinematic parameters are the expansion parameter $\Theta = \frac{3\dot{R}}{R}$, the acceleration $A = \sqrt{\frac{\left(a^2+b^2\right)d^2\dot{R}^2}{a^2b^2R^2}}$, the rotation parameter $\omega = \frac{d}{2abR}$, and the shift $\sigma = \frac{d}{2abR}$.

At large times,

$$\rho \to \frac{3H^2\left(a^2b^2 - d^2(a^2+b^2)\right)}{a^2b^2},\ p \to \frac{3H^2\left(d^2(a^2+b^2) - a^2b^2\right)}{a^2b^2},$$

$$\sigma \to \frac{3H^2\left(d^2(a^2+b^2) - a^2b^2\right)}{a^2b^2},\ \pi \to \frac{3H^2\left(d^2(a^2+b^2) - a^2b^2\right)}{a^2b^2}.$$

Thus, the liquid undergoes asymptotic izotropization at large *t* and also becomes vacuum-like.

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