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ON THE INTERNAL GEOMETRY OF TRAJECTORIES OF CHARGED PARTICLES IN SYMMETRIC EXTERNAL FIELDS

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The curvature and torsion of trajectories of charges in external gauge fields, including fields of magnetic monopoles, have been determined. It has been shown that these quantities are effectively calculated with the help of the equations of motion and first integrals. For a wide class of magnetic fields, their form-invariant combination has been revealed.

Keywords: curvature, torsion, non-Abelian monopole, Wong equations.

1. THE FRENET EQUATIONS

As is well known [1, 2], the internal geometry of the trajectory $\boldsymbol{x} = \boldsymbol{x}(t)$ of a classical particle, like any smooth curve $\boldsymbol{x} = \boldsymbol{x}(\ell)$ in Euclidean space, where ℓ is the arclength along the curve, is prescribed by two scalar parameters – the curvature k and the torsion χ, closely associated with the motion along it of the orthogonal Frenet frame via the corresponding Frenet equations for the tangent vector $\boldsymbol{\tau}$, the normal vector $\boldsymbol{\nu}$, and the binormal vector $\boldsymbol{\beta}$ to this curve (Fig. 1):

$$\boldsymbol{\tau} = \boldsymbol{x}' = \frac{d\boldsymbol{x}}{d\ell}, \; \boldsymbol{x}'' = \boldsymbol{\tau}' = k\boldsymbol{\nu}, \; \boldsymbol{\beta}' = -\chi\boldsymbol{\nu}, \; \boldsymbol{\nu}' = \chi\boldsymbol{\beta} - k\boldsymbol{\tau}, \tag{1}$$

for

$$k = |\boldsymbol{\tau}'|, \; \chi k^2 = \left(\boldsymbol{\tau}\left[\boldsymbol{\tau}' \times \boldsymbol{\tau}''\right]\right), \; \boldsymbol{\beta} = \left[\boldsymbol{\tau} \times \boldsymbol{\nu}\right], \; \boldsymbol{\nu} = \left[\boldsymbol{\beta} \times \boldsymbol{\tau}\right]. \tag{2}$$

Since instead of $\ell = \ell(t)$ it is possible here to use any scalar parameter t increasing along the curve [1, 2], for example, the time of motion, the impression arises [2] that to calculate k and χ it is necessary to know the explicit form of the trajectory $\boldsymbol{x}(t) = \boldsymbol{x}(\ell(t))$. In spherical components for

$$\boldsymbol{x} = r\boldsymbol{n}, \; r = |\boldsymbol{x}|, \; \mathbf{v} = \dot{\boldsymbol{x}} = r\dot{\boldsymbol{n}} + \dot{r}\boldsymbol{n} = \upsilon\boldsymbol{\tau}, \; \upsilon = |\mathbf{v}| = \dot{\ell}(t), \; \boldsymbol{a} = \ddot{\boldsymbol{x}} = \dot{\mathbf{v}} = \upsilon\dot{\boldsymbol{\tau}} + \dot{\upsilon}\boldsymbol{\tau}, \; (\dot{\boldsymbol{n}} \cdot \boldsymbol{n}) \equiv 0 \tag{3}$$

and so on, the curvature and torsion from Eqs. (2) in the nonrelativistic case can be represented in the form [2]

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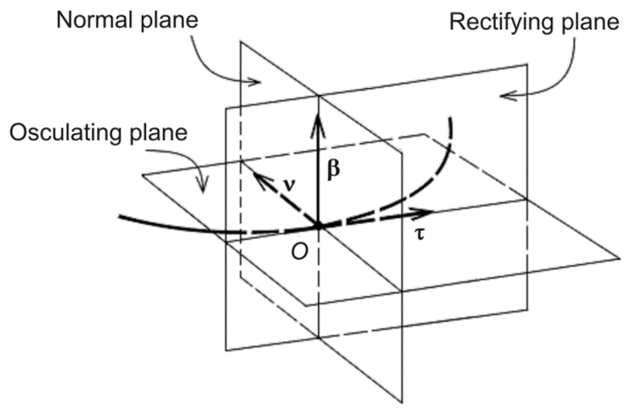

Fig. 1

$$k = \frac{\left|[\mathbf{v}\times \boldsymbol{a}]\right|}{|\mathbf{v}|^3} = \frac{|\dot{\boldsymbol{\tau}}|}{|\mathbf{v}|},\ \chi = \frac{\left(\mathbf{v}[\boldsymbol{a}\times\dot{\boldsymbol{a}}]\right)}{[\mathbf{v}\times\boldsymbol{a}]^2} = \frac{\left(\ddot{\boldsymbol{\tau}}[\boldsymbol{\tau}\times\dot{\boldsymbol{\tau}}]\right)}{|\mathbf{v}|\dot{\boldsymbol{\tau}}^2},\ \text{where } \dot{f} = \frac{df}{dt}. \tag{4}$$

We will show that for motion, including the relativistic case, in some spherically and axially symmetric external fields or magnetic fields, including non-Abelian monopoles, to find the curvature k or some form-invariant combination of k and χ, besides the field it is sufficient to know only the first integrals of the motion.

2. MOTION UNDER THE ACTION OF THE LORENTZ FORCE

As is well known [3–5], the Lorentz force $\boldsymbol{F}$ contains a magnetic component and an electric component: $\boldsymbol{F} = m\ddot{\boldsymbol{x}} = m\boldsymbol{a}$, such that

$$\boldsymbol{a} = \frac{e}{m}\left(\frac{1}{c}[\mathbf{v}\times\boldsymbol{B}] + \boldsymbol{E}\right) \text{ (c is the velocity of light).} \tag{5}$$

Upon substituting this equation of motion into relations (4), after some transformations we obtain

$$k = \frac{|e|}{m|\mathbf{v}|^3}\left|\frac{1}{c}[\mathbf{v}[\mathbf{v}\times\boldsymbol{B}]] + [\mathbf{v}\times\boldsymbol{E}]\right|,\ \chi = \frac{\frac{e}{mc}\left([\mathbf{v}\times\boldsymbol{a}]\{[\boldsymbol{a}\times\boldsymbol{B}] + [\mathbf{v}\times\dot{\boldsymbol{B}}]\}\right) + \frac{e}{m}\left(\mathbf{v}[\boldsymbol{a}\times\dot{\boldsymbol{E}}]\right)}{[\mathbf{v}\times\boldsymbol{a}]^2}. \tag{6}$$

Equation (5) for $\boldsymbol{B} = 0$ in a spherically symmetric potential field takes the form [3]

$$\boldsymbol{a} = -\frac{e}{m}U'(r)\frac{\boldsymbol{x}}{r} \text{ for } \boldsymbol{E} = -\nabla_{\boldsymbol{x}}U(r) = -U'(r)\boldsymbol{n}, \text{ where } \nabla_{\boldsymbol{x}}r = \boldsymbol{n}. \tag{7}$$

Inserting formulas (7) into Eqs. (6) and employing the laws of conservation of energy $2\varepsilon = m\boldsymbol{v}^2 + 2U = m\boldsymbol{v}_0^2$ and angular momentum $\boldsymbol{M} = m[\boldsymbol{x}\times\boldsymbol{v}] = mr^2[\boldsymbol{n}\times\dot{\boldsymbol{n}}]$ with $|\boldsymbol{M}| = m\upsilon_0 b = mr^2|\dot{\boldsymbol{n}}|$, where b is the impact parameter, we find

$$k(r)=\frac{|\boldsymbol{M}||U'(r)|}{r\sqrt{m}(2\varepsilon)^{3/2}}\left(1-\frac{U(r)}{\varepsilon}\right)^{-3/2}=\frac{b|U'(r)|}{2\varepsilon r}\left(1-\frac{U(r)}{\varepsilon}\right)^{-3/2}. \tag{8}$$

Thus, in spherically symmetric potential fields, the curvature of a trajectory as a function of r is uniquely assigned by the potential and the first integrals of motion – the energy and the angular momentum. The curve is planar since for such a potential torsion is absent: $\chi=0$ since $(\mathbf{v}[\boldsymbol{a}\times\dot{\boldsymbol{a}}])=(U'/mr)^2(\mathbf{v}[\boldsymbol{x}\times\boldsymbol{v}])=0$. For example, for the Coulomb potential [3] $U(r)=-\frac{\alpha}{r}$ we find

$$k(r)=\frac{\alpha b}{2\varepsilon r^3}\left(1+\frac{\alpha}{\varepsilon r}\right)^{-\frac{3}{2}}. \tag{9}$$

Let us now consider Eqs. (6) for the case $\boldsymbol{E}=0$ [4, 5]. Since $\boldsymbol{a}\perp\boldsymbol{v}$ and $\boldsymbol{a}\perp\boldsymbol{B}$ for $\boldsymbol{a}=\frac{e}{mc}[\mathbf{v}\times\boldsymbol{B}]$, it follows that

$$k=\frac{|\boldsymbol{a}|}{\mathbf{v}^2}=\frac{|e|}{mc}\frac{|[\mathbf{v}\times\boldsymbol{B}]|}{\mathbf{v}^2},\ \chi=-\frac{e}{mc}\frac{(\mathbf{v}\cdot\boldsymbol{B})}{\mathbf{v}^2}+\left(\frac{e}{mc}\right)^2\frac{(\mathbf{v}[\boldsymbol{B}\times\dot{\boldsymbol{B}}])}{\boldsymbol{a}^2}, \tag{10}$$

from which for $\boldsymbol{B}=|\boldsymbol{B}|\zeta$ with $\zeta^2=1$ we have

$$\chi=-\frac{e}{mc}\frac{(\mathbf{v}\cdot\boldsymbol{B})}{\mathbf{v}^2}+\left(\frac{e}{mc}\right)^2\frac{\boldsymbol{B}^2(\mathbf{v}[\zeta\times\dot{\zeta}])}{\boldsymbol{a}^2}. \tag{11}$$

If $\zeta\mapsto\boldsymbol{n}$, i.e., the magnetic field is, as it were, *spherically symmetric*, then by virtue of Eq. (3) the second term in the expression for the torsion vanishes: $(\mathbf{v}[\boldsymbol{n}\times\dot{\boldsymbol{n}}])=((r\dot{\boldsymbol{n}}+\dot{r}\boldsymbol{n})[\boldsymbol{n}\times\dot{\boldsymbol{n}}])=0$. This also happens if in Eqs. (10) and (11) the field $\boldsymbol{B}$ is everywhere constant in direction $\dot{\zeta}\mapsto 0$. In view of the fact that the force is always perpendicular to the velocity, $(\mathbf{v}\cdot\boldsymbol{a})=0$, the velocity in both cases is constant in magnitude, $m\mathbf{v}^2=2\varepsilon=m\mathbf{v}_0^2$, and the curvature and torsion for $\boldsymbol{p}=m\mathbf{v}$ satisfy the relation

$$k^2+\chi^2=\left(\frac{e}{mc}\right)^2\frac{[\mathbf{v}\times\boldsymbol{B}]^2+(\mathbf{v}\cdot\boldsymbol{B})^2}{(\mathbf{v}^2)^2}=\left(\frac{e}{mc}\right)^2\frac{\boldsymbol{B}^2}{\mathbf{v}^2}=\left(\frac{e}{c}\right)^2\frac{\boldsymbol{B}^2}{\boldsymbol{p}^2}\equiv\frac{1}{\Lambda^2}=\rho_0\frac{4\pi}{\varepsilon}\frac{dW}{dV}, \tag{12}$$

where

$$\rho_0=\frac{e^2}{mc^2},\ \frac{dW}{dV}=\frac{\boldsymbol{B}^2}{8\pi},\ \frac{1}{\Lambda}=\frac{|e|}{mc}\frac{|\boldsymbol{B}|}{|\mathbf{v}|}\equiv\frac{\omega}{|\mathbf{v}|} \tag{12a}$$

are respectively the classical electromagnetic radius of the charge e with mass m, and volume density of the field energy [4, 5], but since ω is the cyclotron frequency, Λ is the radius of the circular trajectory for $\chi=0$. Thus, in the (k,χ) *plane* for $\chi\neq 0$ we have a circle of radius $1/\Lambda$.

3. MOTION IN A FIELD OF MAGNETIC MONOPOLES

The *hedgehog* of the magnetic field of an infinitely heavy monopole with magnetic charge g, located at the origin, can be represented from the classical point of view as outgoing from the initial point of one semi-infinite thin solenoid – a string stretched in some direction (Dirac), or from the initial point for two such symmetric solenoids (Schwinger) [6–9]. The equation of motion (Eq. (5)) of the electric charge e and the field $\boldsymbol{B}$ of the magnetic monopole have the form

$$\boldsymbol{a} = \frac{Q}{mr^3}\left[\mathbf{v}\times\boldsymbol{x}\right] \text{ for } \boldsymbol{B} = \frac{g\boldsymbol{n}}{r^2},\ Q = \frac{eg}{c},\ \tan\theta = \frac{|\boldsymbol{M}|}{|Q|} = \frac{m|\mathbf{v}|b}{|Q|} = \text{const},\ |\mathbf{v}| = \upsilon_0 . \tag{13}$$

Here 2θ is the opening angle of the cone around which the charge trajectory is wound [6, 9], with axis along the conserved total angular momentum vector $\boldsymbol{J} = \boldsymbol{M} - Q\boldsymbol{n} = |\boldsymbol{J}|\boldsymbol{\kappa}$, and b is the conserved (in modulus) impact parameter. Substituting expressions (13) into Eqs. (10) and (11) for the curvature and torsion, we find

$$k(r) = \frac{|Q|\left|[\mathbf{v}\times\boldsymbol{x}]\right|}{mr^3\mathbf{v}^2} = \frac{b^2\cot\theta}{r^3},\ \chi(r) = -\frac{Q(\mathbf{v}\cdot\boldsymbol{x})}{mr^3\mathbf{v}^2} = -\frac{b\cot\theta}{r^2}\sqrt{1-\left(\frac{b}{r}\right)^2} . \tag{14}$$

Squaring these expressions and adding them together, we again obtain in agreement with relations (12) and (13)

$$k^2 + \chi^2 = \left(\frac{Q}{mr^2|\mathbf{v}|}\right)^2 = \left(\frac{e}{c}\right)^2\frac{\boldsymbol{B}^2}{\boldsymbol{p}^2} . \tag{15}$$

By virtue of formula (8), the curvature $\tilde{k}(r)$ of the trajectory for the monopole problem projected onto the plane $\boldsymbol{R}\perp\boldsymbol{\kappa}$ [9] as a function of distance $r = |\boldsymbol{x}| = R = |\boldsymbol{R}|$ in it with $\boldsymbol{R}\sin\theta = \boldsymbol{x} - \boldsymbol{\kappa}(\boldsymbol{x}\cdot\boldsymbol{\kappa})$ falls off more rapidly:

$$\tilde{k}(R) = \frac{|\boldsymbol{J}|Q^2}{R^4(2\varepsilon m)^{3/2}}\left(1+\frac{Q^2}{2\varepsilon mR^2}\right)^{-3/2} \text{ for } m\ddot{\boldsymbol{R}} = -\nabla_{\boldsymbol{R}}U(R) \text{ for } U(R) = -\frac{Q^2}{2mR^2} . \tag{14a}$$

The relativistic generalization of formulas (4) to the case of non-Abelian external fields requires the use of Wong's classical relativistic equations of motion [5, 10–13]. Taking the metric of space $x^\mu = (x^0, \boldsymbol{x})$ as before to be flat and choosing as the parameter $\ell(\tau)$ along the trajectory of a (spinless) particle the proper time τ or the interval ds [4, 5] for $d\boldsymbol{x} = \mathbf{v}dt$

$$(ds)^2 = (cd\tau)^2 = dx^\mu dx_\mu = c^2(dt)^2 - (d\boldsymbol{x})^2 = \frac{c^2(dt)^2}{\gamma^2}, \text{ i.e., } \frac{d}{ds} = \frac{\gamma}{c}\frac{d}{dt} = u^\mu\partial_\mu , \tag{16}$$

for the 4-velocity

$$u^\mu = \dot{x}^\mu = \frac{dx^\mu}{ds} = \frac{p^\mu}{mc} = (u^0, \boldsymbol{u}) = \gamma\left(1, \frac{\mathbf{v}}{c}\right) \text{ for } u^\mu u_\mu = u^2 \equiv 1 , \tag{17}$$

with 4-momentum

$$p^{\mu} = mc\dot{x}^{\mu} = \left(\frac{E}{c}, \boldsymbol{p}\right) \text{ and energy } E = p^0 c = \gamma mc^2 = \sqrt{\left(mc^2\right)^2 + \boldsymbol{p}^2 c^2}\ , \tag{18}$$

and also for the 4-acceleration

$$w^{\mu} = \dot{u}^{\mu} = \left(w^0, \boldsymbol{w}\right) = \frac{du^{\mu}}{ds} = \frac{d^2 x^{\mu}}{ds^2} \equiv \ddot{x}^{\mu} \text{ for } w^{\mu} u_{\mu} \equiv 0\,, \tag{19}$$

we obtain expressions for the curvature and torsion that are analogous to expressions (4) with the substitutions $\mathbf{v} \to \boldsymbol{u}$ and $\boldsymbol{a} \to \boldsymbol{w}$:

$$k = \frac{\left\|\left[\boldsymbol{u} \times \boldsymbol{w}\right]\right\|}{\left|\boldsymbol{u}\right|^3},\ \chi = \frac{\left(\boldsymbol{u}\left[\boldsymbol{w} \times \dot{\boldsymbol{w}}\right]\right)}{\left[\boldsymbol{u} \times \boldsymbol{w}\right]^2}, \text{ i.e., } \chi k^2 = \frac{\left(\boldsymbol{u}\left[\boldsymbol{w} \times \dot{\boldsymbol{w}}\right]\right)}{\left(\boldsymbol{u}^2\right)^3}. \tag{20}$$

The dot in Eqs. (17)–(20) now denotes the operation of differentiation (Eq. (16)) with respect to s, and the transformation to the nonrelativistic limit $c \Rightarrow \infty$ for $\gamma \Rightarrow 1$ is given by the reverse substitution $\boldsymbol{u} \Rightarrow \mathbf{v}$, $\boldsymbol{w} \Rightarrow \boldsymbol{a}$.

Wong's system of equations [5, 10–13] describes motion in an external non-Abelian gauge field A_a^{μ} with tensions $G_a^{\mu\nu}$ both of a classical test particle $x^{\mu}\left(s\right)$ and of its intrinsic classical *color* isospin $\vec{I}\left(s\right)$ with components $\eta_a\left(s\right)$ in the internal *color* isospin space with fixed basis $\vec{e}_a$. In the case of the gauge group *SU*(2) with generators $\left(T^a\right)_{bc} = -i\varepsilon^{abc}$ in the adjoint representation[1] for the covariant derivative $\hat{D}^{\mu} = \mathbf{1}\partial^{\mu} - ie\hat{A}^{\mu}$ with gauge field $\hat{A}^{\mu} = A_a^{\mu} T^a$, scalar field $\hat{h} = h_a T^a$, and unit matrix $\mathbf{1}$ in the same representation [5, 6, 11, 15] for $\hbar = c = 1$ (Latin indices $j,k,l; a,b,c = 1,2,3$ are assumed to be upper and convolve according to the Euclidean metric) and

$$e\hat{G}^{\mu\nu} = eG_a^{\mu\nu} T^a = i\left[\hat{D}^{\mu}, \hat{D}^{\nu}\right], \text{where } G_a^{\mu\nu} = \partial^{\mu} A_a^{\nu} - \partial^{\nu} A_a^{\mu} + e\varepsilon^{abc} A_b^{\mu} A_c^{\nu},\ \partial^{\mu} = \left(\partial^0, -\nabla\right), \tag{21}$$

$$\left(\hat{D}^{\mu}\hat{h}\right)_a \equiv \left[\hat{D}^{\mu}, \hat{h}\right]_a = \partial^{\mu} h_a + e\varepsilon^{abc} A_b^{\mu} h_c, \text{ for } \hat{I} = \eta_a T^a \mapsto \vec{I} = \eta_a \vec{e}_a,\ 2\eta_a = \mathrm{Sp}\left\{\hat{I}\,T^a\right\}, \tag{22}$$

these equations have the form

$$mc^2 w^{\mu} = -eG_a^{\mu\nu} \eta_a u_{\nu},\ u_{\mu}\left(\hat{D}^{\mu}\hat{I}\right)_a = \frac{d\eta_a}{ds} + e\varepsilon^{abc} A_b^{\mu} u_{\mu} \eta_c = 0\,. \tag{23}$$

The gauge $A_a^0\left(\boldsymbol{x}\right) = 0$ is responsible for the absence of a *chromo-electric* field $E_a^j = -G_a^{0j} = 0$ in the stationary case $\partial^0 \to 0$ in the rest frame of reference [16] of this classical field [5, 6, 11]. The *chromo-magnetic* field $G_a^{jk} = -\varepsilon^{jkl} B_a^l$ in the remaining gauge-invariant combination $\breve{\boldsymbol{B}} = \boldsymbol{B}_a \eta_a$ with components of the color isospin (Eqs. (22)) reduces Eqs. (23) to the form [5, 11–13]

$$w^0 = 0\,,\ mc^2 \boldsymbol{w} = e\left[\boldsymbol{u} \times \breve{\boldsymbol{B}}\right],\ \dot{\eta}_a = e\varepsilon^{abc}\left(\boldsymbol{A}_b \cdot \boldsymbol{u}\right)\eta_c, \text{ where } \left(\boldsymbol{w} \cdot \boldsymbol{u}\right) = 0, \text{ i.e., } \boldsymbol{u}^2 = \text{const}\,. \tag{24}$$

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[1] For the general case of compact groups and different representations of the field, see [11, 14–16].

Similarly to the transformation from relations (4) and (6) to relations (10) and (11), we obtain from relations (20) relativistic expressions for k and χ:

$$k=\frac{|\boldsymbol{w}|}{\boldsymbol{u}^2}=\frac{|e|}{mc^2}\frac{\left\|\left[\boldsymbol{u}\times\breve{\boldsymbol{B}}\right]\right\|}{\boldsymbol{u}^2},\ \chi=-\frac{e}{mc^2}\frac{\left(\boldsymbol{u}\cdot\breve{\boldsymbol{B}}\right)}{\boldsymbol{u}^2}+\left(\frac{e}{mc^2}\right)^2\frac{\left(\boldsymbol{u}\cdot\left[\breve{\boldsymbol{B}}\times\dot{\breve{\boldsymbol{B}}}\right]\right)}{\boldsymbol{w}^2}, \tag{25}$$

from which for $\breve{\boldsymbol{B}}=\left|\breve{\boldsymbol{B}}\right|\zeta$ with $\zeta^2=1$ we have

$$\chi=-\frac{e}{mc^2}\frac{\left(\boldsymbol{u}\cdot\breve{\boldsymbol{B}}\right)}{\boldsymbol{u}^2}+\left(\frac{e}{mc^2}\right)^2\frac{\breve{\boldsymbol{B}}^2\left(\boldsymbol{u}\left[\zeta\times\dot{\zeta}\right]\right)}{\boldsymbol{w}^2}. \tag{26}$$

Since the 3-velocity $\boldsymbol{u}$ in Eqs. (17) will be given by the same formula (3) for $\mathbf{v}$, but with the operation of differentiation with respect to s (Eq. (16)), it follows that for both cases, $\dot{\zeta}\mapsto 0$ and $\zeta\mapsto\boldsymbol{n}$, where $\left(\boldsymbol{u}\left[\boldsymbol{n}\times\dot{\boldsymbol{n}}\right]\right)=0$, and taking relation (18) into account, we again arrive at the relativistic non-Abelian generalization of the form-invariant (expression (15)):

$$k^2+\chi^2=\left(\frac{e}{mc^2}\right)^2\frac{\breve{\boldsymbol{B}}^2}{\boldsymbol{u}^2}=\left(\frac{e}{c}\right)^2\frac{\breve{\boldsymbol{B}}^2}{\boldsymbol{p}^2}=\frac{e^2\breve{\boldsymbol{B}}^2}{E^2-m^2c^4},\ \breve{\boldsymbol{B}}=\boldsymbol{B}_a\eta_a. \tag{27}$$

In the nonrelativistic limit $E\Rightarrow mc^2+\varepsilon$ this expression reverts to expression (12) with nonrelativistic 3-momentum $\boldsymbol{p}$ and nonrelativistic energy ε. According to Eqs. (23) and (24), $\vec{I}^2=\eta_a\eta_a=\mathrm{const}$, i.e., the dynamics of the classical *color* angular momentum $x^\mu(s)$, covariantly conserved along the trajectory, reduces to its rotation in color space in the form of precession about some direction with variable frequency and axis of rotation [5, 10–13]. According to [13], this dynamics, tangled up with the trajectory dynamics, nevertheless admits a *decoupling* of the form shown in Eqs. (27), thanks to the effective abelization in Eqs. (24)–(27) for $\zeta\mapsto\boldsymbol{n}$ or $\dot{\zeta}\mapsto 0$ of the interaction of the *color* angular momentum with the *chromo-magnetic* field. In particular, it takes place in the nonrelativistic approximation, where for the 4-velocity (expression (17)) $u^\mu\Rightarrow(1,\mathbf{0})$, $d\tau\Rightarrow dt$, and in the gauge $A_a^0=0$, from Eqs. (23) we obtain

$$\frac{d\eta_a}{dt}=-e\varepsilon^{abc}A_b^0\eta_c\Rightarrow 0,\ \text{i.e., }\eta_a\Rightarrow\mathrm{const}=\delta^{a3}\eta_3=\delta^{a3}\sqrt{\vec{I}^2}, \tag{28}$$

which corresponds to for the effective abelization $e\breve{\boldsymbol{B}}\Rightarrow q\boldsymbol{B}_3$ with the obviously thusly conserved Abelian electric charge $q=e\eta_3$ [5], which in this case is generated by the *color* charge e and the proper *color* isospin of the particle.

Another example of abelization is provided by the Abelian projection of the 't Hooft–Polyakov monopole solution in the Georgi–Glashow model with spontaneous breaking [5, 6, 9, 11–17] of the symmetry arising out of the Lagrangian density, by the corresponding equations of motion and the vacuum mean value:

$$L=-\frac{1}{4}G_a^{\mu\nu}G_{\mu\nu}^a+\frac{1}{2}\left(\hat{D}^\mu\hat{h}\right)_a\left(\hat{D}_\mu\hat{h}\right)_a-\frac{\lambda}{4}\left(h_ah_a-\mathrm{f}^2\right)^2\ \text{for }\langle 0|h_3|0\rangle=\mathrm{f}, \tag{29}$$

$$\left(\hat{D}_\nu\hat{G}^{\mu\nu}\right)_a=e\varepsilon^{abc}h_b\left(\hat{D}^\mu\hat{h}\right)_c,\ \left(\hat{D}_\mu\hat{D}^\mu\hat{h}\right)_a=-\lambda h_a\left(h_bh_b-\mathrm{f}^2\right). \tag{29a}$$

The components h_1 and h_2 of the scalar field are absorbed by the longitudinal components of the charged vector fields A_1^μ and A_2^μ, which thereby acquire mass $M^2 = e^2 \mathrm{f}^2$ [6, 9, 15]. The *photon* field A_3^μ, which remains massless, is responsible for the residual *U*(1) gauge symmetry necessary for the existence in this theory of Abelian magnetic monopoles [6, 14, 15] and associated with the remaining freedom of rotation about the third axis in color space defined by the vacuum mean value in expression (29).

In the gauge $A_a^0 = 0$, $(\boldsymbol{n} \cdot \boldsymbol{A}_a) = 0$, the classical solution is prescribed by the functions $K(r)$ and $H(r)$:

$$\boldsymbol{x} = r\boldsymbol{n}, \; A_a^j(\boldsymbol{x}) = -\varepsilon^{jka}\frac{n^k}{er}\left[1 - K(r)\right], \; h_a(\boldsymbol{x}) = n^a \mathrm{f}\left[1 - S(r)\right] \equiv n^a \frac{H(r)}{er}, \tag{30}$$

$$G_a^{jk} = -\varepsilon^{jkl}B_a^l(\boldsymbol{x}) = \frac{\varepsilon^{jkl}}{er}\left\{\frac{n^l n^a}{r}\left[K^2(r) - 1\right] + \left(\delta^{la} - n^l n^a\right)\frac{dK(r)}{dr}\right\}, \; B_a^l = -\frac{1}{2}\varepsilon^{jkl}G_a^{jk}, \tag{31}$$

which obey well-known equations following from the field equations (Eqs. (29a)) [5, 6, 9, 16, 17]

$$r^2\frac{d^2K}{dr^2} = K\left[K^2 + H^2 - 1\right], \; r^2\frac{d^2H}{dr^2} = H\left\{\frac{\lambda}{e^2}\left[H^2 - (Mr)^2\right] + 2K^2\right\} \tag{32}$$

with boundary conditions in the limit $r \to 0$:

$$K(r) \to 1 + O\left(r^2\right) \text{ and } S(r) \to 1 + O(r), \tag{33}$$

and in the limit $r \to \infty$:

$$K(r) \sim e^{-Mr}, \text{ a } S(r) \sim e^{-2Mr} \text{ if } \lambda > 0, \; S(r) \sim 1/(Mr) \text{ if } \lambda = 0. \tag{34}$$

The corresponding gauge-invariant Abelian magnetic field, acting on the ordinary Abelian electric charge $q \mapsto e$, is determined by the scalar projection $\bar{G}_3^{jk}$ of the tensor assigned by formula (31) onto the direction n^a of the vacuum classical Higgs field (expression (30)) and the 't Hooft added-term [6, 9, 15, 16]:

for
$$\bar{G}_3^{jk} = \frac{h_a G_a^{jk}}{\left(h_b h_b\right)^{1/2}} = n^a G_a^{jk} = -\frac{\varepsilon^{jkl}n^l}{er^2}\left[1 - K^2(r)\right] \underset{r\to\infty}{\to} -\varepsilon^{jkl}\bar{B}^l, \tag{35}$$

$$F^{jk} = \bar{G}_3^{jk} - \frac{\varepsilon^{abc}h_a}{e\left(h_d h_d\right)^{3/2}}\left(\hat{D}^j\hat{h}\right)_b\left(\hat{D}^k\hat{h}\right)_c = -\varepsilon^{jkl}\bar{B}^l, \text{ где } \bar{\boldsymbol{B}}(\boldsymbol{x}) = \frac{\boldsymbol{n}}{er^2}, \tag{36}$$

in complete agreement with Eqs. (13) and (15). For pure Yang–Mills theory with Lagrangian (29) without the scalar field h_a, we obtain the Wu–Yang solution [6], created by the purely longitudinal field B_a^l in Eq. (31) for $K \equiv 0$ and leading to the same form-invariant (15)$\mapsto$(27) with field (13)$\mapsto$(36) for $g = \hbar c/e$. Here, the role of an effective Abelian charge of the test particle instead of the quantity indicated by Eqs. (28) is played by the gauge-invariant quantity q, the latter being the *classical* limit [5] of the corresponding effective charge operator $\hat{q}$ for the quantity $e\breve{\boldsymbol{B}}$, closely associated [6, 9, 11, 12, 15–18] with the limit in Eqs. (22):

$$e\breve{\boldsymbol{B}} = q\bar{\boldsymbol{B}} \text{ for } \hat{q} = en^a T^a \mapsto q = en^a \eta_a \text{ for } 2q = \mathrm{Tr}\left\{\hat{q}\hat{I}\right\}. \tag{37}$$

The stationary **ss** gauge field of general type [15–17] if no additional discrete symmetries removing the last two terms are imposed, is decomposed into three **ss**-structures, mutually orthogonal in both the configuration space and the color space:

$$A_a^j(\boldsymbol{x}) = \varepsilon^{jka} n^k \frac{\gamma(r)}{er} - n^j n^a \frac{\alpha(r)}{er} - \left(\delta^{ja} - n^j n^a\right)\frac{\beta(r)}{er}, \text{ where } \left(\boldsymbol{n}\cdot\boldsymbol{A}_a\right) = -n^a \frac{\alpha(r)}{er} \neq 0, \tag{38}$$

where the second longitudinal structure is orthogonal to the two transverse ones individually in each space. The condition of spherical symmetry of the **ss**-vector field (Eqs. (38)) implies its invariance [11] under simultaneous spatial rotation $\mathrm{R}(\boldsymbol{g})$ and global gauge rotation $\mathrm{V}(\boldsymbol{g})$ with the same rotation parameters from the diagonal subgroup $\boldsymbol{g} \in SU(2)_{\mathrm{d}}$ [17] of rotations in the configuration and color spaces: $\hat{\boldsymbol{A}}(\boldsymbol{x}) = \mathrm{V}(\boldsymbol{g})\left[\mathrm{R}(\boldsymbol{g})\hat{\boldsymbol{A}}\left(\mathrm{R}^{-1}(\boldsymbol{g})\boldsymbol{x}\right)\right]\mathrm{V}^{-1}(\boldsymbol{g})$, and is fulfilled independently for each of these **ss**-structures. In the corresponding orthogonal expansion over them of the strength (Eqs. (31)) of the chromomagnetic field (Eq. (21)), the longitudinal contribution does not depend on the gauge function $\alpha(r)$ and is determined only by the functions on the transverse structures in Eqs. (38), whereas the transverse contributions contain it here and the prime indicates the derivative with respect to r:

$$B_a^j(\boldsymbol{x}) = \frac{n^j n^a}{er^2}\left[1 - (1+\gamma)^2 - \beta^2\right] - \frac{\left(\delta^{ja} - n^j n^a\right)}{er^2}\left[r\gamma' + \alpha\beta\right] - \frac{\varepsilon^{jka} n^k}{er^2}\left[r\beta' - \alpha(1+\gamma)\right]. \tag{39}$$

It would seem that the four functions remaining unknown in the fields assigned by formulas (38), (39), and (30) should be determined by the four equations following from Eqs. (29a). However, gauge invariance leaves one of them arbitrary since for $1+\gamma = Y(r)$ we have the system of equations

$$\begin{gathered}
r^2 \frac{d}{dr}\left(\frac{rY' + \alpha\beta}{r}\right) = Y\left(Y^2 + \beta^2 + H^2 - 1\right) - \alpha\left(r\beta' - \alpha Y\right), \\
r^2 \frac{d}{dr}\left(\frac{r\beta' - \alpha Y}{r}\right) = \beta\left(Y^2 + \beta^2 + H^2 - 1\right) + \alpha\left(rY' + \alpha\beta\right), \\
\beta\left(rY' + \alpha\beta\right) = Y\left(r\beta' - \alpha Y\right), \\
r^2 \frac{d^2 H}{dr^2} = H\left\{\frac{\lambda}{e^2}\left[H^2 - (Mr)^2\right] + 2\left(Y^2 + \beta^2\right)\right\},
\end{gathered} \tag{40}$$

which reverts to Eqs. (32) for α and $\beta = 0$ and can also be obtained by variation of the corresponding energy functional [15]. The third equation immediately gives upon substitution in it of the functions

$$Y(r) = K(r)\cos\omega(r),\ \beta(r) = K(r)\sin\omega(r) \text{ for } Y^2 + \beta^2 = K^2 > 0, \text{ that } r\partial_r\,\omega(r) = \alpha(r). \tag{41}$$

Thus, the fourth and any of the first two of Eqs. (40) lead again to ***the same system of equations*** (Eqs. (32)) for the functions K and H. The *chromo-magnetic* field B_a^j (formula (39)) versus the case $\omega = 0$ (Eqs. (31)) takes the form

$$er^2 B_a^j(\boldsymbol{x}) = n^j n^a\left[1-K^2\right] - \left(\delta^{ja} - n^j n^a\right) rK' \cos\omega - \varepsilon^{jka} n^k rK' \sin\omega\,. \tag{42}$$

It can be seen that exactly the longitudinal contribution, which is of particular interest to us, does not depend on the remaining arbitrary gauge $\omega(r)$, i.e., $\alpha(r)$, and the condition of *complete* spherical symmetry of this field and likewise disappearance of both terms transverse to $n^j n^a$ coincides with the condition of its gauge independence. Thus, Eqs. (31) and (42) lead again to the longitudinal non-Abelian Wu–Yang field with the arbitrary constant $K(r) \to C \geq 0$ and to its subsequent effective abelization (Eqs. (37)) to a field of the form assigned by Eqs. (36), leading thus again to the non-Abelian form-invariant assigned by Eqs. (27), where now

$$B_a^j(\boldsymbol{x}) = \frac{n^j n^a}{er^2}\left[1-C^2\right],\ \bar{\boldsymbol{B}}(\boldsymbol{x}) = \frac{\boldsymbol{n}}{er^2}\left[1-C^2\right] \text{ for } C>0 \tag{43}$$

$$\text{or for } K = C = 0 \text{ when } Y = \beta = 0\,. \tag{44}$$

However, the first of Eqs. (32) for $C>0$ reduces to the condition $H^2 = 1 - C^2$. Thus, the second of Eqs. (32) has meaning only in the Prasad–Sommerfeld limit $\lambda = 0$ [6, 16], where it gives the condition $2C^2\sqrt{1-C^2} = 0$, whose unique solution $C^2 = 1$ assigns the zero strengths $h_a, B_a^j = 0$. The particular solution $K(r) = C = 0$ of the first of Eqs. (32) leads to the particular solution $H(r) = Mr$ of the second of Eqs. (32) with nonzero fields h_a and B_a^j, possessing correct behavior in the limit $r \to \infty$ and correct topological properties [6, 15, 16]. According to the boundary conditions assigned by Eqs. (34) and (35), in this asymptotic limit properties (43), (27), (37), and (36) will be possessed by all of the solutions of Eqs. (32) obeying them.

On the other hand, by virtue of Eqs. (16) and (22), the antisymmetry of the structure constants, and the Wong equation (Eq. (23)), for any operator of the form $\hat{q} = \varphi_a T^a$ the derivative with respect to s of the Abelian charge $q = \varphi_a \eta_a$ is equal to

$$\dot{q} = \frac{dq}{ds} = \frac{1}{2} u_\mu \partial^\mu \mathrm{Tr}\left\{\hat{q}\hat{I}\right\} = \varphi_a u_\mu \left(\hat{D}^\mu \hat{I}\right)_a + \eta_a u_\mu \left(\hat{D}^\mu \hat{q}\right)_a = \frac{1}{2} u_\mu \mathrm{Tr}\left\{\hat{D}^\mu \hat{q}\hat{I}\right\}, \tag{45}$$

and covariant conservation along the trajectory $u_\mu\left(\hat{D}^\mu \hat{q}\right)_a = 0$ entails conservation $\dot{q} = 0$ of the Abelian charge of the particle as a direct generalization of the above-mentioned conservation of the square of its classical color angular momentum $\vec{I}^2$. For the charge operator assigned by Eqs. (37) in the field assigned by formula (38) we have $\varphi_a = en^a$ and $u_\mu\left(\hat{D}^\mu \hat{q}\right)_a \mapsto Y(r)\dot{n}^a + \beta(r)\varepsilon^{abc}\dot{n}^b n^c$, which also independently of the form of the gauge function $\alpha(r)$ vanishes only for $Y = \beta = 0$, again leading to a non-Abelian Wu–Yang monopole (42)$\to$(43) for $K(r) \to C = 0$ [11, 12, 17, 19]. Note that the result of Abelian projection (Eqs. (35) and (36)) with the same effective Abelian magnetic field $\bar{\boldsymbol{B}}(\boldsymbol{x})$ in Eqs. (36) [9, 15] is generally independent of the form of the functions $H(r)$, $\alpha(r)$, $\gamma(r)$, and $\beta(r)$ in the expressions for the fields (expressions (30), (38), and (39)). That is to say, we are talking here about substantially different pathways to abelization.

CONCLUSIONS

In this work we have shown that for motion of a classical particle in some axially and spherically symmetric external fields, to find the curvature k of its trajectory or a form-invariant combination of k and the torsion χ it is sufficient to know, in addition to the external field, only the first integrals of motion. The universal form-invariant combination assigned by Eq. (27) has been revealed in the relativistic case for a wide class of magnetic fields, including fields of non-Abelian monopoles. According to [13], in all of the considered cases an effective abelization of the interaction with the Yang–Mills external field takes place, under which the gauge-invariant effective Abelian charge of the particle q (Eqs. (37)), generally speaking, is not necessarily conserved. We have shown that it is precisely conserved in some solutions of the equations of motion for fields of magnetic monopoles or is, at least, an asymptotically conserved quantity [9, 11, 12]. We emphasize that this Abelian electric charge is generated here by the initial *color* charge e and the proper *color* isospin of the particle. Therefore, quantization of this charge is closely associated with quantization of its *color* angular momentum [9].

Taking into account that the second Wong equation (Eqs. (23)) for $\hat{I}(s)$ coincides in essence with the equation for the Wilson line [17, 20] in the adjoint representation, one may hope for the existence of a more universal approach to *decoupling* the trajectory $x^{\mu}(s)$ dynamics from the internal $\eta^{a}(s)$ dynamics of the classical motion in non-Abelian external fields by using closed Wilson lines [16] traced out by the classical color angular momentum vector on the sphere $\vec{I}^{\,2}=\mathrm{const}$, for which $\vec{I}(s)=\eta^{a}(s)\vec{e}_{a}=\vec{I}(s+\Delta\overline{s})$ for some $\Delta\overline{s}$. But then $\hat{I}(s+\Delta\overline{s})=\hat{I}(s)$, transforming gaugewise only globally – the same as for the field strength (Eqs. (21)): $\hat{I}(s)\to \mathrm{V}(x(s))\hat{I}(s)\mathrm{V}^{-1}(x(s))$ with an arbitrary point $x(s)$ from a segment of the particle trajectory corresponding to any interval $\Delta\overline{s}$, meaning from an arbitrary point on the trajectory $x(s)$ if, for given initial conditions on the Wong equations (Eqs. (23) and (24)), $\Delta\overline{s}$ exists as a unique finite period of such a kind.

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