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ELEMENTARY PARTICLE PHYSICS AND FIELD THEORY

MASSLESS HIGHER SPIN SUPERMULTIPLETS WITH EXTENDED SUPERSYMMETRY

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We develop the Lagrangian description for the free N-extended massless higher spin supermultiplets in the four-dimensional anti-de Sitter space. The massless supermultiplets are formulated for $N \leq 4k$, where k is a maximal integer or half-integer spin in the multiplet. We found supertransformations connecting bosonic and fermionic components of the supermultiplets. It is shown that their algebra is closed on-shell. The supersymmetric invariant Lagrangians are constructed.

Keywords: massless fields, higher symmetry spins, anti-de Sitter space, supersymmetry.

INTRODUCTION

The theory of higher spin fields attracts considerable attention, because it enables new methods in classical and quantum field theory to be developed and is closely connected to superstring theory and AdS conformity. It is expected that in the context of the theory of higher spin fields, it would be possible to develop new approaches to a study of problems of combining fundamental interactions and quantum gravitation (for example, see reviews [1–3] and the references therein). Recently there has been a growing interest in the study of supersymmetric models of higher spins fields [4–17]. The present work solves the general problem of constructing the component Lagrangian formulation of free *N*-extended massless higher spin supermultiplets in the four-dimensional anti-de Sitter (4D AdS) space.

It is well known that in four dimensions, the *N*-extended massless supermultiplets with maximal spin $k = 1$ are restricted by the condition $N \leq 4$, and the supermultiplets with maximal spin $k = 2$ are restricted by the condition $N \leq 8$. The supermultiplets with $N > 8$ must contain higher spins $k > 2$. To be more precise, there is a specific relationship between the parameter N and the highest spin in the supermultiplet, $N \leq 4k$ (for example, see [18]).

For the case of simple $N = 1$ supersymmetry, the component Lagrangian formulation of higher spin supermultiplets in the Minkowski space has been known for a long time [19, 20], and the component approach has been generalized and studied in [21–24]. In particular, supertransformations were found that leave invariant the sum of Lagrangians for massless fields with spins k and $k + 1/2$. The off-shell Lagrangian formulation for such models was constructed within the framework of the superfield approach [25, 26]. Completely off-shell Lagrangian formulation for $N = 1$ highest superspins in the 4D AdS space has been developed in [27] in the superfield formalism, and its component form was derived from superfield theory. Quantization of this theory has been given in [28]. The $N = 2$ supersymmetric higher spin models both in the Minkovsky and AdS spaces were discussed in [29], the universal superfield approach to higher spin fields in the four-dimensional $N = 1$ AdS superspace has been developed in [30].

Recently, the on-shell superfield Lagrangian realization has been constructed for the extended $N = 1$ massless supermultiplets in the framework of the light-cone gauge formalism [16]. The extension of this approach to the on-shell

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N-extended supermultiplets has been given in [17] under the condition that $N = 4n$, where n is a natural number. In the present work, we give an explicit component Lagrangian construction of arbitrary N-extended mussless higher spin on-shell supermultiplets in the 4D AdS space.

Our construction is based on the frame-like approach for higher spin fields which generalizes the frame formulation of gravity. The generic scheme of the Lagrangian formulation for free bosonic and fermionic higher spin fields in this approach was developed in [31]. We find supertransformations for N-extended mussless higher spin supermultiplets that leave invariant the free Lagrangian system with integer and half-integer spins and demonstrate that the algebra of these supertransformations is on-shell closed.

In Section 1, we describe the basic elements of the frame-like Lagrangian formulation for free massless higher spin fields in the 4D AdS space and the 4D multispinor technique. In section 2, we present the minimal massless $N = 1$ supermultiplets [14] which are subsequently used as building blocks to construct the N-extended supermultiplets. Section 3 is devoted to the construction of arbitrary N-extended mussless higher spin supermultiplets in the 4D AdS space. In the Conclusion, the main results of our work are formulated.

1. FREE HIGHER SPIN FIELDS

In the frame-like approach, the massless fields with integer spin $k \geq 2$ are described by the dynamical 1-form $f^{\alpha(k-1)\dot{\alpha}(k-1)}$ and the auxiliary 1-form $\Omega^{\alpha(k)\dot{\alpha}(k-2)}$, $\Omega^{\alpha(k-2)\dot{\alpha}(k)}$ (all notations see in [32]). These fields are totally symmetric with respect to the dotted and undotted indices and generalize the tetrad field and the Lorentz connection in the frame formulation of gravity. We choose them to be real valued, that is, satisfying the rules of the Hermitian conjugation:

$$(f^{\alpha(k-1)\dot{\alpha}(k-1)})^{\dagger} = f^{\alpha(k-1)\dot{\alpha}(k-1)},$$

$$(\Omega^{\alpha(k)\dot{\alpha}(k-2)})^{\dagger} = \Omega^{\alpha(k-2)\dot{\alpha}(k)}.$$

The Lagrangian, being the differential 4-form in the 4D AdS space, is written in the form

$$\frac{(-1)^k}{i} L_k = k\Omega^{\alpha(k-1)\beta\dot{\alpha}(k-2)} E_{\beta}{}^{\gamma}\Omega_{\alpha(k-1)\gamma\dot{\alpha}(k-2)} - (k-2)\Omega^{\alpha(k)\dot{\alpha}(k-3)\dot{\beta}} E_{\dot{\beta}}{}^{\dot{\gamma}}\Omega_{\alpha(k)\dot{\alpha}(k-3)\dot{\gamma}}$$
$$+2\Omega^{\alpha(k-1)\beta\dot{\alpha}(k-2)} e_{\beta}{}^{\dot{\beta}} Df_{\alpha(k-1)\dot{\alpha}(k-2)\dot{\beta}}$$
$$+2k\lambda^2 f^{\alpha(k-2)\beta\dot{\alpha}(k-1)} E_{\beta}{}^{\gamma} f_{\alpha(k-2)\gamma\dot{\alpha}(k-1)} + \text{h.c.} \tag{1}$$

Here the 1-form $e^{\alpha\dot{\alpha}}$ is the AdS background tetrad, D is the AdS covariant derivative $De^{\alpha\dot{\alpha}} = 0$, and $E^{\alpha\beta}$, $E^{\dot{\alpha}\dot{\beta}}$ are double products of $e^{\alpha\dot{\alpha}}$ (see [32] for more details). The form of Lagrangian (1) is determined by the invariance under gauge transformations

$$\delta f^{\alpha(k-1)\dot{\alpha}(k-1)} = D\xi^{\alpha(k-1)\dot{\alpha}(k-1)} + e_{\beta}{}^{\dot{\alpha}}\eta^{\alpha(k-1)\beta\dot{\alpha}(k-2)} + e^{\alpha}{}_{\dot{\beta}}\eta^{\alpha(k-2)\dot{\alpha}(k-1)\dot{\beta}},$$

$$\delta\Omega^{\alpha(k),\dot{\alpha}(k-2)} = D\eta^{\alpha(k),\dot{\alpha}(k-2)} + \lambda^2 e^{\alpha}{}_{\dot{\beta}}\xi^{\alpha(k-1)\dot{\alpha}(k-2)\dot{\beta}},$$

$$\delta\Omega^{\alpha(k-2),\dot{\alpha}(k)} = D\eta^{\alpha(k-2),\dot{\alpha}(k)} + \lambda^2 e_{\beta}{}^{\dot{\alpha}}\xi^{\alpha(k-2)\beta\dot{\alpha}(k-1)}.$$

The remarkable property of the frame-like formulation is that it enables one to construct the gauge invariant objects which generalize the curvature and torsion in the theory of gravity:

$$T^{\alpha(k-1)\dot\alpha(k-1)} = Df^{\alpha(k-1)\dot\alpha(k-1)} + e_\beta{}^{\dot\alpha}\Omega^{\alpha(k-1)\beta\dot\alpha(k-2)} + e^\alpha{}_{\dot\beta}\Omega^{\alpha(k-2)\dot\alpha(k-1)\dot\beta},$$

$$R^{\alpha(k),\dot\alpha(k-2)} = D\Omega^{\alpha(k),\dot\alpha(k-2)} + \lambda^2 e^\alpha{}_{\dot\beta} f^{\alpha(k-1)\dot\alpha(k-2)\dot\beta},$$

$$R^{\alpha(k-2),\dot\alpha(k)} = D\Omega^{\alpha(k-2),\dot\alpha(k)} + \lambda^2 e_\beta{}^{\dot\alpha} f^{\alpha(k-2)\beta\dot\alpha(k-1)}.$$

To simplify the construction of the supermultiplets, we do not introduce any supertransformations for the auxiliary fields Ω. Instead, all calculations are performed up to the terms proportional to the auxiliary field equations of motion. As a matter of fact, this is equivalent to the following *zero torsion conditions*:

$$T^{a(k-1)} \approx 0 \quad \Rightarrow \quad e_\beta{}^{\dot\alpha} R^{\alpha(k-1)\beta\dot\alpha(k-2)} + e^\alpha{}_{\dot\beta} R^{\alpha(k-2)\dot\alpha(k-1)\dot\beta} \approx 0 . \tag{2}$$

As to the supertransformations for the dynamic fields f, the corresponding variation of the Lagrangian can be written compactly as follows:

$$(-1)^k \delta L_k = -i2R^{\alpha(k-1)\beta\dot\alpha(k-2)} e_\beta{}^{\dot\beta} \delta f_{\alpha(k-1)\dot\alpha(k-2)\dot\beta} + \text{h.c.}$$

Now let us turn to massless fields with half-integer spin $k+1/2 \geq 3/2$ described by the 1-form $\Phi^{\alpha(k)\dot\alpha(k-1)}$, $\Phi^{\alpha(k-1)\dot\alpha(k)}$. To be Majorana spinors, these fields should be real:

$$(\Phi^{\alpha(k)\dot\alpha(k-1)})^\dagger = \Phi^{\alpha(k-1)\dot\alpha(k)}.$$

The corresponding Lagrangian has the form

$$(-1)^k L_{k+\frac{1}{2}} = \Phi_{\alpha(k-1)\beta\dot\alpha(k-1)} e^\beta{}_{\dot\beta} D\Phi^{\alpha(k-1)\dot\alpha(k-1)\dot\beta} + \varepsilon_{k+\frac{1}{2}} \frac{\lambda}{2}\Big[(k+1)\Phi_{\alpha(k-1)\beta\dot\alpha(k-1)} E^\beta{}_\gamma \Phi^{\alpha(k-1)\gamma\dot\alpha(k-1)}$$

$$-(k-1)\Phi_{\alpha(k)\dot\alpha(k-2)\dot\beta} E^{\dot\beta}{}_{\dot\gamma} \Phi^{\alpha(k)\dot\alpha(k-2)\dot\gamma} + \text{h.c.}\Big]. \tag{3}$$

It can be shown that the Lagrangian is invariant under the gauge transformations

$$\delta\Phi^{\alpha(k)\dot\alpha(k-1)} = D\xi^{\alpha(k)\dot\alpha(k-1)} + e_\beta{}^{\dot\alpha}\eta^{\alpha(k)\beta\dot\alpha(k-2)} + \varepsilon_{k+\frac{1}{2}}\lambda e^\alpha{}_{\dot\beta}\xi^{\alpha(k-1)\dot\alpha(k-1)\dot\beta},$$

$$\delta\Phi^{\alpha(k-1)\dot\alpha(k)} = D\xi^{\alpha(k-1)\dot\alpha(k)} + e^\alpha{}_{\dot\beta}\eta^{\alpha(k-2)\dot\alpha(k)\dot\beta} + \varepsilon_{k+\frac{1}{2}}\lambda e_\beta{}^{\dot\alpha}\xi^{\alpha(k-1)\beta\dot\alpha(k-1)},$$

where $\varepsilon_{k+\frac{1}{2}} = \pm 1$. Note that the above consideration does not fix the sign of $\varepsilon_{k+\frac{1}{2}}$. As in the case of the integer spin, we can construct the gauge invariant curvatures of the form

$$F^{\alpha(k)\dot\alpha(k-1)} = D\Phi^{\alpha(k)\dot\alpha(k-1)} + \varepsilon_{k+\frac{1}{2}}\lambda e^{\alpha}{}_{\dot\beta}\Phi^{\alpha(k-1)\dot\alpha(k-1)\dot\beta},$$

$$F^{\alpha(k-1)\dot\alpha(k)} = D\Phi^{\alpha(k-1)\dot\alpha(k)} + \varepsilon_{k+\frac{1}{2}}\lambda e_{\beta}{}^{\dot\alpha}\Phi^{\alpha(k-1)\beta\dot\alpha(k-1)}.$$

Direct calculations show that the variation of Lagrangian (3) can be written as follows:

$$(-1)^k \delta L_{k+\frac{1}{2}} = -F_{\alpha(k-1)\beta\dot\alpha(k-1)} e^{\beta}{}_{\dot\beta}\delta\Phi^{\alpha(k-1)\dot\alpha(k-1)\dot\beta} + \text{h.c.}$$

Both in the bosonic and fermionic cases, the variation of the free higher spin field Lagrangian is completely expressed in geometric terms.

2. MINIMAL $N = 1$ SUPERMULTIPLETS

In this Section, we present the minimal massless $N = 1$ higher spin supermultiplets in the 4D AdS space. In the next Sections, they play the role of building blocks for the construction of the extended supermultiplets.

S**upermultiplet** $(k+1/2, k)$ contains two massless fields, one with spin k and another with spin $k+1/2$. They are described by the fields

$$f^{\alpha(k-1)\dot\alpha(k-1)}, \quad \Omega^{\alpha(k)\dot\alpha(k-2)}, \quad \Omega^{\alpha(k-2)\dot\alpha(k)} \text{ and } \Phi^{\alpha(k)\dot\alpha(k-1)}, \quad \Phi^{\alpha(k-1)\dot\alpha(k)},$$

respectively. The supertransformations that connect these fields are written in the form

$$\delta f^{\alpha(k-1)\dot\alpha(k-1)} = \alpha\Phi^{\alpha(k-1)\beta\dot\alpha(k-1)}\zeta_{\beta} - \bar\alpha\Phi^{\alpha(k-1)\dot\alpha(k-1)\dot\beta}\zeta_{\dot\beta},$$

$$\delta\Phi^{\alpha(k)\dot\alpha(k-1)} = \beta\Omega^{\alpha(k)\dot\alpha(k-2)}\zeta^{\dot\alpha} + \gamma f^{\alpha(k-1)\dot\alpha(k-1)}\zeta^{\alpha},$$

$$\delta\Phi^{\alpha(k-1)\dot\alpha(k)} = \bar\beta\Omega^{\alpha(k-2)\dot\alpha(k)}\zeta^{\alpha} + \bar\gamma f^{\alpha(k-1)\dot\alpha(k-1)}\zeta^{\dot\alpha},$$

where $\alpha, \beta,$ and γ are complex parameters, and the parameters $\zeta^{\alpha}, \zeta^{\dot\alpha}$ of the $N = 1$ supertransformations satisfy to the conditions

$$D\zeta^{\alpha} = -\lambda e^{\alpha}{}_{\dot\beta}\zeta^{\dot\beta}, \quad D\zeta^{\dot\alpha} = -\lambda e_{\beta}{}^{\dot\alpha}\zeta^{\beta}. \tag{4}$$

Note that here and below we do not introduce any supertransformations for the auxiliary field Ω since calculations are performed up to equations of motion (2) for this field.

The invariance of the Lagrangian under supertransformations, $\delta\left(L_k + L_{k+\frac{1}{2}}\right) = 0$, requires restrictions on the coefficients $\alpha, \beta,$ and γ:

$$\alpha = i\frac{(k-1)}{4}\bar\beta, \quad \gamma = \lambda\beta, \quad \beta = \varepsilon_{k+\frac{1}{2}}\bar\beta, \quad \varepsilon_{k+\frac{1}{2}} = \pm 1.$$

The remaining free parameter β can be either purely real or purely imaginary. In the AdS space, it relates the sign of the mass-like term for the fermionic field and the parity of the bosonic field. Two cases $\varepsilon_{k+\frac{1}{2}} = +1/-1$ correspond to different $N=1$ massless supermultiplets with parity even/odd boson. To fix the parameter β, we calculate the commutator of two supertransformations on the bosonic field

$$\frac{1}{\rho}[\delta_1,\delta_2]f^{\alpha(k-1)\dot{\alpha}(k-1)} = \Omega^{\alpha(k-1)\beta\dot{\alpha}(k-2)}\xi_\beta{}^{\dot{\alpha}} + \Omega^{\alpha(k-2)\dot{\alpha}(k-1)\dot{\beta}}\xi^\alpha{}_{\dot{\beta}}$$

$$+\lambda(f^{\alpha(k-2)\beta\dot{\alpha}(k-1)}\eta^\alpha{}_\beta + f^{\alpha(k-1)\dot{\alpha}(k-2)\dot{\beta}}\eta^{\dot{\alpha}}{}_{\dot{\beta}}), \quad (5)$$

where

$$\xi_\beta{}^{\dot{\alpha}} = i(\zeta_1^{\dot{\alpha}}\zeta_{2\beta} - \zeta_2^{\dot{\alpha}}\zeta_{1\beta}), \quad \eta^\alpha{}_\beta = i(\zeta_1^\alpha\zeta_{2\beta} - \zeta_2^\alpha\zeta_{1\beta}), \quad (6)$$

$$\rho = \frac{(k-1)}{4}\bar{\beta}\beta .$$

We can see that the commutator of these supertransformations is a combination of the translation with the parameter $\xi^{\alpha\dot{\alpha}}$ and of the Lorentz rotation with the parameter $\eta^{\alpha\beta}$, $\eta^{\dot{\alpha}\dot{\beta}}$. This means that two corresponding supercharges Q_α, $Q_{\dot{\alpha}}$ satisfy to the commutation relations of the $N=1$, AdS superalgebra

$$\{Q_\alpha, Q_{\dot{\beta}}\} \sim P_{\alpha\dot{\beta}},$$

$$\{Q_\alpha, Q_\beta\} \sim \lambda M_{\alpha\beta},$$

$$\{Q_{\dot{\alpha}}, Q_{\dot{\beta}}\} \sim \lambda M_{\dot{\alpha}\dot{\beta}},$$

where $P_{\alpha\dot{\alpha}}, M_{\alpha(2)}$, and $M_{\dot{\alpha}(2)}$ are the AdS generators.

Supermultiplet $(k, k-1/2)$ contains massless fields with integer spin k and half-integer spin $k-1/2$. The corresponding fields are

$$f^{\alpha(k-1)\dot{\alpha}(k-1)}, \quad \Omega^{\alpha(k)\dot{\alpha}(k-2)}, \quad \Omega^{\alpha(k-2)\dot{\alpha}(k)}$$

and

$$\Phi^{\alpha(k-1)\dot{\alpha}(k-2)}, \quad \Phi^{\alpha(k-2)\dot{\alpha}(k-1)}.$$

Supertransformations under the equations of motion for the auxiliary field Ω given by formula (2) are written in the form

$$\delta f^{\alpha(k-1)\dot{\alpha}(k-1)} = \alpha'\Phi^{\alpha(k-1)\dot{\alpha}(k-2)}\zeta^{\dot{\alpha}} - \bar{\alpha}'\Phi^{\alpha(k-2)\dot{\alpha}(k-1)}\zeta^\alpha,$$

$$\delta\Psi^{\alpha(k-1)\dot{\alpha}(k-2)} = \beta'\Omega^{\alpha(k-1)\beta\dot{\alpha}(k-2)}\zeta_\beta + \gamma' f^{\alpha(k-1)\dot{\alpha}(k-2)\dot{\beta}}\zeta_{\dot{\beta}},$$

$$\delta\Psi^{\alpha(k-2)\dot\alpha(k-1)} = \bar\beta'\Omega^{\alpha(k-2)\dot\alpha(k-1)\dot\beta}\zeta_{\dot\beta} + \bar\gamma' f^{\alpha(k-2)\beta\dot\alpha(k-1)}\zeta_\beta .$$

The Lagrangian invariance under these transformations, $\delta\left(L_k + L_{k-\frac{1}{2}}\right) = 0$, gives

$$\alpha' = \frac{i}{4(k-1)}\bar\beta', \quad \gamma' = \lambda\beta', \quad \beta' = \varepsilon_{k-\frac{1}{2}}\bar\beta', \quad \varepsilon_{k-\frac{1}{2}} = \pm 1 .$$

Again the free parameter β' can be either real or purely imaginary. This corresponds to two different $N = 1$ massless supermultiplets with parity even/odd boson. Calculating the commutator of two supertransformations and requiring algebra (5), we fix β':

$$\rho = \frac{1}{4(k-1)}\bar\beta'\beta' .$$

3. *N*-EXTENDED SUPERMULTIPLETS

In this Section, we consider the massless *N*-extended higher spin supermultiplets in the 4D AdS space. As pointed out in the Introduction, the parameter N satisfies to the condition $N \le 4k$ for the given maximal integer or half-integer spin k in the supermultiplet. For each spin k, we describe the field contents in the supermultiplet and introduce the corresponding field variables. Then we derive the supertransformations and determine the Lagrangian as a sum of the Lagrangians for all fields with integer and half-integer spins in the supermultiplet. It is shown that such Lagrangian is invariant under these supertransformations. Finally, we proof that the constructed supertransformations form the on-shell closed *N*-extended 4D AdS superalgebra.

We restrict ourselves to the consideration of the case $N \le 2k-3$ when the extended supermultiplet contains the massless fields with the following spins:

$$k, k-\frac{1}{2}, k-1, \ldots, k-\frac{N-1}{2}, k-\frac{N}{2},$$

where *k* is an integer or half-integer. We will write it compactly as $k-\frac{m}{2}, \quad m = 0,1,\ldots,N$. The number of massless fields with the given spin $k-\frac{m}{2}$ is equal to $\frac{N!}{m!(N-m)!}$. It is easy to note that the minimal spin equals to $\frac{3}{2}$ in the boundary case $N = 2k-3$. So all massless fields entering into the extended supermultiplet have been uniformly described in Section 1.

Let us introduce the bosonic field variables

$$f_{k-\frac{m}{2},i[m]}^{\alpha\left(k-\frac{m+2}{2}\right)\dot\alpha k\left(-\frac{m+2}{2}\right)}, \quad \Omega_{k-\frac{m}{2},i[m]}^{\alpha k\left(-\frac{m}{2}\right)\dot\alpha\left(k-\frac{m+4}{2}\right)}$$

and the fermionic variables

$$\Phi_{k-\frac{m}{2},i[m]}^{\alpha\left(k-\frac{m+1}{2}\right)\dot{\alpha}\left(k-\frac{m+3}{2}\right)}.$$

Here the first subscript denotes the spin of the field, the compact index $i[m]=[i_1 i_2 ... i_m]$ denotes the antisymmetric combination of indexes $i=1, 2, ..., N$ and corresponds to the antisymmetric representation of the internal symmetry group $SO(N)$. If the maximal spin k is integer, then m takes even values $0, 2, ..., 2\left[\frac{N}{2}\right]$ for the bosonic fields and odd values $1, 3, ..., 2\left[\frac{N-1}{2}\right]+1$ for the fermionic fields. In the case of maximal half-integer spin k, the parameter m takes even values for the fermions and odd values for bosons.

The generic ansatz for the linear supertransformations with a set of arbitrary complex coefficients $\alpha_m, \alpha'_m, \beta_m, \beta'_m, \gamma_m$, and γ'_m is chosen in the following form:

$$\delta f_{k-\frac{m}{2},i[m]}^{\alpha\left(k-\frac{m+2}{2}\right)\dot{\alpha}\left(k-\frac{m+2}{2}\right)} = \alpha'_m \Phi_{k-\frac{m+1}{2},i[m]j}^{\alpha\left(k-\frac{m+2}{2}\right)\dot{\alpha}\left(k-\frac{m+4}{2}\right)} \zeta^{j\dot{\alpha}} - \bar{\alpha}'_m \Phi_{k-\frac{m+1}{2},i[m]j}^{\alpha\left(k-\frac{m+4}{2}\right)\dot{\alpha}\left(k-\frac{m+2}{2}\right)} \zeta^{j\alpha}$$

$$+\alpha_m \Phi_{k-\frac{m-1}{2},i[m-1]}^{\alpha\left(k-\frac{m+2}{2}\right)\beta\dot{\alpha}\left(k-\frac{m+2}{2}\right)} \zeta_{i\beta} - \bar{\alpha}_m \Phi_{k-\frac{m-1}{2},i[m-1]}^{\alpha\left(k-\frac{m+2}{2}\right)\dot{\alpha}\left(k-\frac{m+2}{2}\right)\dot{\beta}} \zeta_{i\dot{\beta}}, \qquad (7)$$

$$\delta\Phi_{k-\frac{m}{2},i[m]}^{\alpha\left(k-\frac{m+1}{2}\right)\dot{\alpha}\left(k-\frac{m+3}{2}\right)} = \beta_m \Omega_{k-\frac{m+1}{2},i[m]j}^{\alpha\left(k-\frac{m+1}{2}\right)\dot{\alpha}\left(k-\frac{m+5}{2}\right)} \zeta^{j,\dot{\alpha}} + \beta'_m \Omega_{k-\frac{m-1}{2},i[m-1]}^{\alpha\left(k-\frac{m+1}{2}\right)\beta\dot{\alpha}\left(k-\frac{m+3}{2}\right)} \zeta_{i,\beta}$$

$$+\gamma_m f_{k-\frac{m+1}{2},i[m]j}^{\alpha\left(k-\frac{m+3}{2}\right)\dot{\alpha}\left(k-\frac{m+3}{2}\right)} \zeta^{j,\alpha} + \gamma'_m f_{k-\frac{m-1}{2},i[m-1]}^{\alpha\left(k-\frac{m+1}{2}\right)\dot{\alpha}\left(k-\frac{m+3}{2}\right)\dot{\beta}} \zeta_{i,\dot{\beta}}. \qquad (8)$$

Here ζ_i^{α}, $\zeta_i^{\dot{\alpha}}$ are the parameters of the extended supertransformations satisfying conditions (4). The Lagrangian is defined as $L=\sum_m L_{k-\frac{m}{2}}$, where $L_{k-\frac{m}{2}}$ is the Lagrangian for the free field with spin $k-\frac{m}{2}$. The invariance of the Lagrangian under these supertransformations leads to restrictions on the arbitrary parameters

$$\alpha_m = \frac{i\left(k-\frac{m+2}{2}\right)}{4}\bar{\beta}_{m-1}, \quad \gamma_m = \lambda\beta_m, \quad \beta_m = \varepsilon_{k-\frac{m}{2}}\bar{\beta}_m,$$

$$\alpha'_m = \frac{i}{4\left(k-\frac{m+2}{2}\right)}\bar{\beta}'_{m+1}, \quad \gamma'_m = \lambda\beta'_m, \quad \beta'_m = \varepsilon_{k-\frac{m}{2}}\bar{\beta}'_m,$$

entering into formulas (7) and (8). In these relations, $\varepsilon_{k-\frac{m}{2}}=+1$ or $\varepsilon_{k-\frac{m}{2}}=-1$ for any m depending on the parity of the corresponding bosonic fields. Two families of free parameters β_m and β'_m remain. To relate them with each other, we require the closure of the algebra of supertransformations (7) and (8). This yields the condition

$$\alpha_m\beta_{m-1}-\alpha'_m\beta'_{m+1}=0\,. \tag{9}$$

Calculation of the commutator for two supertransformations (7) yields the following result:

$$\frac{1}{\rho}[\delta_1,\delta_2]f_{k-\frac{m}{2},i[m]}^{\alpha\left(k-\frac{m+2}{2}\right)\dot\alpha\left(k-\frac{m+2}{2}\right)}=\Omega_{k-\frac{m}{2},i[m]}^{\alpha\left(k-\frac{m+2}{2}\right)\beta\dot\alpha\left(k-\frac{m+4}{2}\right)}\xi_\beta{}^{\dot\alpha}+\Omega_{k-\frac{m}{2},i[m]}^{\alpha(k-\frac{m+4}{2})\dot\alpha(k-\frac{m+2}{2})\dot\beta}\xi^\alpha{}_{\dot\beta}$$

$$+\lambda\left(f_{k-\frac{m}{2},i[m]}^{\alpha(k-\frac{m+2}{2})\dot\alpha(k-\frac{m+4}{2})\dot\beta}\eta_{\dot\beta}{}^{\dot\alpha}+f_{k-\frac{m}{2},i[m]}^{\alpha(k-\frac{m+4}{2})\beta\dot\alpha(k-\frac{m+2}{2})}\eta_\beta{}^\alpha\right)+\lambda f_{k-\frac{m}{2},i[m-1]j}^{\alpha(k-\frac{m+2}{2})\dot\alpha(k-\frac{m+2}{2})}z^j{}_i\,, \tag{10}$$

where

$$\xi_\beta{}^{\dot\alpha}=i(\zeta_{1j,\beta}\zeta_2^{j\dot\alpha}-\zeta_{2j,\beta}\zeta_1^{j\dot\alpha}),\quad \eta_{\dot\beta}{}^{\dot\alpha}=i(\zeta_{1j,\dot\beta}\zeta_2^{j\dot\alpha}-\zeta_{2j,\dot\beta}\zeta_1^{j\dot\alpha})\,, \tag{11}$$

$$z^j{}_i=i(\zeta_1^{j,\beta}\zeta_{2i\beta}-\zeta_2^{j,\beta}\zeta_{1i\beta}+\zeta_1^{j,\dot\beta}\zeta_{2i\dot\beta}-\zeta_2^{j,\dot\beta}\zeta_{1i\dot\beta}),\quad z^{ij}=-z^{ji}\,. \tag{12}$$

It is possible to demonstrate that the right-hand side of commutator (10) represents the combination of the Lorentz rotation and internal $SO(N)$ transformations with the parameters $\xi^{\alpha\dot\alpha}$, $\eta^{\alpha\beta}$, and z^{ij}, respectively. From (9) and (10) we have restriction on the parameters

$$\bar\beta_{m-1}\beta_{m-1}=\frac{4\rho}{\left(k-\dfrac{m+2}{2}\right)},\quad \bar\beta'_{m+1}\beta'_{m+1}=4\rho\left(k-\frac{m+2}{2}\right).$$

The form of the above commutator demonstrates that the supercharges $Q_\alpha{}^i$, $Q^i_{\dot\alpha}$ corresponding to supertransformations (7) and (8) satisfy to the commutation relations of the on-shell extended AdS superalgebra

$$\{Q^i_\alpha,Q^j_{\dot\beta}\}\sim\delta^{ij}P_{\alpha\dot\beta}\,,$$

$$\{Q^i_\alpha,Q^j_\beta\}\sim\lambda\left(\delta^{ij}M_{\alpha\beta}+\frac{1}{2}\varepsilon_{\alpha\beta}T^{ij}\right), \tag{13}$$

$$\{Q^i_{\dot\alpha},Q^j_{\dot\beta}\}\sim\lambda\left(\delta^{ij}M_{\dot\alpha\dot\beta}+\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}T^{ij}\right),$$

where $P_{\alpha\dot\alpha}, M_{\alpha(2)}$, and $M_{\dot\alpha(2)}$ are the AdS generators and $T^{ij}=-T^{ji}$ are the generators of internal $SO(N)$ group symmetry.

CONCLUSIONS

Let us briefly formulate the main results of the work. We have constructed the field realizations of arbitrary *N*-extended mussless supermultiplets in the four-dimensional anti-de-Sitter space. For arbitrary highest integer or half-integer spin k fields entering the supermultiplet, we realized the on-shell supersymmetric component Lagrangian under the condition $N < 2k - 3$. The corresponding supertransformations were found, and it was shown that they form the on-shell closed superalgebra and leave invariant the Lagrangian. It was shown that the commutators of two such supertransformations form the *N*-extended AdS superalgebra, that is, they are combinations of the translations, Lorentz rotations, and internal $SO(N)$ transformations.

To go beyond the framework of the above-considered cases for $N > 2k - 3$, the massless lowest spin fields should also be included into the supermultiplet. For example, for $N = 2k - 2$, it would be sufficient to add massless spins 1, and for $N = 2k - 1$ and $N = 2k$, to add the massless fields with spin $\frac{1}{2}$ and a set of complex fields with massless spin 0. An explicit realization for such supermultiplets requires separate consideration (see [32]).

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