DOI 10.1007/s11182-021-02435-x
Russian Physics Journal, Vol. 64, No. 6, October, 2021 (Russian Original No. 6, June, 2021)

NEW APPROACH TO CALCULATING THE DOUBLE COAXIAL RESONATOR WITH A SHORTENING CAPACITANCE

V. N. Egorov and Le Quang Tuyen

UDC 621.317.335

The double coaxial resonator with a shortening capacitance is calculated by the partial volume method. The double resonator is represented by two single coaxial resonators with the same resonant frequencies equal to the frequency of the double resonator. The convergence of the calculated resonant frequency is investigated for different numbers of eigenmodes. The accuracy of calculating the resonant frequency is estimated compared to the experimental resonant frequency of the double coaxial resonator with exact internal sizes including the capacitive gap size.

Keywords: coaxial resonator with a shortening capacitance, resonant frequency, partial volume method, eigenmodes.

The coaxial resonator with a shortening capacitance (CRSC) belongs to the class of quasi-stationary resonators (Fig. 1*a*) also called toroidal resonators. In the decimeter wavelength range, it has compact sizes compared to a hollow-volume resonator and is widely used, in particular, for dielectric measurements [1, 2]. The CRSC is called single if its capacitive (shortening) gap is placed between faces of the plane metal resonator and the central electrode (Fig. 1*b*). The gap in the double CRSC is in the central electrode gap (Fig. 1*a*) [3]. To solve experimental problems with CRSC application, its resonant frequency and other characteristics should be calculated, and the accuracy of the calculation model should be estimated.

In calculations based on the quasi-stationary approximation, the CRSC gap is considered to be a capacitor [3]. This approach is simple, but does not provide the accuracy required for the CRSC application for measurements. In a more strict electrodynamic approach – the partial volume (PV) method – the resonant volume is subdivided into sub-volumes, and the PV field is described by the spectrum of its eigenmodes with boundary conditions imposed on the conductive resonator walls and the boundaries of the introduced PVs. The given approach was developed in [1, 4–7]. A number of investigations were carried out by numerical methods [8, 9].

Common for the exact electrodynamic CRSC models in different approaches is the need for field representation by a large number of eigenmodes that leads to the occurrence of the high dimensional matrix determinant. Elucidation and attraction of additional physical conditions imposed on the field behavior in the resonator, in particular, inside the gap volume, can concretize the eigenfunctions and limit their number without loss of accuracy. To estimate the accuracy of the CRSC model, the exact internal sizes of a real resonator are required. The shortening gap height most strongly affects the resonant frequency. Its exact measurement inside of the assembled resonator is a separate problem.

In the present work, a new approach to calculation of the resonant frequency of the double CRSC (Fig. 1*a*) is proposed by reducing it to two single resonators with different shortening gaps, the same resonant frequencies equal to

[1]East-Siberian Branch of the All-Russian Scientific Research Institute for Physical-Engineering and Radiotechnical Metrology, Irkutsk, Russia, e-mail: egorov@vniiftri-irk.ru; [2]Irkutsk National Research Technical University, Irkutsk, Russia, email: lequangtuyen1402@gmail.com. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Fizika, No. 6, pp. 164–169, June, 2021. Original article submitted November 11, 2020; revision submitted December 4, 2020.

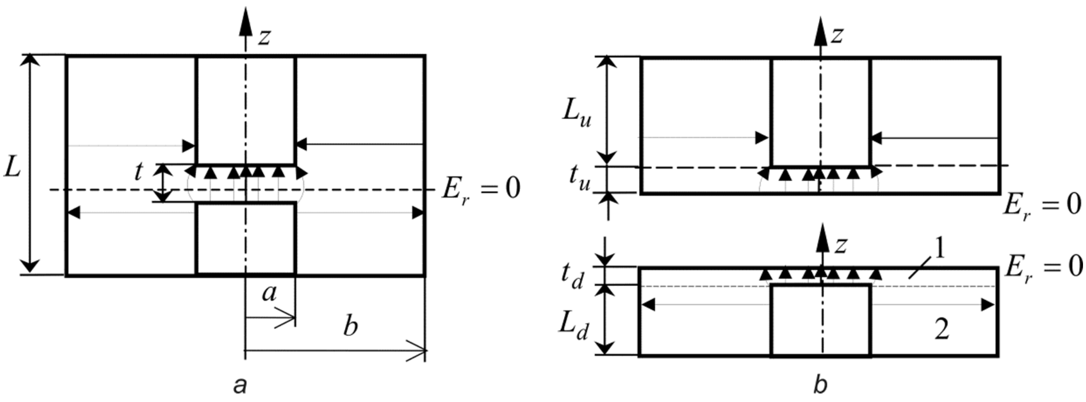

Fig. 1. Double CRSC (*a*) and its representation by two single CRSCs (*b*).

the double CRSC frequency, and the total height of the gaps of the single resonators equal to the height of the gap of the double CRSC (Fig. 1*b*). The PV method was used for calculation. The sizes of the experimental double CRSCs were measured, including the shortening gap sizes, and the calculated and experimental resonant frequencies were compared.

In the first stage of CRSC calculations, the quasi-stationary approach considerably reduced the volume of calculations and the number of iterations. In this approach, the double CRSC is represented by two single CRSC in the coordinates r, φ, z (Fig. 1). Coaxial line segments with lengths of L_d and L_u and the *TEM* mode short-circuited at one end and opened at the other end are separated by the gap of height t. In the approximation of the ideal conductivity of the resonator walls, the input impedances of the lower and upper coaxial segments of the resonator in planes $z = L_d$ and $z = L_d + t$ are $Z_d = jX_d = j\rho \tan(kL_d)$ and $Z_u = jX_u = j\rho\tan(kL_u)$, respectively, where $\rho = 60\sqrt{\mu/\varepsilon}\ln(b/a)$ is the wave impedance of the coaxial resonator segments, a and b are radii of the central electrode and housing, $k = \omega\sqrt{\varepsilon_0\varepsilon\mu_0\mu} = 2\pi/\lambda$ is the wave number, λ is the *TEM* wavelength, $\omega = 2\pi f$ is the frequency, ε and μ are the relative dielectric permittivity and the magnetic permeability of the medium in the resonator. Below we consider that the medium is non-magnetic ($\mu = 1$ and $\varepsilon \geq 1$).

When L_d and L_u are less than $\lambda/4$, the impedance of the coaxial segments is inductive in character. The coaxial resonator segments are connected through the gap of height t and capacitive impedance $jX_C = -j/(\omega C_\Sigma)$, where C_Σ is the total capacitance of the gap volume including the so-called *lateral* capacitance. The resonant frequency is determined by the condition $X_d + X_u + X_C = 0$ that gives

$$\rho\left[\tan(kL_d) + \tan(kL_u)\right] - 1/(\omega C_\Sigma) = 0\,. \quad (1)$$

The fields of the lowest *TEM* modes in the coaxial sections are in antiphase, and in the cross-section of the resonator in the gap volume at the distance $z_d = L_d + t_d$ ($t - t_d = t_u > 0$) there is the plane – an *electric* wall – with the radial electric field component $E_r(z_d) = 0$ in the entire plane for $0 \leq r \leq b$ (Fig. 1). This boundary condition on the *virtual* surface allows us to divide the double CRSC with the lowest *TEM* mode into two single CRSCs. The condition $E_r(z_d) = 0$ concretizes the field dependence on the coordinate z and limits the form of the functions suitable for field representation. For the fundamental *TEM* mode in the initial double resonator, the gap can be represented in the form of two capacitors-gaps in each single resonator connected in series. Their total capacitance is determined by the relation

$$C_\Sigma^{-1} = C_d^{-1}(t_d) + C_u^{-1}(t_u)\,, \quad (2)$$

where $C_d(t_d)$ and $C_u(t_u)$ are capacitances of gaps with heights t_d and t_u, including *lateral* capacitances. The gap sizes t_d and $t_u = t - t_d$ are determined by the equality of the resonant frequencies of two single CRSCs and the frequency of the initial double CRSC. Substitution of formula (2) into Eq. (1) taking into account the condition $E_r(z_d) = 0$ on the *electric* wall leads to the equations

$$\rho \tan\left[k(\omega) L_{d,u}\right] - 1\big/\left[\omega C_{d,u}(t_d)\right] = 0 \tag{3}$$

for two single CRSCs. The system of equations (3) gives the resonant frequency ω the same for the single CRSCs, the gap size t_d, and $t_u = t - t_d$. In addition to the capacities of the flat capacitors, the capacitances $C_{d,u}$ also include the *lateral* capacitances [3]

$$C_{d,u} = \varepsilon_0 \varepsilon \left[\frac{\pi a^2}{t_{d,u}} + 2a \ln\left(\frac{2b - 2a}{t_{d,u}}\right)\right]. \tag{4}$$

We preformed electrodynamic calculations of the resonant frequency of the double CRSCs by the PV method. First we considered the single CRSCs with the coaxial sections of lengths L_d and L_u corresponding to the gap heights t_d and t_u. The frequencies of the single identical CRSCs monotonically increased with gaps; moreover, the decrease of $t_u(t_d) = t - t_d$ corresponded to the increase of t_d. The change of the t_d value was accompanied by opposite changes of the frequencies $f_d(t_d)$ and $f_u(t_u(t_d))$ of the lower and upper single CRSCs; they coincided for a certain gap size t_{d0}. The point $f_d(t_{d0}) = f_u(t_u(t_{d0})) = f_0$ gives the resonant frequency of the double CRSC f_0 and the t_{d0} value, that is, the wall coordinate $z_{d0} = L_d + t_{d0}$. The gap height t of the double real CRSC should be determined from separate measurements.

Thus, calculation of the double CRSC is reduced to calculation of two single CRSCs with different lengths of the coaxial sections and gap sizes by the same program. The eigenfunctions of the field representation must satisfy to the boundary condition imposed on the conductive resonator walls and to the condition $E_r = 0$ on the partition plane $z_d = L_d + t_d$ the position of which is determined in the process of problem solution.

Let us consider for definiteness the lower section of the double CRSC – the single CRSC with the height L_d of the central electrode and the gap height t_d (see Fig. 1*b*). By analogy with [4], we take advantage of the electromagnetic field representation for the single CRSC. Owing to the independence of the field of the azimuthal angle, the function φ used to represent the field should depend only on the coordinates (r, z). We divide the single CRSC by the plane $z = L_d$ into two PVs: waveguide 1 and coaxial 2 (Fig. 1*b*).

We represent the field in waveguide volume 1 for $L_d \le z \le L_d + t_d$, $0 \le r \le b$, by the spectrum of waves E_{0q}. The resonant frequency of the fundamental *TEM* mode for the resonators used in practice lies below the critical frequencies of waveguide modes E_{0q}, that is, waveguide volume 1 at the resonant frequency is the postcritical waveguide for them. The field components in volume 1 are described by the functions

$$E_{z,1} = \sum_{q=1}^{\infty} B_q J_0(u_q r) \cosh\left[\beta_q \left(z - L_d - t_d\right)\right], \tag{5}$$

$$H_{\varphi,1} = j \sum_{q=1}^{\infty} \frac{k}{u_q} B_q J_1(u_q r) \cosh\left[\beta_q \left(z - L_d - t_d\right)\right], \tag{6}$$

$$E_{r,1} = -\sum_{q=1}^{\infty} \frac{\beta_q}{u_q} B_q J_1(u_q r) \sinh\left[\beta_q \left(z - L_d - t_d\right)\right], \tag{7}$$

where B_q are the amplitudes; u_q and $\beta_q = \sqrt{u_q^{\,2} - k^2}$ are the transverse and longitudinal wave numbers in volume 1, $q = 1, 2, 3, \ldots$; and $J_0\left(u_q r\right)$ and $J_1\left(u_q r\right)$ are the zero and first order Bessel functions. The boundary condition $E_{z,1}\left(b\right) = 0$ applied to Eq. (5) gives the equation $J_0\left(u_q b\right) = 0$ the roots of which define the transverse wave numbers u_q. The functions in Eqs. (5)–(7) satisfy to the boundary condition $E_{r,1}\left(z\right) = 0$ introduced at $z = L_d + t_d$. In each of Eqs. (5)–(7), the dependence on z without this condition should contain even functions $\cosh(\cdot)$ and odd functions $\sinh(\cdot)$ for the double CRSC.

We represent the field in coaxial volume 2 as the sum of the *TEM* modes and the spectrum of higher waveguide E_{0s}-modes of the coaxial line:

$$E_{z,2} = \sum_{s=1}^{\infty} C_s Z_0(\chi_s r) \cosh(\gamma_s z), \tag{8}$$

$$H_{\varphi,2} = jC_0 \frac{a \cos(kz)}{2\pi r} + j \sum_{s=1}^{\infty} \frac{k}{\chi_s} C_s Z_1(\chi_s r) \cosh(\gamma_s z), \tag{9}$$

$$E_{r,2} = C_0 \frac{a \sin(kz)}{2\pi r} - \sum_{s=1}^{\infty} \frac{\gamma_s}{\chi_s} C_s Z_1(\chi_s r) \sinh(\gamma_s z), \tag{10}$$

where C_0 and C_s are the amplitudes; χ_s and $\gamma_s = \sqrt{\chi_s^{\,2} - k^2}$ are the transverse and longitudinal wave numbers in volume 2;

$$Z_0\left(\chi_s r\right) = J_0\left(\chi_s r\right) - \frac{J_0\left(\chi_s a\right)}{N_0\left(\chi_s a\right)} N_0\left(\chi_s r\right),\ Z_1\left(\chi_s r\right) = J_1\left(\chi_s r\right) - \frac{J_0\left(\chi_s a\right)}{N_0\left(\chi_s a\right)} N_1\left(\chi_s r\right),\ s = 1, 2, 3, \ldots;$$

and $N_0\left(u_q r\right)$ and $N_1\left(u_q r\right)$ are the zero and first order Neumann functions. The boundary condition $E_{z,2}\left(b\right) = 0$ applied to Eq. (8) gives for χ_s the equation

$$Z_0\left(\chi_s b\right) = J_0\left(\chi_s b\right) - \frac{J_0\left(\chi_s a\right)}{N_0\left(\chi_s a\right)} N_0\left(\chi_s b\right) = 0.$$

The number of eigenwaves in calculation of the CRSC should be limited. We take Q eigenwaves for waveguide PV 1 and S eigenwaves (except the *TEM* wave) for coaxial PV 2. At $z = L_d$, the tangential field components in PVs 1 and 2 should be continuous: $E_{r,1}\left(L_d\right) = E_{r,2}\left(L_d\right)$ and $H_{\varphi,1}\left(L_d\right) = H_{\varphi,2}\left(L_d\right)$; then from Eqs. (6), (7), (9), and (10) we obtain the equations

$$\sum_{q=1}^{Q} \frac{\beta_q}{u_q} B_q J_1(u_q r) \sinh\left(\beta_q t_d\right) - C_0 \frac{a \sin\left(kL_d\right)}{2\pi r} + \sum_{s=1}^{S} \frac{\gamma_s}{\chi_s} C_s Z_1(\chi_s r) \sinh(\gamma_s L_d) = 0, \tag{11}$$

$$\sum_{q=1}^{Q}\frac{k}{u_q}B_qJ_1(u_qr)\cosh\left(\beta_qt_d\right)-C_0\frac{a\cos\left(kL_d\right)}{2\pi r}-\sum_{s=1}^{S}\frac{k}{\chi_s}C_sZ_1(\chi_sr)\cosh\left(\gamma_sL_d\right)=0\,. \tag{12}$$

Let us multiply Eq. (11) by $rJ_1\left(u_mr\right)$ ($m=1,2,\ldots,Q$) and integrate it over r within the limits $\left(0,b\right)$. We multiply Eq. (12) by $r\cdot 1/r=1$ corresponding to the *TEM* wave and integrate it over r within the limits $\left(a,b\right)$. Then we multiply Eq. (12) by $rZ_1\left(\chi_nr\right)$ ($n=1,2,\ldots,S$) and integrate it over r within the limits $\left(a,b\right)$. After integration, from Eq. (11) we obtain Q homogeneous linear equations, and from Eq. (12) we obtain 1+S linear homogeneous equations. Thus, Eqs. (11) and (12) lead to the system of $M=Q+1+S$ linear homogeneous equations for the amplitudes of the B_q, C_0, and C_s modes used to represent the field in PVs 1 and 2 of the single CRSCs. We designate $Q+1=p$; then the elements of the main matrix of the system assume the form

$$D_{i,j}=\begin{cases}\left(\beta_i/u_i\right)\sinh\left(\beta_it_d\right)\left[J_1(u_ib)\right]^2/2\,, & i=j,\\ 0\,, & i\neq j,\end{cases}\quad i=1,\ldots,Q\,,\; j=1,\ldots,Q\,,$$

$$D_{i,j}=-\frac{a\sin\left(kL_d\right)}{2\pi b}\frac{J_0\left(u_ia\right)}{u_ib}\,,\; i=1,\ldots,Q\,,\; j=p\,,$$

$$D_{i,j}=\left(\gamma_{j-p}/\chi_{j-p}\right)\sinh\left(\gamma_{j-p}L_d\right)\frac{\left(u_ia\right)J_0\left(u_ia\right)Z_1\left(\chi_{j-p}a\right)}{\left(u_ib\right)^2-\left(\chi_{j-p}b\right)^2}\,,\; i=1,\ldots,Q\,,\; j=p+1,\ldots,M\,,$$

$$D_{i,j}=\left(k/u_j\right)\cosh\left(\beta_jt_d\right)\frac{J_0\left(u_ja\right)}{u_jb}\,,\; i=p\,,\; j=1,\ldots,Q\,,$$

$$D_{i,j}=\frac{a\cos\left(kL_d\right)}{2\pi b}\ln\left(a/b\right)\,,\; i=p\,,\; j=p\,,$$

$$D_{i,j}=0\,,\; i=p\,,\; j=p+1,\ldots,M\;\text{ and }\; i=p+1,\ldots,M\,,\; j=p\,,$$

$$D_{i,j}=\left(k/u_{i-p}\right)\cosh\left(\beta_{i-p}t_d\right)\frac{\left(u_{i-p}a\right)J_0\left(u_{i-p}a\right)Z_1\left(\chi_ja\right)}{\left(u_{i-p}b\right)^2-\left(\chi_jb\right)^2}\,,\; i=p+1,\ldots,M\,,\; j=1,\ldots,Q\,,$$

$$D_{i,j}=\begin{cases}\left(k/\chi_{j-p}\right)\cosh\left(\gamma_{j-p}L_d\right)\dfrac{\left[Z_1\left(\chi_{j-p}a\right)\right]^2-\left[Z_1\left(\chi_{j-p}b\right)\right]^2}{2}, & i=j,\\ 0, & i\neq j,\end{cases}\quad i=p+1,\ldots,M\,,\; j=p+1,\ldots,M\,.$$

The determinant of the matrix D is a function of the frequency and vanishes at the resonant frequency. To calculate the double CRSC, it is necessary to find the resonant frequencies f_d and f_u (zeros of the determinants) of

TABLE 1. Sizes of the Double CRSC, mm

$2b$	$2a$	L_d	L_u	t	$L = L_d + t + L_u$
152.167	38.029	25.520	39.391	2.159	67.070

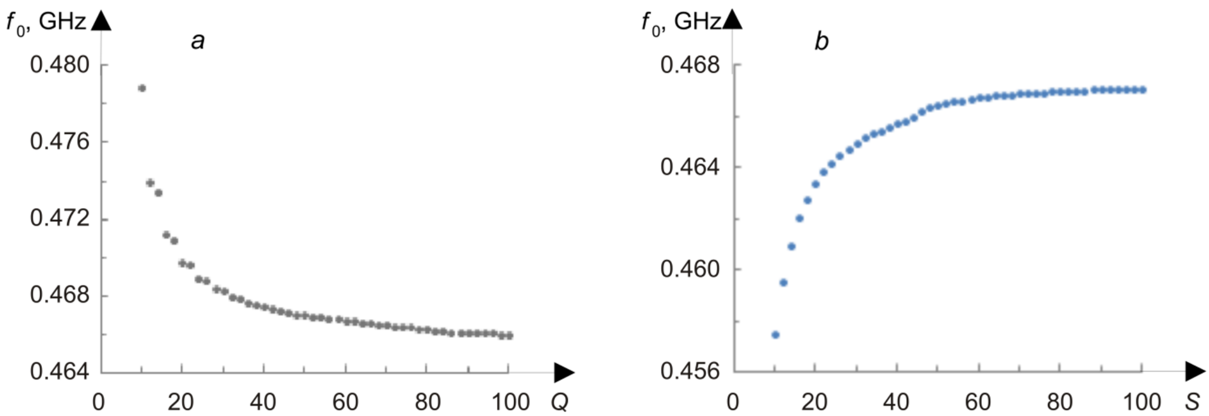

Fig. 2. Dependences of the resonant frequency on the number Q of waves in waveguide volume 1 (*a*) and on the number S of waves in the coaxial volume (*b*).

the single CRSCs and the t_d value at which $f_d = f_u$ .The results of quasi-stationary calculations are used as initial approximations.

The problem on the accuracy of the CRSC model is inseparable from the accuracy of the initial data – the internal sizes of the real CRSC, in particular, of the gap size t that most strongly affects the resonant frequency. The diameter $2a$ and the lengths L_d and L_u of the cylinders-electrodes were measured precisely with a micrometer for the disassembled resonator and remained unchanged after its assemblage. The internal diameter $2b$ of the cylindrical housing was determined from the spectrum of resonant frequencies of the volume resonator – initial CRSC – without central electrodes [10]. The most difficult and critical was direct measurement of the height of the gap between the electrodes in the assembled resonator. Separate measurements of L, L_d , and L_u in the disassembled resonator and calculation of $t = L - L_d - L_u$ for the assembled design did not provide the required accuracy. For the experimental assembled CRSC model, the upper electrode could be descended until it rested against the lower electrode ($t = 0$), and electrode lifting was accompanied by measuring its motion and fixing in the upper position. The gap t was measured with an electronic displacement meter with resolution of 1 μm. The sizes of the double CRSC are presented in Table 1.

The convergence of the calculated resonant frequency of the single CRSC was studied for the constant number S = 60 of waves in PV 2 and increasing number Q of waves in PV 1 (Fig. 2*a*) and for the number Q = 60 of waves in PV 1 and increasing number S of waves in PV 2 (Fig. 2*b*).

The calculated resonant frequencies of the single resonators f_d and f_u and their differences $\Delta f = f_d - f_u$ during measurements of the gap height t_d are given in Table 2. The experimental resonant frequency and the results of calculations by the quasi-stationary and PV methods are presented in Table 3.

From Tabes 2 and 3 it follows that there are points of equal frequencies of the single CRSCs with different L_d and L_u values and gaps t_d and $t_u = t - t_d$ for quasi-stationary approximation and electrodynamic analysis. The steepness of the resonant frequency was $2\,\Delta f_d / \Delta t_d \approx 25$ MHz/mm, which corresponded to the relative frequency shift $\Delta f_d / (f_d \cdot \Delta t_d) \approx 0.05\%$ and showed the importance of exact measurement of the gap size for estimation of the real accuracy of the calculation model. The difference between the calculated and experimental frequencies δf = 0.2% was mainly determined by the error in determining the t value.

TABLE 2. Dependence of the Calculated Frequencies f_d and f_u of the Single CRSCs on the Gap Height t_d ($Q = S = 70$)

t_d, mm	0.790976	0.790977	0.790978	0.790979	0.790980
f_d, GHz	0.4667079	0.4667081	0.4667084	0.4667086	0.4667088
f_u, GHz	0.4667086	0.4667085	0.4667083	0.4667082	0.4667081
Δf, kHz	–0.7	–0.4	0.1	0.4	0.7

TABLE 3. Experimental and Calculated Resonant Frequencies $f_{\exp}$ and f_{calc} together with Calculated t_{d0} and t_{u0} Values

$f_{\exp}$, GHz	Model	f_{calc}, GHz	δf, %	t_{d0}, mm	t_{u0}, mm
0.467648	Quasi-stationary	0.495473	5.9	0.801	1.358
	Multi-mode waveguide with PV	0.466708	0.2	0.79098	1.36802

The approach considered above with application to the double CRSC with a dielectric sample will allow the accuracy of measuring the dielectric parameters to be increased up to technical restrictions on the accuracy of the sample sizes and its position in the resonator. The reduction of the eigenmode number without loss of accuracy is connected with inclusion of the modes with continuous spectrum in the gap volume.

REFERENCES

1. J. Baker-Javis and B. F. Riddle, Dielectric Measurement Using Reentrant Cavity, NIST Technical Note 1384, (1996).
2. V. N. Egorov, Ya. O. Zuev, V. V. Kostromin, *et al*., Cables and Wires, No. 6, 12–14 (2017).
3. S. I. Orlov, Calculation and Design of Coaxial Resonators [in Russian], Radio i Svyaz', Moscow (1970).
4. T. S. Ryzhakova, Calculation and Investigation of Microwave Resonators Based on Cylindrical Stepped Structures, Publishing House of Nizhny Novgorod State Technical University, Nizhny Novgorod (2002).
5. R. G. Carter, J. Feng, and U. Becker, IEEE Trans. Microw. Theory Tech., **55**, No. 12, 2531–2538 (2007).
6. F. L. Penaranda-Foix, J. M. Catala-Civera, A. J. Canos-Martin, and B. Garcia-Banos, in: IEEE MTT-S Microwave Symp. Digest MTT' 09, Boston (2009), pp. 1309–1312.
7. D. Marques-Villarroya, F. Penaranda-Foix, B. Garcia-Banos, *et al*., Proc. 47th European Microwave Conf. EuMC 2017, Nuremberg (2017), pp. 440–443.
8. K. Yasushi, T. Tsukamoto, M. Miyakawa, and Y. Kashiwa, IEEE Trans. Microw. Theor. Techn., **36**, No. 4, 1750–1753 (2000).
9. W. Kazuki, K. Misunori, K. Kazuo, *et al*., in: Proc. IEEE 7th Int. Symp. on Medical Information and Communication Technology (ISMICT), IEEE (2013), pp. 238–241.
10. V. N. Egorov, E. Yu. Tokareva, and Le Quang Tuyen, Izmer. Tekh., No. 10, 65–72 (2020).